11.4 Comparison Tests

We have two big classes of examples of series that we know converge/diverge:

<u>p-series</u> $\sum_{n=1}^{\infty} \frac{1}{n^p}$ and <u>geometric series</u>: $\sum_{n=0}^{\infty} ar^n$ or $\sum_{n=1}^{\infty} ar^{n-1}$.

What about a series which is <u>almost a p-series or geometric series</u>?

<u>Ex</u>: Does $\sum_{n=1}^{\infty} \frac{1}{n^3+2n}$ converge or diverge? • $\frac{1}{n^3+2n} \approx \frac{1}{n^3}$, which converges, so we guess yes.

More precisely: $\frac{1}{n^3+3n} < \frac{1}{n^3}$, since it has a larger denominator. So for all $N \geq 1$, $\sum_{n=1}^{N} \frac{1}{n^3+3n} < \sum_{n=1}^{N} \frac{1}{n^3}$.

So $\lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n^3+3n} \leq \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n^3} < \infty$. So $\sum_{n=1}^{\infty} \frac{1}{n^3+2n}$ converges.

<u>Ex</u>: $\sum_{n=5}^{\infty} \frac{n}{n^2-3}$. Converge or diverge?

Well, $\frac{n}{n^2-3} > \frac{n}{n^2} = \frac{1}{n}$. So $\sum_{n=5}^{N} \frac{n}{n^2-3} > \sum_{n=5}^{N} \frac{n}{n^2}$, so $\sum_{n=5}^{\infty} \frac{n}{n^2-3} \geq \sum_{n=5}^{\infty} \frac{1}{n} = \infty$. Thus $\sum_{n=5}^{\infty} \frac{n}{n^2-3}$ diverges to ∞.

~~Ex: $\sum_{n=1}^{\infty} \frac{n}{3^n}$. Converge or diverge~~

<u>Comparison Test</u>: Consider $\sum a_n$, $\sum b_n$ where $a_n \geq b_n \geq 0$ eventually.

Then 1) if $\sum a_n$ converges, so does $\sum b_n$.

2) if $\sum b_n$ diverges, so does $\sum a_n$.

<u>Ex</u>: $\sum_{n=3}^{\infty} \frac{\ln(n)}{n}$. We could use the Integral Test here, but it is easier to compare.

$\frac{\ln(n)}{n} \geq 0$ for $n \geq 3$, $\frac{\ln(n)}{n} > \frac{1}{n}$ and $\sum_{n=3}^{\infty} \frac{1}{n}$ diverges, so $\sum_{n=3}^{\infty} \frac{\ln(n)}{n}$ also diverges.

<u>Ex</u>:
What about $\sum_{n=1}^{\infty} \frac{1}{3^n-1}$?

For large n, $3^n \gg 1$, so ~~the~~ $\frac{1}{3^n-1} \approx \frac{1}{3^n}$, so we guess this series converges.

But $\frac{1}{3^n-1} > \frac{1}{3^n}$, so the comparison test isn't useful.

Limit Comparison Test:

Suppose $\sum a_n$, $\sum b_n$ are series with $a_n, b_n \geq 0$ eventually. Let $c = \lim_{n\to\infty} \frac{a_n}{b_n}$.

If $c=0$ and $\sum b_n$ converges, $\sum a_n$ also converges.

If $c=\infty$ and $\sum a_n$ converges, $\sum b_n$ also converges.

If $0<c<\infty$, then $\sum a_n$ converges if and only if $\sum b_n$ converges.

Back to $\sum_{n=1}^{\infty} \frac{1}{3^n-1}$: Use LCT with $a_n = \frac{1}{3^n-1}$, $b_n = \frac{1}{3^n}$.

$\lim_{n\to\infty} \frac{\frac{1}{3^n-1}}{\frac{1}{3^n}} = \lim_{n\to\infty} \frac{3^n}{3^n-1} = \lim_{n\to\infty} \frac{1}{1-\frac{1}{3^n}} = 1$. Since $\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges, this means $\sum_{n=1}^{\infty} \frac{1}{3^n-1}$ also converges.

More examples:

$\sum_{n=2}^{\infty} \frac{n^2}{n^4-n-1}$ LCT w/ $\frac{1}{n^2}$, $\sum_{n=3}^{\infty} \frac{1}{\sqrt{n^2+4}}$ LCT w $\frac{1}{n}$, $\sum_{m=1}^{\infty} \frac{4}{m!+4^m}$ CT w/ $\frac{4}{4^m}$.