§7.7: Approx. Integration

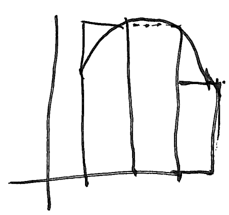
Some integrals cannot be evaluated using FTC because ~~we~~ not all functions have closed-form antiderivatives, e.g. $\int_0^1 e^{x^2}\,dx$ or $\int_{-1}^{1}\sqrt{1+x^3}\,dx$. ~~&~~ Alternately, we have only data points rather than a function, but still want to perform integration.

Recall from Calc I that $\int_a^b f(x)\,dx$ is a limit of Riemann sums $\sum_{i=1}^{n} f(x_i^*)\Delta x$ where $x_i^* \in [x_{i-1}, x_i]$. This means that any Riemann sum can be used as an approx of $\int_a^b f(x)\,dx$ and as n increases, the error from our approximation decreases.

We will ~~concern ou~~ primarily work with 4 of these ~~inte~~ approximations:

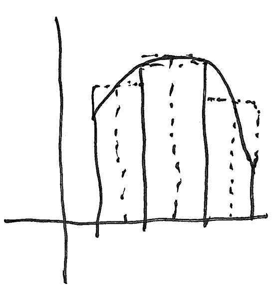
1) Left Endpoint, L_n: Choose $x_i^* = x_{i-1}$. $\int_a^b f(x)\,dx \approx L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x$

2) Right Endpoint, R_n: Choose $x_i^* = x_i$. $\int_a^b f(x)\,dx \approx R_n = \sum_{i=1}^{n} f(x_i)\Delta x$



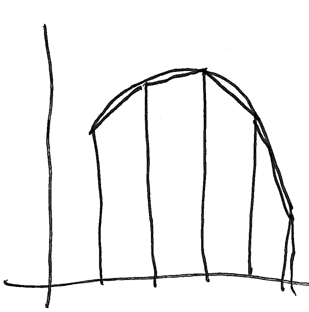
3) Midpoint, M_n: Choose $x_i^* = \frac{x_{i-1}+x_i}{2} = \bar{x}_i$ = midpoint of $[x_{i-1}, x_i]$.

$$\int_a^b f(x)\,dx \approx M_n = \sum_{i=1}^{n} f(\bar{x}_i)\Delta x$$



4) Trapezoid Rule, T_n: Average of L_n and R_n

$$\begin{aligned}\int_a^b f(x)\,dx &\approx \frac{1}{2}\left(\sum_{i=1}^{n} f(x_{i-1})\Delta x + \sum_{i=1}^{n} f(x_i)\Delta x\right)\\ &= \frac{\Delta x}{2}\sum_{i=1}^{n} f(x_{i-1}) + f(x_i)\\ &= \frac{\Delta x}{2}\Big([f(x_0) + f(x_1)) + (f(x_1) + f(x_2)) + \dots + (f(x_{n-1}) + f(x_n)\Big)\\ &= \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n)\right]\end{aligned}$$



Additionally, we can approximate with parabolas rather than straight lines, which gives

<u>Simpson's Rule</u>, S_n:

- Divide $[a,b]$ into n subint. of length $\Delta x = \frac{b-a}{n}$, <u>n must be even.</u>

Then $\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\dots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\right]$.

- It turns out $S_{2n} = \frac{1}{3}T_n + \frac{2}{3}M_n$

<u>Ex</u>: Use T, M (L, R), $n=5$ to approx $\int_1^2 \frac{1}{x}\,dx$ 0.745635 ~~S_n: 0.658201~~

0.695635 0.691908 0.645635

Corresponding errors: L_5 R_5 T_5 M_5

error −0.052488 0.047512 −0.002488 0.001239

<u>Notice</u>: 1) L, R errors have opposite sign

2) S_n, T_n, M_n much more accurate than L_n, R_n

3) T, M errors opposite in sign

4) M error $\approx \frac{1}{2}$ T_n error.

These hold true in general: ~~In fact, the following bounds on the~~.

<u>Ex</u>: S_{10} for $\int_1^2 (\frac{1}{x})\,dx \approx 0.693150$

- <u>Next Question</u>: What if we don't know $\int_a^b f(x)\,dx$? Can we still determine accuracy?

<u>A</u>: Yes. $|f''(x)| \leq K$:

$$|E_T| < \frac{K(b-a)^3}{12n^2}, \quad |E_M| \leq \frac{K(b-a)^3}{24n^2},$$

$|f^{(4)}(x)| \leq K$,

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}$$

We can also use these bounds to find n large enough to guarantee small error.

<u>Ex</u>: T_n for $\int_1^2 \frac{1}{x}dx$ accurate within 0.0001. First, $|f''(x)| = \left|\frac{2}{x^3}\right|$ so $|f''(x)| \leq 2$ on $[1,2]$

$$|E_T| \leq \frac{2(2-1)^3}{12n^2} < 0.0001$$

Solve for n: $n^2 > \frac{2}{12(.0001)}$

$n > 40.8$

<u>$n = 41$</u>