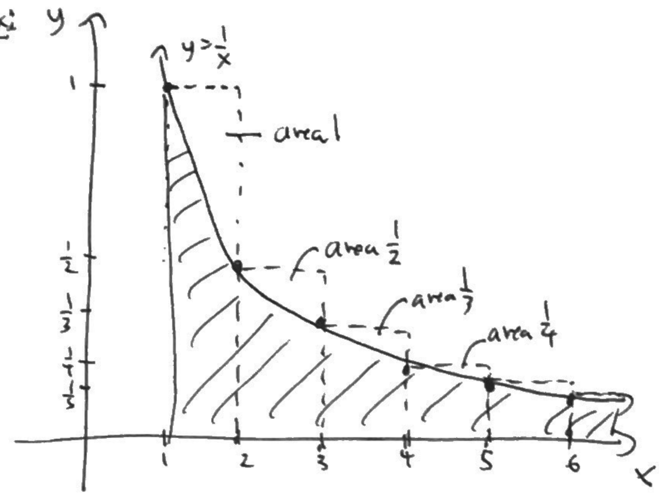
6/25/18 Lecture – 11.3 Integral Test

• Begin with quick recap of Friday, Q from webwork
Worksheet 9a (geom series, telescoping, partial sums, div test)

Last time, we said $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Let's see why:

Ex:



$/// = \int_1^{\infty} \frac{1}{x}\,dx \Rightarrow \infty$

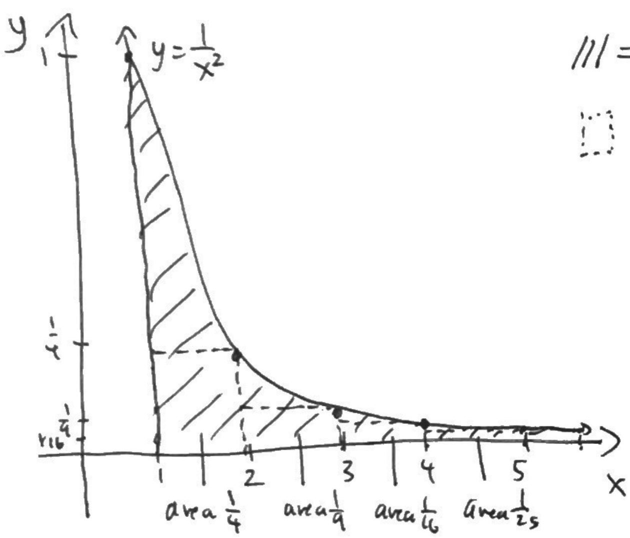
$[\,] = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots = \sum_{n=1}^{\infty} \frac{1}{n}.$

Notice that the area of the rectangles is greater than the area under the graph, which is infinite.

So $\sum_{n=1}^{\infty} \frac{1}{n}$ must also be infinite.

The idea of comparing to an integral is also useful for convergent series:

Ex: Consider $\sum_{n=1}^{\infty} \frac{1}{n^2}$:



$/// = \int_1^{\infty} \frac{1}{x^2}\,dx = 1$

$[\,] = \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots = \sum_{n=2}^{\infty} \frac{1}{n^2}$

From the picture, $\sum_{n=2}^{\infty} \frac{1}{n^2} < \int_1^{\infty} \frac{1}{x^2}\,dx$.

So $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^2} < 1 + \int_1^{\infty} \frac{1}{x^2}\,dx = 1+1=2.$

So $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Notes: • $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, but this argument does not tell us what it converges to. (in fact $\frac{\pi^2}{6}$) (by Euler & others)

• This also illustrates the useful idea that only the tail of a series matters for convergence.

We ignored the n=1 term at first and added it back after we found that the other terms converged.

Integral Test: Consider $\sum_{n=k}^{\infty} a_n$. ~~If $a_n \geq 0$ and $a_{n+1} \leq a_n$ eventually, then and $a_n = f(n)$, where f is cts,~~ If $a_n = f(n)$, where f is cts, positive, and decreasing (eventually), then $\sum_{n=k}^{\infty} a_n$ converges if and only if $\int_k^{\infty} f(x)\,dx$ converges.

Ex: Does $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2} = \frac{1}{2^2} + \frac{2}{5^2} + \frac{3}{10^2} + \dots$ converge?

Let's use the Integral Test: $f(x) = \frac{x}{(x^2+1)^2}$ is positive and continuous if $x \geq 1$.

$$f'(x) = (x^2+1)^{-2} - 2x(x^2+1)^{-3} \cdot 2x = \frac{x^2+1-4x^2}{(x^2+1)^3} = \frac{1-3x^2}{(x^2+1)^3} < 0 \text{ if } x \geq 1, \text{ so } f \text{ is also decreasing.}$$

So ~~we the~~ we can use the Integral Test:

$$\int_1^{\infty} \frac{x}{(x^2+1)^2}\,dx = \lim_{R\to\infty} \int_1^R \frac{x}{(x^2+1)^2}\,dx = \lim_{R\to\infty} \frac{1}{2}\int_2^{R^2+1} \frac{du}{u^2} \overset{\lim_{R\to\infty}}{=} \frac{-1}{2u}\Big|_2^{R^2+1} = \lim_{R\to\infty} \frac{1}{4} - \frac{1}{2(R^2+1)} = \frac{1}{4}.$$

So $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$ also converges.

p-series: ~~If~~ We saw that $\int_1^{\infty} \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1} & \text{if } p>1 \\ \infty & \text{if } p\leq 1 \end{cases}$. The Integral Test tells us that the same is true for series: the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p>1$ and diverges if $p \leq 1$.

Ex: Does $\sum_{n=0}^{\infty} n e^{-n^2}$ converge or diverge?

$f(x) = xe^{-x^2}$ is positive and cts if $x \geq 0$.

$f'(x) = e^{-x^2} - 2x^2e^{-x^2} = e^{-x^2}(1-2x^2)$. Notice that $f' > 0$ for x in $[0, \frac{1}{\sqrt{2}})$, but eventually (for $x > \frac{1}{\sqrt{2}}$) $f'(x) < 0$ and so f is decreasing.

So we can use the Integral Test.

$$\int_0^{\infty} xe^{-x^2}\,dx = \cancel{\lim_{R\to\infty} \int_0^R xe} = \lim_{R\to\infty} \int_0^R xe^{-x^2}\,dx = \lim_{R\to\infty} -\frac{1}{2}e^{-x^2}\Big|_0^R = \lim_{R\to\infty} 1 - \frac{1}{2}e^{-R^2} = 1.$$

So $\sum_{n=0}^{\infty} ne^{-n^2}$ also converges.

Worksheet 4b: Integral Test

End w/ ~20 min for exam 1 discussion.