ALGEBRA PRELIM REVIEW

Group Theory

(1) (N53) Let G be an abelian group and let $a, b \in G$ be elements of finite order m and n, respectively.
   (a) Show that $\mathrm{ord}(ab) \le e$, where e is the positive least common multiple of m and n.
   (b) Decide whether equality is always true in (a).
   (c) If m and n are relatively prime, argue that $\mathrm{ord}(ab) = mn$.

(2) (N67) Let H be a subgroup of a group G, and denote by $P(G)$ the set of all subsets of G. Show:
   (a) G acts on $P(G)$ by conjugation, i.e., $G \times P(G) \to P(G)$ is given by $(g, M) \mapsto gMg^{-1}$. (For every non-empty subset $M \subset G$, the stabilizer of M with respect to this action is the *normalizer* $N_G(M)$ of M in G).
   (b) $H \lhd N_G(H) < G$.
   (c) $H \lhd G$ if and only if $N_G(H) = G$.
   (d) If G is finite, then $[G : N_G(H)]$ is the number of subgroups of G that are conjugates of H.

(3) (N68, 6/16, 6/11) Prove that a group G is abelian if $G/Z(G)$ is a cyclic group.

(4) (DF4.2.8) Prove that if H has finite index n, then there is a normal subgroup K of G with $K \le H$ and $|G : K| \le n!$.

(5) (DF4.2.14) Let G be a finite group of composite order n with the property that G has a subgroup of order k for each positive integer k dividing n. Prove that G is not simple.

(6) (DF4.3.34) Prove that if p is a prime and P is a subgroup of S_p of order p, then $|N_{S_p}(P)| = p(p-1)$. [Hint: Argue that every conjugate of P contains exactly $p-1$ p-cycles and use the formula for the number of p-cycles to compute the index of N_{S_p} in S_p].

(7) (DF4.4.2) Prove that if G is an abelian group of order pq, where p and q are distinct primes, then G is cyclic.

(8) (DF4.4.13) Let G be a group of order 203. Prove that if H is a normal subgroup of order 7 in G then $H \le Z(G)$. Deduce that G is abelian in this case.

(9) (DF4.5.15) Prove that a group of order 351 has a normal Sylow p-subgroup for some prime p dividing its order.

(10) (DF4.5.35) Let $P \in Syl_p(G)$ and let $H \leq G$. Prove that $gPg_{-1} \cap H$ is a Sylow p-subgroup of H for some $g \in G$. Give an explicit example showing that $hPh^{-1} \cap H$ is not necessarily a Sylow p-subgroup of H for any $h \in H$ (in particular, we cannot always take $g = 1$ in the first part of this problem, as we could when H was normal in G.