11.2 - Series

Goals: 1) What is a series? 2) In particular, recognize & deal with geometric & telescoping series. 3) The harmonic series diverges. 4) Divergence Test

Ex: $\frac{1}{9} = 0.1111111\ldots$ What does this really mean?

$$\frac{1}{9} \text{"="} \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \frac{1}{10{,}000} + \frac{1}{100{,}000} + \ldots$$

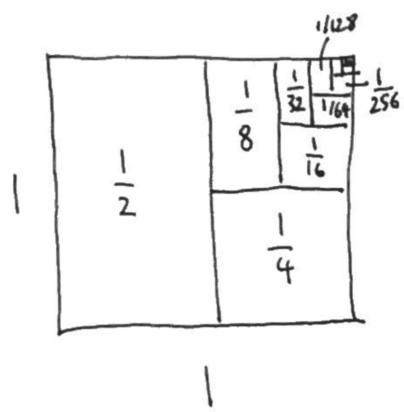
What does it mean to add infinitely many numbers here?

A: The sequence of partial sums: (add the first n terms)

$$S_1 = \frac{1}{10},\quad S_2 = \frac{1}{10} + \frac{1}{100},\quad S_3 = \frac{1}{10} + \frac{1}{100} + \frac{1}{1000},\quad \ldots,\quad S_n = \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \ldots + \frac{1}{10^n}$$

approaches $\frac{1}{9}$ as $n \to \infty$.

Ex: What meaning can we give to the infinite sum $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots$?



This sum approaches 1 as we add more terms!

So we say $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$

More generally, these are examples of geometric series:

$$a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} + \ldots \qquad (a = \tfrac{1}{10} = r,\ a = \tfrac{1}{2} = r)$$

The n^{th} partial sum is

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1}$$

$$rS_n = ar + ar^2 + \ldots + ar^{n-1} + ar^n$$

$$S_n - rS_n = a - ar^n$$

$$S_n(1-r) = a(1-r^n)$$

$$S_n = a\left(\frac{1-r^n}{1-r}\right)$$

• What happens as $n \to \infty$?

A: Depends on r.

If $r=1$: $a+a+a+a+\ldots+a = na \to \infty$ as $n \to \infty$

If $r=-1$: $a-a+a-a+\ldots+(-1)^{n-1}a$ oscillates btwn a & 0.

If $|r|>1$: $\frac{1-r^n}{1-r} \to \pm\infty$ as $n \to \infty$ (oscillates if $r<0$)

If $|r|<1$: $\frac{1-r^n}{1-r} \to 0$ as $n \to \infty$

$-1<r<1$

Generalizing, if we have an infinite series $S = \sum_{i=1}^{\infty} a_i$, we say that <u>S converges</u> if the sequence of partial sums $S_n = \sum_{i=1}^{n} a_i$ converges to a limit.

Our Question for the next two weeks: When does a series $\sum_{i=1}^{\infty} a_i$ converge?

<u>Ex</u>: $\sum_{i=1}^{\infty} i$. $\quad S_n = \frac{n(n+1)}{2}$ (Gauss), $\lim_{n\to\infty} S_n = \infty$, so this series <u>diverges</u>.

<u>Ex</u>: $\sum_{i=1}^{\infty} \frac{1}{i(i+1)}$ $\quad S_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n}\left(\frac{1}{i} - \frac{1}{i+1}\right)$

$S_1 = 1 - \frac{1}{2} \qquad S_2 = (1-\frac{1}{2}) + (\frac{1}{2}-\frac{1}{3}) = 1 - \frac{1}{3} \qquad S_3 = (1-\frac{1}{2}) + (\frac{1}{2}-\frac{1}{3}) + (\frac{1}{3}-\frac{1}{4}) = 1-\frac{1}{4}$

$S_n = 1 - \frac{1}{n+1}$

So $\lim_{n\to\infty} S_n = 1$! We say $\sum_{i=1}^{\infty} \frac{1}{i(i+1)} = 1$.

<u>Ex</u>: $\sum_{i=1}^{\infty} \frac{1}{n}$ $\quad S_1 = 1 \quad S_2 = 1 + \frac{1}{2} \quad S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1+1$

$S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = 1 + \frac{3}{2}$

If we keep doing this, $S_{2^n} > 1 + \frac{n}{2}$, so $S_n \to \infty$ as $n \to \infty$, hence the harmonic series diverges.

<u>Divergence Test</u>: If $\sum_{i=1}^{\infty} a_i$ converges, then $\lim_{i\to\infty} a_i = 0$.

So, if $\lim_{i\to\infty} a_i \neq 0$, the series $\sum_{i=1}^{\infty} a_i$ <u>must</u> diverge.

<u>Warning!!</u> It is <u>false</u> that if $\lim_{i\to\infty} a_i = 0$ the series $\sum_{i=1}^{\infty} a_i$ converges. See harmonic series

- If time, ~~aster~~ problems.