§7.3 Trig Substitutions

(Review sec x / tan x trig integrals).

- Integrals showing up with circles/ellipses/arc length/surface area.
  - $\sqrt{x^2+a^2}$, $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$

$1+\tan^2\theta=\sec^2\theta$	$1-\sin^2\theta=\cos^2\theta$	$\sec^2\theta-1=\tan^2\theta$
Sub: $x=a\tan\theta$	$x=a\sin\theta$	$x=a\sec\theta$

This is ok if we can undo the substitution (may need to restrict domain)

Ex 1: $\int x\sqrt{x^2+4}\,dx$ (note: u-sub is easier)

w/ u-sub: $u=x^2+4$, $du=2x\,dx$

$= \int \frac{1}{2}\sqrt{u}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2}+C$

$= \frac{1}{3}(x^2+4)^{3/2}+C$

w/ trig sub: $x=2\tan\theta$ $dx=2\sec^2\theta\,d\theta$

$= \int 2\tan\theta\sqrt{4\tan^2\theta+4}\;2\sec^2\theta\,d\theta$

$= \int 2\tan\theta\cdot 2\sqrt{\sec^2\theta}\cdot 2\sec^2\theta\,d\theta$

$= 8\int \tan\theta\sec^3\theta\,d\theta$

$u=\sec\theta$ $du=\sec\theta\tan\theta\,d\theta$

$= 8\int u^2\,du$

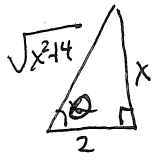
$= \frac{8}{3}u^3+C$

$= \frac{8}{3}\sec^3\theta+C$

$= \frac{8}{3}\left(\frac{(x^2+4)^{1/2}}{2}\right)^3+C$

$= \frac{1}{3}(x^2+4)^{3/2}+C$

Need to put $\sec\theta$ in terms of x



$\tan\theta=\frac{x}{2}$

$\sec\theta=\frac{hyp}{adj}=\frac{\sqrt{x^2+4}}{2}$

Ex 2: $\int\sqrt{1-x^2}\,dx$ • no u-sub here! $x=\sin\theta$ $dx=\cos\theta\,d\theta$

• $\sin(2\theta)=2\sin\theta\cos\theta$

$= \int\sqrt{1-\sin^2\theta}\cos\theta\,d\theta$

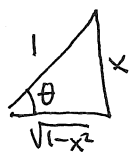
$= \int\cos^2\theta\,d\theta$

$= \int\frac{1}{2}(1+\cos(2\theta))\,d\theta$

$= \frac{1}{2}\theta+\frac{1}{4}\sin(2\theta)+C$

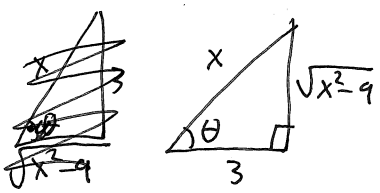
$= \frac{1}{2}\arcsin(x)+\frac{1}{2}\sin\theta\cos\theta$

$= \frac{1}{2}\arcsin(x)+\frac{1}{2}x\sqrt{1-x^2}+C$



Ex: $\int \frac{dx}{x^2\sqrt{x^2-9}}$ $x = 3\sec\theta \quad dx = 3\sec\theta\tan\theta\, d\theta$

$$= \int \frac{3\sec\theta\tan\theta\, d\theta}{9\sec^2\theta\sqrt{9\sec^2\theta - 9}}$$



$$= \int \frac{\tan\theta\, d\theta}{9\sec\theta \cdot 3\sqrt{\tan^2\theta}}$$

$$= \frac{1}{9}\int \frac{d\theta}{\sec\theta}$$

$$= \frac{1}{9}\int \cos\theta\, d\theta$$

$$= \frac{1}{9}\sin\theta + C$$

$$= \frac{1}{9}\frac{\sqrt{x^2-9}}{x} + C$$

- Sometimes need to complete the square. See Worksheet 4.
- Do worksheet 4.