ALGEBRA PRELIM REVIEW

Ring Theory

(1) (N18) Find all ring homomorphisms $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$.

(2) (N20) Convince yourself that the set $R := \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is differentiable}\}$ is a commutative ring with the usual sum and product of functions. Prove:
  (a) The set $I := \{f \in R \mid f(5) = f'(5) = 0\}$ is an ideal of R.
  (b) $\mathbb{R}[x]/(x^2) = \{(a + bx) \pmod{x^2} \mid a, b \in \mathbb{R}\}$.
  (c) The rings R/I and $\mathbb{R}[x]/(x^2)$ are isomorphic.

(3) (N21) Let I and J be ideals of a ring R and let $\pi : R \to R/I$ be the canonical epimorphism. Show:
  (a) $\pi(J)$ is an ideal in R/I (it is denoted $(J + I)/I$).
  (b) π induces an inclusion-preserving bijection

$$\{J \mid J \text{ is an ideal of } R \text{ such that } I \subset J\} \to \{\text{ideals of } R/I\}.$$

(4) (N25) Let I be an ideal in a ring R, and let $\varphi : R \to S$ be a ring homomorphism. Prove that φ factors through the canonical epimorphism $\pi : R \to R/I$, i.e., there is a ring homomorphism $\psi : R/I \to S$ such that $\varphi = \psi \circ \pi$, if and only if $I \subset \ker(\varphi)$.

(5) (N27) Let p be a fixed prime and consider the set

$$R := \left\{\frac{a}{b} \mid a, b \in \mathbb{Z} \text{ are coprime}, \ p \nmid b\right\}.$$

  (a) Show that R is a subring of $\mathbb{Q}$ and not a field.
  (b) Determine the set of units $R^\times$.
  (c) Show that each non-zero ideal of R is principal and of the form (p^e) for some $e \in \mathbb{N}_0$.

(6) (N31) Let I be an ideal of a ring R. If R is a PID, show that every ideal of R/I is principal. Is the converse true?

(7) (N32) Consider the ring $R := \mathbb{Q}[x]/(x - 1)(x + 2)$.
  (a) Determine all ideals of R.
  (b) Find (up to isomorphism) all rings S such that there is a surjective ring homomorphism $R \to S$.

(8) (N38) Let R be a factorial domain with the property that every ideal that is generated by two elements is a principal ideal. Prove the R must be a PID.

(9) (N39,41) Show that the following polynomials are irreducible in the given ring:
  (a) $2x^4 + 200x^3 + 40x^2 + 2000x + 20 \in \mathbb{Q}[x]$
  (b) $(y+8)^2x^3 - x^2 + (y+7)(y+8) - y - 12 \in \mathbb{Q}[x, y]$
  (c) $x^2y + xy^2 - x - y + 1 \in \mathbb{Q}[x, y]$
  (d) $2 + i \in \mathbb{Z}[i]$ (as an element)
  (e) $x^n - 2 - i \in \mathbb{Q}(i)[x]$ for every positive integer n (you may use that the quotient field of $\mathbb{Z}[i]$ is $\mathbb{Q}(i)$).

(10) (5/15) Consider the ring $\mathbb{Z}[i]$ if Gaussian integers and let f be the ring homomorphism

$$f : \mathbb{Z} \to \mathbb{Z}[i]/(3 + 2i),\ c \mapsto c + (3 + 2i).$$

Show the following:
  (a) f is surjective.
  (b) $\ker(f) = 13\mathbb{Z}$.
  (c) $|\mathbb{Z}[i]/(3 + 2i)| = 13$.

(11) (DF 7.6.3, 6/16) Let R and S be commutative rings with identity. Prove that every ideal of $R \times S$ is of the form $I \times J$ where I and J are ideals of R and S, respectively.

(12) (DF 7.6.4) Prove that if R and S are nonzero rings, then $R \times S$ is never a field.