§7.4 Integration of Rational Functions

Warm up: 1) Find $f'(x)$ for $f(x) = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)$. 2) Add $\frac{1}{x}$ and $\frac{-1}{x+1}$. 3) Do long division: $\frac{x^3+x}{x-1}$.

A: 1) $f'(x) = \frac{1}{2}\frac{1}{\left(\frac{x}{2}\right)^2+1}\cdot\frac{1}{2} = \frac{1}{4}\frac{1}{\frac{x^2}{4}+1} = \frac{1}{x^2+4}$ 2) $\frac{1}{x^2+x}$ 3)

$$\begin{array}{r} x^2+x+2 \quad r2 \\ x-1 \overline{)\, x^3+0x^2+x+0} \\ -(x^3-x^2) \\ \hline x^2+x \\ -(x^2-x) \\ \hline 2x+0 \\ -(2x-2) \\ \hline 2 \end{array}$$

$: \; x^2+x+2+\frac{2}{x-1}$.

• Goal: Integrate rational functions $f(x) = \frac{P(x)}{Q(x)}$ where P, Q are polynomials

We'll start with simple rational functions.

Recall: a) $\int \frac{1}{2x+5}\,dx = \int \frac{1}{u}\frac{du}{2} = \frac{1}{2}\ln|2x+5| + C$

$u = 2x+5$
$du = 2dx$

b) $\int \frac{6x}{3x^2+4}\,dx = \int \frac{1}{u}\,du = \ln|3x^2+4| + C$

$u = 3x^2+4$
$du = 6x\,dx$

c) $\int \frac{1}{x^2+4}\,dx = \int \frac{1}{4}\cdot\frac{1}{\frac{x^2}{4}+1}\,dx$

$= \frac{1}{4}\int \frac{1}{\left(\frac{x}{2}\right)^2+1}\,dx$

$= \frac{1}{4}\cdot 2\int \frac{1}{u^2+1}\,du$ $\quad u = \frac{x}{2}$, $du = \frac{1}{2}dx$

$= \frac{1}{2}\tan^{-1}(u) + C$

$= \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)$.

Plan: Write complicated rational functions as sums of these.

Ex: $\int \frac{x^3+x}{x-1}\,dx = \int x^2+x+2+\frac{2}{x-1}\,dx = \frac{1}{3}x^3 + \frac{x^2}{2} + 2x + 2\ln|x-1| + C$.

This function is improper, $\deg Q(x) \geq \deg P(x)$. First we do division to write as a proper fraction. (warmup)
In this case that was enough!

Ex: $\int \frac{1}{x^2+x}\,dx = \int \frac{1}{x(x+1)}\,dx \underset{\text{by warmup}}{=} \int \frac{1}{x} - \frac{1}{x+1}\,dx = \ln|x| - \ln|x+1| + C = \ln\left|\frac{x}{x+1}\right| + C$

• already proper
• Next step: factor denominator!

Worksheet 1A

- note: we will only look at cases where $Q(x)$ factors into linear factors or irreducible (no rational/real roots) quadratic factors

~~[illegible]~~

Partial Fraction Decomposition

We are ready to put everything together. Want to "reverse addition" of ~~fr~~

~~Ex~~: For each linear term: $ax+b$ we get a term $\frac{A_i}{ax+b}$ in PFD.

For each irr. quadratic: ax^2+bx+c we get a term $\frac{A_1x+B_1}{ax^2+bx+c}$ in PFD.

Ex: $\frac{1}{x^2-1} = \frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} = \boxed{\frac{1}{2}\cdot\frac{1}{x-1} - \frac{1}{2}\frac{1}{x+1}}$ Now we could integrate $\frac{1}{x^2-1}$!

Solve for A, B: ~~6~~ $1 = A(x-1) + B(x+1)$ Plug in $x=1$: $1 = A\cdot 0 + B\cdot 2 \Rightarrow B = \frac{1}{2}$

Plug in $x=-1$: $1 = A\cdot(-2) + B\cdot 0 \Rightarrow A = -\frac{1}{2}$

Ex: $\int \frac{1}{x^2-7x+10}\,dx$

$\frac{1}{(x-5)(x-2)} = \frac{A}{x-5} + \frac{B}{x-2} \Rightarrow 1 = A(x-2) + B(x-5) \Rightarrow$ $x=2$: $1 = -3B$, $B = -\frac{1}{3}$

$x=5$: $1 = 3A$ $A = \frac{1}{3}$

$= \int\left(\frac{1}{3}\cdot\frac{1}{x-5} - \frac{1}{3}\cdot\frac{1}{x-2}\right)dx = \boxed{\frac{1}{3}\ln|x-5| - \frac{1}{3}\ln|x-2| + C}$

Ex: $\int \frac{3x-9}{(x-1)(x+2)^2}\,dx$

$\frac{3x-9}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2} \longrightarrow 3x-9 = A(x+2)^2 + B(x-1)(x+2) + C(x-1)$

repeated linear term

$x=1$: $-6 = 9A$, $A = -\frac{2}{3}$

$x=-2$: $-15 = -3C$, $C = 5$

$x=0$: $-9 = 4A - 2B - C$

$-9 = -\frac{8}{3} - 2B - 5$

$-4 + \frac{8}{3} = -2B$

$-\frac{4}{3} = -2B$

$B = \frac{2}{3}$

$\int \frac{3x-9}{(x-1)(x+2)^2}\,dx = \int -\frac{2}{3}\cdot\frac{1}{x-1} + \frac{2}{3}\cdot\frac{1}{x+2} + 5\cdot\frac{1}{(x+2)^2}\,dx$

$= \boxed{-\frac{2}{3}\ln|x-1| + \frac{2}{3}\ln|x+2| - \frac{5}{x+2} + C}$

Ex: Find the PFD of $\frac{18}{(x+3)(x^2+9)}$.

$\frac{18}{(x+3)(x^2+9)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+9} \Rightarrow 18 = A(x^2+9) + Bx(x+3) + C(x+3)$

$x=-3$: $18 = 18A$

$A = 1$

So $18 = x^2+9 + Bx^2 + 3Bx + Cx + 3C$

$= (B+1)x^2 + (3B+C)x + (9+3C)$

$\rightarrow B+1=0$ and $9+3C=18$ and $3B+C=0$

$B=-1$ $3C = 9$ $-3+3=0$ ✓

$C=3$