So far, most of the series we have looked at have only positive terms, and our three tests: Integral Test, Comp. Test, Limit Comp. Test require positive terms.

Start lecture with $\sum_{k=1}^{\infty} \frac{n^2-4}{n^{7/2}}$

What about series with alternating negative & positive terms?

<u>Ex</u>: Does $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$ converge or diverge? (alternating harmonic series)

Let's find partial sums

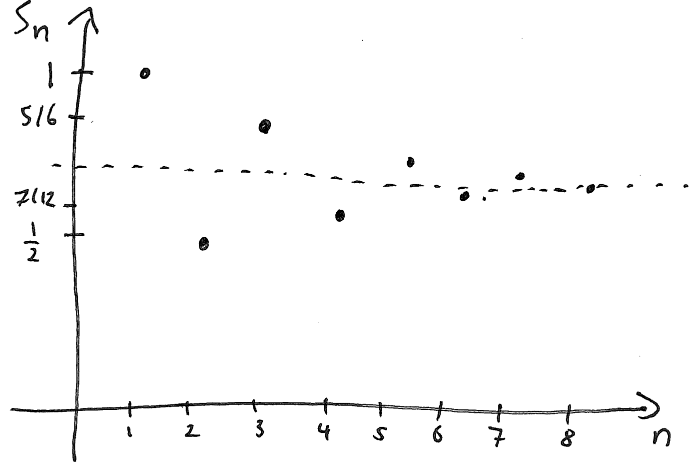
$S_1 = 1$ $\quad S_2 = 1 - \frac{1}{2} = \frac{1}{2}$

$S_3 = 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ $\quad S_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12}$

$S_5 \approx 0.78333$ $\quad S_6 \approx 0.616667$

$S_7 \approx 0.759524$ $\quad S_8 \approx 0.635524$

$S_{15} \approx 0.725372$ $\quad S_{16} \approx 0.662872$



(see Desmos graph)

Odd partial sums are decreasing, even partial sums are increasing, towards each other.

Think about this as at each step the "correction" goes the other way, gets smaller, going to 0.
So this series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ will converge. C.

Alternating Series Test

Consider $\sum_{n=1}^{\infty} (-1)^n b_n$. If $b_n \geq b_{n+1}$ eventually and $\lim_{n\to\infty} b_n = 0$, then the series converges.

<u>ex</u>: $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$: $b_n = \frac{1}{n}$. $\{b_n\}$ is decreasing, and $\lim_{n\to\infty} \frac{1}{n} = 0$.

<u>ex</u>: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$: $b_n = \frac{1}{n^2}$ is decreasing, $\lim_{n\to\infty} \frac{1}{n^2} = 0$, so by the AST $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ converges.

<u>ex</u>: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$: $b_n = \frac{1}{\sqrt{n}}$ is decreasing, $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, so $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ converges by AST.

<u>ex</u>: $\sum_{n=1}^{\infty} (-1)^{n+3} \frac{\sqrt{n}}{n+4}$: $(-1)^{n+3}$ is ok, $b_n = \frac{\sqrt{n}}{n+4} \to 0$ as $n \to \infty$, $f(x) = \frac{\sqrt{x}}{x+4} \Rightarrow f'(x) = \frac{4-x}{2\sqrt{x}(x+4)^2} < 0$ if $x > 4$

so eventually b_n is decreasing

Some cautions:

1) We need $\lim\limits_{n\to\infty} b_n = 0$: otherwise Div Test applies. e.g. $\sum\limits_{n=1}^{\infty} (-1)^n \frac{n+1}{n}$

~~2) We need alternating?~~

~~$\sum\limits_{n=1}^{\infty} \frac{\cos(n)}{n}$ has both positive and negative terms, but does not alternate.~~

2) We need $b_n \le b_{n+1}$ eventually. (posted on Canvas thanks to Jared Antrobus)

3) We need alternating: $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$ diverges.

group every 3 terms: $b_k = \frac{1}{3k-2} + \frac{1}{3k-1} - \frac{1}{3k} = \frac{3k(3k-1) + 3k(3k-2) - (3k-1)(3k-2)}{3k(3k-1)(3k-2)}$

$= \frac{9k^2 - 3k + 9k^2 - 6k - 9k^2 + 9k - 2}{3k(3k-1)(3k-2)}$

$= \frac{9k^2 - 2}{3k(3k-1)(3k-2)} \sim \frac{1}{3k}$

so div by LCT w/ $\sum \frac{1}{3k}$.

→ Group work?

Estimating alternating series

Looking at our picture from earlier, at each step we get closer to the limit.

This is always true for AST series: if $\sum\limits_{n=1}^{\infty} (-1)^{n-1} b_n$, $b_n \ge b_{n+1}$, $b_n \to 0$, then

$$|S_N - S| \le b_{N+1}$$

So it is fairly easy to ~~use~~ tell how good approximations of alt. series with only a few terms are.

ex: What is the max error in estimating $S = \sum\limits_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ ~~with~~ using the partial sum $S_{20} = \sum\limits_{n=1}^{20} (-1)^{n-1} \frac{1}{n}$?

A: $|S_{20} - S| \le \frac{1}{21}$

ex: How many terms should we use to estimate the sum $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ within 0.0001?

$|S_N - S| \le \frac{1}{\sqrt{N}}$ so find N such that $\frac{1}{\sqrt{N}} < 0.0001$

$\sqrt{N} > 10000$

$N > 100$