Ring Theory Review

Monday, May 18, 2020 3:26 PM

7) $R = \mathbb{Q}[x]/(x-1)(x+2)$

$$I$$
$$\parallel$$

a) Ideals in R? $\longleftrightarrow$ J in $\mathbb{Q}[x]$ containing $\langle(x-1)(x+2)\rangle$

- $\mathbb{Q}[x]$ is a E.D, so a PID

so $J = \langle p(x) \rangle$

Since $\langle (x-1)(x+2) \rangle \subseteq \langle p(x) \rangle$

$p(x) \mid (x-1)(x+2)$

$J = \mathbb{Q}[x]$, $J = I$, $J = \langle x-1 \rangle$, $J = \langle x+2 \rangle$.

Ideals in R: R, 0, $\langle x-1 \rangle + I$, $\langle x+2 \rangle + I$

b) Find S, s.t. $\varphi: R \twoheadrightarrow S$.

1st isom thm: $R/\ker\varphi \cong S$

S determined by $\ker\varphi$: $\ker\varphi = R$: $\varphi = 0$, $S = 0$

$\ker\varphi = 0$: $\varphi = id$, $S = R$

$\ker\varphi = \langle x-1 \rangle + I$: $S \cong \dfrac{\mathbb{Q}[x]/(x-1)(x+2)}{(x-1)/(x-1)(x+2)} \cong \mathbb{Q}[x]/(x-1) \cong \mathbb{Q}$

(3rd isom thm)

$\ker\varphi = \langle x+2 \rangle + I$: $S \cong \mathbb{Q}$ (same as above)

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8) R is UFD, if $\underline{I = \langle a, b \rangle}$, then $I = \langle c \rangle$.

WTS: R is PID

Idea: $c = \gcd(a, b)$.

Goal: If $I \subset R$, show $I = \langle a, b \rangle$.

- $I = \langle a_1, \dots, a_n \rangle$. Induct on n.

$I = \langle a_1, a_2 \rangle + \langle a_3, \dots, a_n \rangle$

By hypothesis $\langle a_1, a_2 \rangle = \langle c \rangle$, so $I = \langle c, a_3, \dots, a_n \rangle, \dots$, so I is principal.

Q: Does I have to be fin. gen.?

A: No. $R = \mathbb{Q}[x_1, \dots, x_n, \dots]$

$I = (x_2, x_3, \dots)$, but any $f \in R$ lies $\mathbb{Q}[x_1, \dots, x_n]$

not fin. gen

a UFD

- this does not have 2^{nd} prop, see (x_1, x_2)

Hint: UFD's satisfy ACC for principal ideals

$(x_1) \subset (x_2) \subset (x_3) \subset \dots \subset (x_n) \subset \dots$

then $(\cancel{x_n}) = (x_{n+k})$ $\forall k$

Suppose $I = (a_1, \dots, a_n, \dots)$

$(a_1) \subseteq (a_1, a_2) \subseteq (a_1, a_2, a_3) \subseteq \dots \subseteq I$

By above, each fin gen ideal is principle

$(c_1) \subseteq (c_2) \subseteq (c_3) \subseteq \dots \subseteq I$

So $\exists n$ where the chain stabilizes: $I = (c_n) = (\gcd(a_1, \dots, a_n))$

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10) a) $\mathbb{Z}[i]$ is an ED w/ norm $a^2 + b^2$

It is enough to show $\bar{i} = \inf$, since then $(i = f(x))$

$\overline{a+bi} = f(a) + b \cdot f(x) = f(a + bx)$

$\overline{a+bi} = f(a) + b \cdot f(x) = f(a+bx)$

Need $x - i \in (3+2i)$, so $13 \mid x^2+1$. In particular, $x = -8$ works:

$13 \mid 65$ and $-8-i = (3+2i)(-2+i)$

so $f(-8) = i$,

so $\overline{a+bi} = f(a-8b)$.

this says for $8-i, -8-i$ either $3+2i$ or $3-2i$ is a divisor

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b) $13\mathbb{Z} \subset \ker f$, b/c $13 = (3+2i)(3-2i)$

if $a \in \ker f$, $a \in (3+2i)$, so $3+2i \mid a$,

so $N(3+2i) \mid N(a)$,

$\Downarrow$

13

$N(a) \in 13\mathbb{Z}$, $N(a) = a^2$, so $13 \mid a$, so $a \in 13\mathbb{Z}$

c) Use 1st Isom:

Alt: do b) as above, then c):

$\exists q, r \in \mathbb{Z}[i]$ s.t. $(a+bi) = q \cdot (3+2i) + r$, $N(r) < 13$

r is unique

We know $\mathbb{Z}/13\mathbb{Z} \cong \operatorname{im} f \subseteq \mathbb{Z}[i]/(3+2i)$.

$N(a+bi) < 13$: up to units $(\pm 1, \pm i)$

not ideal b/c doesn't generalize well.

There are exactly 13 $a+bi$ w/

$0, 1, 2, 3, i, 2i, 3i, 1+i, 1+2i, 1+3i, 2+2i, 2+i, 3+i$

Since every coset has a unique rep w/ $N(r) < 13$, we have $|\mathbb{Z}[i]/(3+2i)| = 13$,

... ...ctive

Since every coset has a unique rep w/ $N(r) < 13$, we

a) So $\operatorname{im} f = \mathbb{Z}[i]/(3+2i)$, so f is surjective.