Group Theory Review

Tuesday, May 12, 2020 11:03 AM

$|G| = 203 = 7 \cdot 29$, $H \trianglelefteq G$, $|H| = 7$

a) H is the unique Sylow 7-subgroup of G (Sylow 2)

$n_{29} \equiv 1 \bmod 29$, $n \mid 7 \Rightarrow 1 = n_{29}$

$\Rightarrow \exists J$ unique $\overset{\text{normal}}{\downarrow}$ Sylow 29-subgroup

$HJ = H \times J$ $\quad |HJ| = \dfrac{|H||J|}{\underbrace{|H \cap J|}_{=1}} = 203 \Rightarrow G = H \times J$

$\Rightarrow H, J$ are cyclic b/c prime order

$\Rightarrow G$ is cyclic

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b) $|G| = 203 = 7 \cdot 29$, $H \trianglelefteq G$, $|H| = 7$

WTS: $H \leq Z(G)$

$G \curvearrowright H$ by conj.: B/c $H \trianglelefteq G$, $gHg^{-1} = H$,

$g \mapsto \sigma_g : H \to H$

$h \mapsto ghg^{-1}$

$\in Aut(H)$

$|Aut(H)| = 6$

$ker(\sigma) = C_G(H)$

$ghg^{-1} = h \Leftrightarrow gh = hg$

$G/C_G(H) \cong K \leq \mathrm{Aut}(H)$

$\Rightarrow |G/C_G(H)| \mid G \Rightarrow G = C_G(H)$

$\Leftrightarrow H \leq Z(G)$

Claim: $|G| = pq \Rightarrow |Z(G)| = 1$ or pq

if $|Z(G)| == p$

$\Rightarrow |G/Z(G)| = q \Rightarrow G/Z(G)$ cyclic

$\Rightarrow G$ abelian

$\Rightarrow |Z(G)| \neq p$

Find # elts of order p^2 in $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p^2\mathbb{Z} \times \mathbb{Z}/p^3\mathbb{Z}$

$$|(a, b, c)| = \mathrm{lcm}(|a|, |b|, |c|)$$

$\Rightarrow p^2 = \mathrm{lcm}(\sim\sim)$

$|a| = 1, p$

$|b| = 1, p, p^2$

$|c| = 1, p, p^2$

At least one w/ order p^2.

— a can be any elt of $\mathbb{Z}/p\mathbb{Z}$

if $|b| = p^2$: $\varphi(p^2) = p(p-1)$ elts of order p^2

$\varphi(p^n) = p^{n-1}(p-1)$

elts in $\mathbb{Z}/p^3\mathbb{Z}$ w/ order not p^3: $p^3 - (p^3 - p^2) = p^2$

$$\boxed{p \times p(p-1) \times p^2}$$

if $|b| = 1$ or p, $|c| = p^2$:

elts of $\mathbb{Z}/p^2\mathbb{Z}$ of order not p^2: $p^2 - (p^2 - p) = p$

elts of order p^2 in $\mathbb{Z}/p^2\mathbb{Z}$: $p^2 - p = \varphi(p^2)$

$$\boxed{p \times p \times p(p-1)}$$

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elts in $\mathbb{Z}/p^k\mathbb{Z}$ of order at most p^n $= \begin{cases} p^k, & k \leq n \\ p^n, & n < k \end{cases}$

b/c there is a unique subgp of order p^n and all elts have order p^n

p^2: $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p^2\mathbb{Z} \times \mathbb{Z}/p^3\mathbb{Z} \times \mathbb{Z}/p^4\mathbb{Z}$

$p \cdot p^2 \cdot p^2 \cdot p^2$ elts at most order p^2

$p \cdot p \cdot p \cdot p$ elts at most order p

$p \cdot p^2 \cdot p^2 \cdot p^2$ elts at most order p^2

$p \cdot p \cdot p \cdot p$ elts at most order p

$p^7 - p^4$ elts of order exactly p^2.

$$\prod_{i=1}^{\infty} \mathbb{Z} \neq \bigoplus_{i=1}^{\infty} \mathbb{Z}$$

infinite tuples

infinite tuples w/ finitely many nonzero entries

$g, h \in G$, $|g| = m$, $|h| = n$. $\gcd(m,n) = 1$

Q: $\langle g \rangle \cap \langle h \rangle = \{e\}$?

Yes: $|\langle g \rangle \cap \langle h \rangle| \mid \gcd(m,n) = 1$