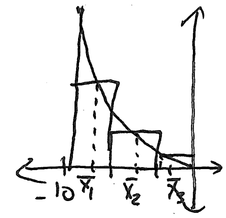
Notes for 6/15, MA114

Ex (7.7) Approximate $\int_{-10}^{0} 7x^2\,dx$ using the Midpoint rule with $n=3$.



$$\Delta x = \frac{0-(-10)}{3} = \frac{10}{3}$$

$x_0 = -10,\ x_1 = -10 + \frac{10}{3} = -6\frac{2}{3},\ x_2 = -3\frac{1}{3},\ x_3 = 0.$

$$\bar{x}_1 = \frac{x_1 + x_0}{2} = \frac{-10 + -6\frac{2}{3}}{2} = -5 - 3\frac{1}{3} = -8\frac{1}{3}$$

$$\bar{x}_2 = \frac{-6\frac{2}{3} - 3\frac{1}{3}}{2} = -5$$

$$\bar{x}_3 = \frac{-3\frac{1}{3} + 0}{2} = -\frac{5}{3}$$

$$M_3 = \Delta x\left[f(\bar{x}_1) + f(\bar{x}_2) + f(\bar{x}_3)\right] = \frac{10}{3}\left[7\left(-\frac{25}{3}\right)^2 + 7(-5)^2 + 7\left(-\frac{5}{3}\right)^2\right]$$

$$\approx 2268.5$$

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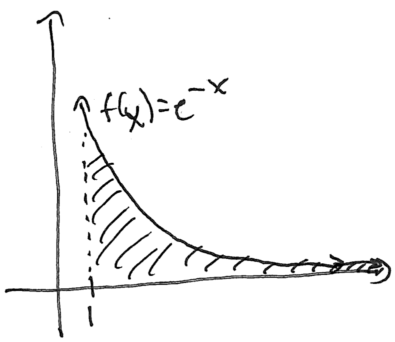
§7.8 Improper Integrals

- So far, in order to integrate $f(x)$, we need f to be continuous on $[a,b]$.

Ex What is the area under the graph of $f(x) = e^{-x}$ to the right of $x=1$? This is an infinitely long region, but we can still assign a meaningful idea of area!



We will use the notation:

$$\int_1^{\infty} e^{-x}\,dx = \lim_{r\to\infty} \int_1^r e^{-x}\,dx, \text{ if this limit exists.}$$

$$= \lim_{r\to\infty} -e^{-x}\Big|_1^r$$

$$= \lim_{r\to\infty} -e^{-r} + e^{-1}$$

$$= 0 + \frac{1}{e}$$

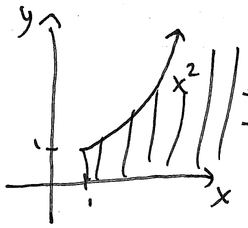
$$= \boxed{\frac{1}{e}}$$

- We are finding the area under f between 1 and r for larger and larger values of r!

So the area under e^{-x} between 1 and r approaches $\frac{1}{e}$ as $r \to \infty$. Hence $\int_1^{\infty} e^{-x}\,dx = \frac{1}{e} =$ area under f to the right of $x=1$.

More examples:

Ex) $\int_1^\infty x^2\,dx = \lim_{r\to\infty} \int_1^r x^2\,dx = \lim_{r\to\infty} \frac{1}{3}r^3 - \frac{1}{3} = \underline{\infty}.$



From the graph, we should expect there to be infinite area.

- We say that $\int_a^\infty f(x)\,dx$ converges if $\lim_{r\to\infty} \int_a^r f(x)\,dx$ exists and diverges otherwise.

- $\int_1^\infty e^{-x}dx$ converges, while $\int_1^\infty x^2 dx$ diverges.
- If $\lim_{x\to\infty} f(x) \neq 0$, then $\int_a^\infty f(x)dx$ diverges. (see prev example)
  -> if $\lim_{x\to\infty} f(x) = 0$, $\int_a^\infty f(x)dx$ still might diverge.

Ex) $\int_1^\infty \frac{1}{x}dx = \lim_{r\to\infty} \int_1^r \frac{1}{x}dx = \lim_{r\to\infty} \ln|x|\Big|_1^r = \lim_{r\to\infty} \ln|r| = \infty.$

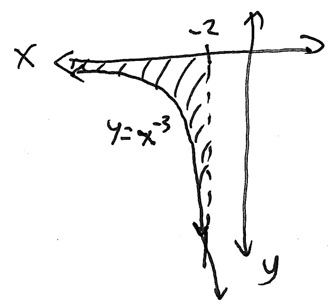
- We can also consider areas to the left of a given x-value:

Ex) $\int_{-\infty}^{-2} \frac{1}{x^3}dx = \lim_{r\to-\infty} \int_r^{-2} \frac{1}{x^3}dx = \lim_{r\to-\infty} -\frac{1}{2}x^{-2}\Big|_r^{-2}$

$= \lim_{r\to-\infty} -\frac{1}{8} + \frac{1}{2r^2}$

$= -\frac{1}{8} + 0$

$= \boxed{-\frac{1}{8}}$



Ex) $\int_0^\infty xe^{-x}dx = \lim_{r\to\infty} \int_0^r xe^{-x}dx = \lim_{r\to\infty} \left[-xe^{-x}\Big|_0^r + \int_0^r e^{-x}dx\right]$ (by parts with $u=x$ $dv=e^{-x}dx$)

$= \lim_{r\to\infty} \left[-xe^{-x} - e^{-x}\Big|_0^r\right]$

$= \lim_{r\to\infty} \left((-re^{-r} - e^{-r}) - (0 - e^0)\right)$

$= \lim_{r\to\infty} \left(-(r+1)e^{-r} + 1\right)$

What is $\lim_{r\to\infty} -(r+1)e^{-r}$? Use L'Hospital's rule: if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$ or $\frac{0}{0}$, $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$

$= \lim_{r\to\infty} \frac{-(r+1)}{e^r} \overset{L'H}{=} \lim_{r\to\infty} \frac{-1}{e^r} = 0.$ So $\boxed{\int_0^\infty xe^{-x}dx = 0 + 1 = 1}$

($\frac{-\infty}{\infty}$)

Worksheet 6 on Monday & more