Intro to Differential Eq (9.1)

Def: A <u>differential equation</u> is an equation that contains an unknown function and one or more of its derivatives.

The <u>order</u> of a differential equation is the order of the highest derivative that occurs.

$x(y')^2 = y + x$, 1st, nonlin.

Ex: $y' = x$, $y' = \sin x$, $y' = e^x + 72\cos 3x$, $y' = xy$, $y'' + 3y' + 2y = 0$, $y'' = \frac{-y}{x} + y'$

1st, lin   1st, lin   1st, lin   1st, lin   2nd, lin   2nd, lin

Linear: can write as $a_n(x)y^{(n)} + \dots + a_1(x)y' + a_0(x)y = b(x)$

i.e. there are no non-linear expressions involving y or its derivatives

Why do we care?

- Lots and lots and lots of problems in physics/engineering/biology/statistics/chemistry are modeled by differential equations.

• <u>Initial Value Problem</u>: a differential equation together with an <u>initial condition</u> that the solution or its derivatives must satisfy.

Ex: Population Growth

One model is $P'(t) = k \cdot P(t)$, $P(0) = P_0$

↑ pop. at time $t=0$, k the "rate constant"

- i.e. growth is proportional to the current population.

(Actually, we already know solutions to this model)

$P(t) = Ce^{kt}$ satisfies the diff. eq., i.e. plugging in $P'(t)$ and $P(t)$ yields a true statement.
or solves

$P'(t) = k\,Ce^{kt} = k \cdot P(t)$ ✓

- In general, differential eq. are "hard" to solve, but "easy" to check solutions for.

(Sim to finding roots of a polynomial: finding the roots is hard, but its easy to check if a particular value is a root)

Ex: Cooling

From Newton, the rate of cooling of an object at initial temp T_0 in a room at temp T^* is

$$\frac{dT}{dt} = -k\,(T(t) - T^*), \quad T(0) = T_0,$$

k determined by?

If $T(t) = A + Be^{-kt}$, can we find A & B in terms of the known constants?

$T(0) = T_0 = A + B$

As $t \to \infty$, $T(t) \to T^*$ (the object cools to match the room)

$T^* = \lim_{t\to\infty} T(t) = \lim_{t\to\infty} A + Be^{-kt} = A$

so A is T^* and B is the diff. between T_0 and T^*.

ex: Find the values of r for which $y=e^{rx}$ is a solution to $y''-7y'+12y=0$.

Plug in $y=e^{rx}$, $y'=re^{rx}$, $y''=r^2e^{rx}$:

$$r^2e^{rx}-7re^{rx}+12e^{rx}=0$$

$$e^{rx}\underbrace{(r^2-7r+12)}_{\text{characteristic polynomial}}=0$$

$e^{rx}\neq 0$, so we need $r^2-7r+12=0$

$$(r-4)(r-3)=0$$

So $y=e^{rx}$ is a solution of the D.E.

if $r=3, r=4$.

ex: Show that $y=ce^{-2x}+e^{-x}$ is a solution of $y'+2y=e^{-x}$ for any constant c.
Find the constant c which corresponds to the solution satisfying I.C. $y(1)=10$.

ex: Verify that $y=t^{-\frac{3}{2}}$ is a solution ~~for~~ to $4t^2y''+12ty'+3y=0$ for $t>0$.

$y'(t)=-\frac{3}{2}t^{-\frac{5}{2}}$, $y''(t)=\frac{15}{4}t^{-\frac{7}{2}}$

Substitute: $4t^2\left(\frac{15}{4}t^{-\frac{7}{2}}\right)+12t\left(-\frac{3}{2}t^{-\frac{5}{2}}\right)+3\left(t^{-\frac{3}{2}}\right)=0$

$$15t^{-\frac{3}{2}} - 18t^{-\frac{3}{2}}+3t^{-\frac{3}{2}}=0$$

$$0=0.$$

So $y=t^{-\frac{3}{2}}$ is a solution.

Note: $y=t^{-\frac{1}{2}}$, $y=-4t^{-\frac{3}{2}}$, $y=5t^{-\frac{1}{2}}$, $y=-4t^{-\frac{3}{2}}+5t^{-\frac{1}{2}}$ are also solutions

- If we add the initial conditions $y(4)=\frac{1}{8}$, $y'(4)=-\frac{3}{64}$, then
  $y=t^{-3/2}$ is still a solution since $y(4)=4^{-3/2}=8^{-1}=\frac{1}{8}$ and
  $y'(4)=-\frac{3}{2}(4)^{-\frac{5}{2}}=\frac{-3}{2^6}=-\frac{3}{64}$.

In fact, this is now the unique solution. (second order ODE $\Rightarrow$ 2 conditions needed)