ALGEBRA PRELIM REVIEW

Field and Galois Theory

Note: We denote by $\mathrm{Gal}(E, K)$ the Galois group of E/K, that is, the subgroup of $\mathrm{Aut}(E)$ consisting of K-homomorphisms.

(1) (N45) Let E/K be a field extension of degree 2 and assume that the characteristic of K is not 2. Show that:
  (a) E/K is a simple field extension.
  (b) There are exactly two K-automorphisms of E.
  (c) If $f \in K[x]$ is irreducible and has a root in E, then f splits over E.

(2) (N60) Let K be a prime field of finite order p and let $f \in K[x]$ be an irreducible polynomial. For every positive integer n, show that $\deg(f) \mid n$ iff $f \mid (x^{p^n} - x)$.

(3) (N55) Let E/K be a Galois field extension and let L_1, L_2 be subfields of E that contain K. Prove:
  (a) $\mathrm{Gal}(E, L_1L_2) = \mathrm{Gal}(E, L_1) \cap \mathrm{Gal}(E, L_2)$
  (b) $\mathrm{Gal}(E, L_1 \cap L_2)$ is the subgroup of $\mathrm{Gal}(E, K)$ that is generated by $\mathrm{Gal}(E, L_1)$ and $\mathrm{Gal}(E, L_2)$.

(4) (N57) Prove that $[\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}) : \mathbb{Q}] = 8$.

(5) (N58) Let E/K be any field extension where E is finite. Show:
  (a) E/K is a Galois extension.
  (b) The Galois group $\mathrm{Gal}(E, K)$ is cyclic. Specify a map that generates the group.

(6) (N73) Let E be a splitting field of an irreducible and separable polynomial $f \in K[x]$ over a field K. Denote by $\alpha_1, \dots, \alpha_n \in E$ the roots of f. Assume that the Galois group $\mathrm{Gal}(E, K)$ is abelian. Show that $E = K(\alpha_i)$ for each $i = 1, \dots, n$, and so $[E : K] = \deg(f)$.

(7) (N87) Determine the Galois groups of the polynomials $f := x^3 + 6x^2 + 11x + 7$ and $g := x^3 + 3x^2 - 1$ in $\mathbb{Q}[x]$. Besides giving the isomorphism type of the group, describe the automorphisms explicitly.

(8) (6/14) Let $f(x) = (x^3 - 5)(x^5 - 7) \in \mathbb{Q}[x]$, and let K be a splitting field of $f(x)$ over $\mathbb{Q}$. Let $n = [K : \mathbb{Q}]$.
  (a) Argue that n is divisible by 15.
  (b) Show that K must contain a primitive 15th root of unity over $\mathbb{Q}$ which satisfies a monic polynomial of degree 8.
  (c) Deduce that $n = 120$.

(9) (1/13) Let $n \geq 3$ and let ζ be a primitive nth root of unity over $\mathbb{Q}$. Recall that $[\mathbb{Q}(\zeta) : \mathbb{Q}] = \varphi(n)$, where φ is the Euler φ-function. Prove that $\alpha := \zeta + \zeta^{-1}$ is algebraic over $\mathbb{Q}$ of degree $\varphi(n)/2$. [Hint: It will be useful to note that $\alpha \in \mathbb{R}$. If you want to use this fact then you also have to prove it].

(10) (5/15) Consider the field extension $\mathbb{F}_{5^4}|\mathbb{F}_5$.
  (a) Determine the number of elements $a \in \mathbb{F}_{5^4}$ satisfying $\mathbb{F}_{5^4} = \mathbb{F}_5(a)$.
  (b) Determine the number of irreducible polynomials of degree 4 in $\mathbb{F}_5[x]$.

(11) Decide whether the following pairs of fields are isomorphic:
  (a) $\mathbb{Q}(\sqrt[4]{2})$ and $\mathbb{Q}(i\sqrt[4]{2})$
  (b) $\mathbb{Q}(\sqrt[3]{1+\sqrt{3}})$ and $\mathbb{Q}(\sqrt[3]{1-\sqrt{3}})$
  (c) $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$