Field Theory Review

Wednesday, May 20, 2020 4:00 PM

8) $f = \underbrace{(x^3 - 5)}_{g}\underbrace{(x^5 - 7)}_{h}$, K = splitting field, $n = [K:\mathbb{Q}]$

a) $3 \mid n$, $5 \mid n$ b/c $\mathbb{Q}(\alpha), \mathbb{Q}(\beta) \subseteq K$, α a root of g
β " " " h.

b) L_1 the splitting field of g:

$L_1 = \mathbb{Q}(\sqrt[3]{5}, \zeta_3)$ ← deg 6 b/c rel prime

$\subseteq$ ✓

? $[\mathbb{Q}(\sqrt[3]{5}) : \mathbb{Q}] = 3$

$[\mathbb{Q}(\zeta_3) : \mathbb{Q}] = \varphi(3) = 2$

L_2 splitting field of H

$L_2 = \mathbb{Q}(\sqrt[5]{7}, \zeta_5)$ deg$/\mathbb{Q}$ = 20

$K \subseteq L_1 L_2 = \mathbb{Q}(\sqrt[3]{5}, \sqrt[5]{7}, \zeta_3, \zeta_5)$

$$\begin{tikzcd} & K \arrow[dl, dash] \arrow[dr, dash] & \\ L_1 \arrow[dr, dash] & & L_2 \arrow[dl, dash] \\ & \mathbb{Q} & \end{tikzcd}$$

$\zeta_3, \zeta_5 \in K$

$\boxed{z = \zeta_3 \zeta_5 \in K}$

(What is $|z|$? $|z| = 15$, satisfying $\Phi_{15}(x)$
$\deg \Phi_{15} = \varphi(15)$)

What is $|z|$? $|z| = 15$, so [illegible] Φ_{15}

$\deg \Phi_{15} = \varphi(15) = 8$

Claim: $K \overset{=}{\underset{\subseteq}{}} \mathbb{Q}(\sqrt[3]{5}, \sqrt[5]{7}, z)$

$$\underbrace{\mathbb{Q}(\sqrt[3]{5}, \sqrt[5]{7}) \subseteq \underbrace{K \subseteq \mathbb{Q}}_{\{1,2,4\}}(\sqrt[3]{5}, \sqrt[5]{7}, z)}_{8}$$

But Φ_3 is not reducible over $\mathbb{Q}(\underbrace{\sqrt[3]{5}, \sqrt[5]{7}}, \zeta_5)$

$\iff \boxed{\mathbb{Q}(\zeta_3, \zeta_5) \neq \mathbb{Q}(\zeta_5)}$

So must have $K = \mathbb{Q}(\sqrt[3]{5}, \sqrt[5]{7}, z)$

$n = 120$