Direction Fields

Let's say we have a first order ~~Eqn~~ diff. eq.

$y' = F(x,y)$.

This says that at any (x,y), the slope of ~~curve~~ the solution curve is $F(x,y)$.

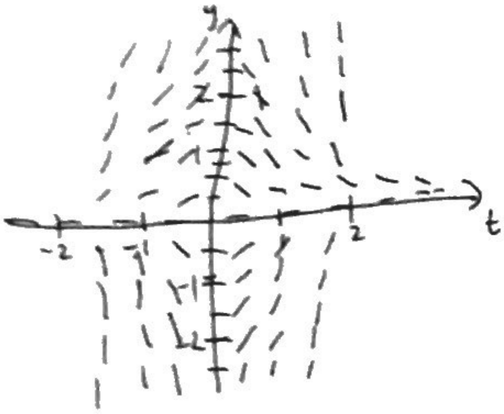
We can approximate near (x,y) with a straight line. If we sketch such lines on a graph, we get the slope field or direction field for the diff. eq.

$y' - xy = -x^2$

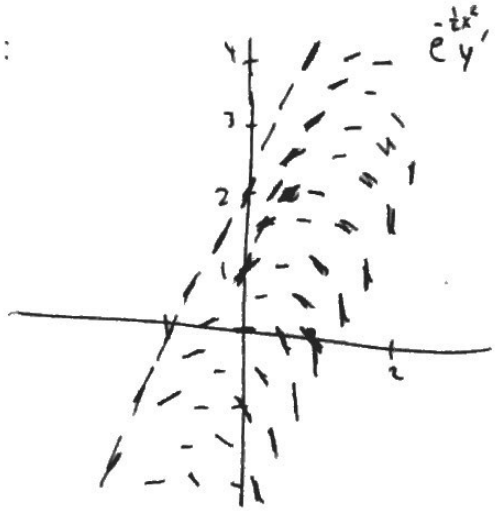
$e^{-\frac{1}{2}x^2} y' - x e^{-\frac{1}{2}x^2} y = e^{-\frac{1}{2}x^2} \cdot x^2$

Ex: $y' = -ty$

Here $F(t,y) = -ty$. Let's sketch the direction field:



$\frac{dy}{dx} = y - 2x$



Ex: Draw the slope field for $y' = y - t$ and sketch the solution curves satisfying the initial conditions a) $y(0) = 1$ and b) $y(1) = -2$

We can sketch the slope field using isoclines: $F(t,y) = c$.

We can sketch the slope field using isoclines: $F(t,y)=c$.

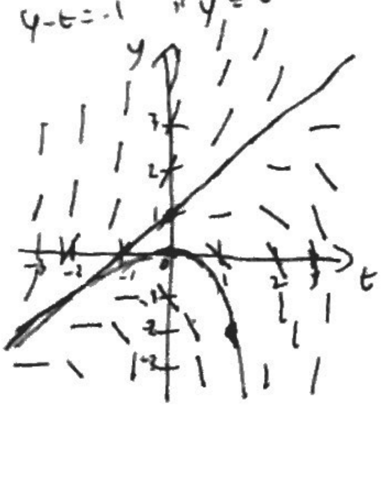
We can sketch the slope field: so we have segments of slope 0 along $y=t$

$c=0$: $y - t = 0$ so $y = t$ " " " " " " 1 along $y = t+1$

$c=1$: $y - t = 1$ so $y = t+1$ " " " " " " 2 " $y = t+2$

$c=2$: $y - t = 2$ " $y = t+2$ " " " " " " -1 along $y = t-1$

$c=-1$: $y - t = -1$ " $y = t-1$



We can get information about asymptotic ($t \to \infty$) behavior of solutions.

Autonomous

A first-order diff. eq. is <u>autonomous</u> if ~~x~~ x does not appear.

Ex: $y' = y^2(1-y)(y-2)$

$y' = (2+y)(3-2y)$

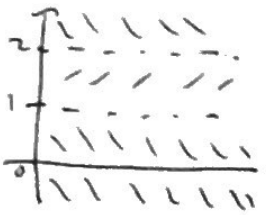
$y' = e^y$

We can determine the ~~limiting~~ asymptotic behavior of ~~autonomous~~ what to autonomous D.E.

$\to y' = y^2(1-y)(y-2)$

Ex:

- eq. sols at y = 0, 1, 2



if $y > 2$, y' is $(+)(-)(+) = (-)$

$1 < y < 2$, y' is $(+)(-)(-) = (+)$

$0 < y < 1$, y' is $(+)(+)(-) = (-)$

$y < 0$, y' is $(+)(+)(-) = (-)$

So, if a solution satisfies $y(0) > 2$, $\lim_{x\to\infty} y(x) = 2$. ~~etc~~

What is $\lim_{x\to\infty} y(x)$ for the solution to this diff. eq. that satisfies $y(0) = \frac{1}{2}$?

~~Linear Diff Eq~~ Sep. of Vars - Logistic Equation

Pre class: Find PFD of $\frac{1}{y(1-\frac{y}{M})}$: $\frac{A}{y} + \frac{B}{1-\frac{y}{M}} = \frac{A}{y} + \frac{MB}{M-y}$

$= 1$

$A(M-y) + MBy = 1$

$\frac{1}{y} + \frac{1}{M-y} = \frac{1}{y} - \frac{1}{y-M}$

$A(1-\frac{y}{M}) + By = 1$

$B = \frac{1}{M}$ $A = 1$

$y = M$: $B = \frac{1}{M^2}$

$y = 0$: $A = \frac{1}{M}$

Recall that one model for population growth is the <u>logistic equation</u>: $\frac{dy}{dt} = ky\left(1 - \frac{y}{M}\right)$,

where $k > 0$ is a growth constant and $M > 0$ is the carrying capacity.

Let's solve this equation.

$$\frac{dy}{y(1-y/M)} = k\,dt$$

$$\int\left(\frac{1}{y} - \frac{1}{y-M}\right)dy = \int k\,dt$$

$$\ln|y| - \ln|y-M| = kt + C$$

$$\ln\left|\frac{y}{y-M}\right| = e^{kt+C}$$

$$\frac{y}{y-M} = Ae^{kt}$$

(Note, $A = \frac{y_0}{y_0 - M}$ by subbing $y(0) = y_0$)

$$y = (y-M)Ae^{kt}$$

$$y - yAe^{kt} = -MAe^{kt}$$

$$y(1 - Ae^{kt}) = -MAe^{kt}$$

$$y = \frac{MAe^{kt}}{Ae^{kt} - 1}$$

$$\boxed{y = \frac{M}{1 - \frac{1}{A}e^{-kt}}}$$

$t \to \infty$, $y \to M$ ✓