ALGEBRA PRELIM REVIEW

Linear Algebra

(1) Let V be an n-dimensional K-vector space, and let $U \subset V$ be a subspace of positive dimension $r < n$. Let $\varphi : V \to V$ be a K-linear map with $\varphi(U) \subset U$. Argue that:
   (a) There is an ordered basis B of V such that the coordinate matrix $A_\varphi^{B,B}$ has the form
   $$A_\varphi^{B,B} = \begin{bmatrix} M & N \\ 0 & P \end{bmatrix}$$
   where $M \in K^{r\times r}$ and $P \in K(n-r) \times (n-r)$.
   (b) If $\mathrm{im}(\varphi) \subset U$, then one can choose B such that $P = 0$.

(2) Consider the map $\varphi : \mathbb{Q}^{2\times 2} \to \mathbb{Q}^{2\times 2}$ given by $A \mapsto AM - MA$, where
$$M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \mathbb{Q}^{2\times 2}.$$
   (a) Show that φ is a $\mathbb{Q}$-linear map.
   (b) Find a basis for $ker(\varphi)$ and determine its dimension.

(3) Consider a matrix $A \in K^{n\times n}$, where K is a field. Show:
   (a) If $\mathrm{rk}(A) < n$, then there is a matrix $0 \neq B \in K^{n\times n}$ such that $A \cdot B = 0$
   (b) If $A^k = 0$ for some $k \in \mathbb{N}$, then A is not invertible.

(4) Let V be a 4-dimensional vector space, and let $U_1, U_2 \subset V$ be two 3-dimensional subspaces of V.
   (a) Determine the possible values of $m = \dim(U_1 \cap U_2)$.
   (b) For each value of m in (a), give an example of subspaces $U_1, U_2 \subset V = \mathbb{Q}^4$ whose intersection has dimension m.

(5) Let $U \subset K^n$ be a subspace, and consider the set
$$V := \{\varphi \in \mathrm{Hom}(K^n, K^m) \mid U \subset ker(\varphi)\}$$
Prove:
   (a) V is a K-vector space.
   (b) $\dim(V) = m(n - \dim(U))$.

(6) (6/16) In the real vector space of continuous real-valued functions on $\mathbb{R}$, consider the functions $p_i, i = 0, 1, 2$ and exp defined by $p_i(x) = x^i$ and $exp(x) = e^x$ for $x \in \mathbb{R}$. Set $V := \langle p_0, p_1, p_2, exp\rangle_{\mathbb{R}}$ and consider the endomorphism $\sigma : V \to V$ defined by $(\sigma f)(x) := f(x - 1)$ for $x \in \mathbb{R}$.
  (a) Give the matrix representation of σ with respect to the basis $\{p_0, p_1, p_2, exp\}$.
  (b) Determine all the eigenvalues and find the bases of all eigenspaces of σ.
  (c) Is σ diagonalizable?
  (d) Determine the minimal polynomial of σ.

(7) Let $A \in \mathcal{M}_{m\times n}$ and $B \in \mathcal{M}_{n\times p}$. Show the following:
  (a) $\mathrm{rk}(AB) \leq \min\{\mathrm{rk}(A), \mathrm{rk}(B)\}$.

  (b) $\mathrm{rk}(AB) \geq \mathrm{rk}(A) + \mathrm{rk}(B) - n$.

(8) Let A, B, C be square matrices over a field K such that $A = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$. Argue that their minimal polynomials satisfy $\mu_A = lcm(\mu_B, \mu_C)$, where lcm denotes the least common multiple.

(9) (6/14) Consider a linear transformation T on a vector space V of dimension 4 over $\mathbb{R}$. On a basis e_1, e_2, e_3, e_4 of V, the transformation is defined by $T(e_1) = e_2$, $T(e_2) = e_1$, $T(e_3) = 2e_3 + e_4$, and $T(e_4) = e_3 - 2e_4$.
  (a) Construct the matrix A of the transformation with respect to the given basis.
  (b) Determine the characteristic polynomial, eigenvalues, and eigenspaces of A.
  (c) Determine the kernel and image of the transformation defined by the matrix $A^2 - I$ on $\mathbb{R}^4$.
  (d) Is A diagonalizable? Would you answer differently if $\mathbb{R}$ was replaced by $\mathbb{Q}$?

(10) (1/09) Let $n \in \mathbb{N}$ and F be a field. Suppose that $T : F \to F^n$ is a linear transformation. Show that T is injective if and only if T is not the zero map.

(11) (6/11) Let $A \in M_{n\times n}(\mathbb{C})$ be a Hermitian matrix. Prove or disprove (with a counterexample) the following statements:
  (a) $\det(A) \in \mathbb{R}$.
  (b) $|\det(A)| = 1$.
  (c) If A has exactly one eigenvalue, then A is a real matrix.
  (d) If $v = (v_1, \ldots, v_n)^T$ is an eigenvector of A, then $\overline{v} = (\overline{v_1}, \ldots, \overline{v_n})^T$ is also an eigenvector of A (where $\overline{v_i}$ denotes the complex conjugate of v_i).