§7.2 Trig Integrals

• Review Quiz 1, questions from PFD

Goal: Integrate powers & products of the trig functions. (electromagnetism)

~~Toolkit~~

Ex 1: Compute $\int \sin^3(x)\,dx$.

How do we attack this? A u-sub for $u=\sin(x)$ would be nice, but there is no $\cos(x)$ for $du=\cos(x)\,dx$.

Use the Pyth. Thm: $\sin^2(x)+\cos^2(x)=1$!

$$\int \sin^3(x)\,dx = \int \sin^2(x)\cdot\sin(x)\,dx = \int (1-\cos^2(x))\sin(x)\,dx \qquad u=\cos(x)\quad du=-\sin(x)\,dx$$
$$= -\int (1-u^2)\,du$$
$$= -u+\tfrac{1}{3}u^3+C$$
$$= \underline{-\cos(x)+\tfrac{1}{3}\cos^3(x)+C}$$

Ex 2: Compute $\int \sin^3(x)\cos^8(x)\,dx$.

$$\int \sin^3(x)\cos^8(x)\,dx = \int \sin(x)\,(1-\cos^2(x))\cos^8(x)\,dx \qquad u=\cos(x)$$
$$= -\int (1-u^2)u^8\,du$$
$$= \int u^{10}-u^8\,du$$
$$= \tfrac{1}{11}u^{11}-\tfrac{1}{9}u^9+C$$
$$= \underline{\tfrac{1}{11}\cos^{11}(x)-\tfrac{1}{9}\cos^9(x)+C}$$

Useful Tools

- $\sin^2(x)+\cos^2(x)=1$
- $\sin^2(x)=\frac{1}{2}-\frac{1}{2}\cos(2x)$
- $\cos^2(x)=\frac{1}{2}+\frac{1}{2}\cos(2x)$
- $\tan^2(x)+1=\sec^2(x)$
- $\int \sec(x)=\ln|\sec(x)+\tan(x)|$
- $\int \tan(x)=\ln|\cos(x)|$

~~Group~~: You do it: Find $\int \cos^3(x)\sin^2(x)\,dx$.

A: $\int \cos(x)\,(1-\sin^2(x))\sin^2(x)\,dx = \underline{\tfrac{1}{3}\sin^3(x)-\tfrac{1}{5}\sin^5(x)+C}$

Ex 3: Compute $\int_0^\pi \cos^2(x)\,dx$.

$$\int_0^\pi \cos^2(x)\,dx = \int_0^\pi \tfrac{1}{2}(1+\cos(2x))\,dx = \tfrac{1}{2}\left(x+\tfrac{1}{2}\sin(2x)\right)\Big|_0^\pi = \tfrac{1}{2}\left(\pi+\tfrac{1}{2}\cancel{\sin(2\pi)}^{0}\right)-\tfrac{1}{2}\left(\cancel{0+\tfrac{1}{2}\sin(0)}^{0}\right)$$
$$= \underline{\tfrac{\pi}{2}}.$$

Use ~~half~~ ~~double~~ half angle formula

u-sub

• Worksheet 3a.

Part 2: $\sec(x), \tan(x)$.

Recall: $\int \tan(x)\,dx = \ln|\sec(x)| + c$, $\int \sec(x)\,dx = \ln|\sec(x)+\tan(x)| + C$

$\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$, $\frac{d}{dx}(\tan(x)) = \sec^2(x)$.

Ex: 1 $\int \tan^3\theta \sec^4\theta \, d\theta$

$$\begin{aligned}\int \tan^3\theta \sec^4(\theta)\,d\theta &= \int \tan^3(\theta)\sec^2(\theta)\sec^2(\theta)\,d\theta \\ &= \int \tan^3(\theta)(1+\tan^2(\theta))\sec^2(\theta)\,d\theta \qquad u=\tan\theta,\ du=\sec^2(\theta)\,d\theta \\ &= \int u^3(1+u^2)\,du \\ &= \frac{1}{4}u^4 + \frac{1}{6}u^6 + C \\ &= \underline{\frac{1}{4}\tan^4(\theta) + \frac{1}{6}\tan^6(\theta) + C}\end{aligned}$$

Ex: $\int \tan^3\theta \sec^3\theta\,d\theta = \int \tan\theta \tan^2\theta \sec^2\theta \sec\theta\,d\theta$

$$\begin{aligned}&= \int (\sec^2\theta - 1)\sec^2\theta \sec\theta\tan\theta\,d\theta \qquad u=\sec\theta \\ &= \int (u^2-1)u^2\,du \\ &= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C \\ &= \underline{\frac{1}{5}\sec^5(\theta) - \frac{1}{3}\sec^3(\theta) + C}\end{aligned}$$

Other cases are hard:

Ex: $\int \sec^3 x\,dx = \sec x\tan x - \int \sec x \tan^2 x\,dx$ $\qquad u=\sec x$, $du=\sec x\tan x$, $dv=\sec^2 x$, $v=\tan x$

$$\begin{aligned}&= \sec x\tan x - \int \sec x(\sec^2 x - 1)\,dx \\ &= \sec x\tan x - \int \sec^3 x + \int \sec x\,dx\end{aligned}$$

$$2\int \sec^3 x\,dx = \sec x\tan x + \ln|\sec x + \tan x| + C$$

$$\underline{\int \sec^3(x)\,dx = \frac{1}{2}\left[\sec x\tan x + \ln|\sec x + \tan x|\right] + C}$$

Worksheet 3b: