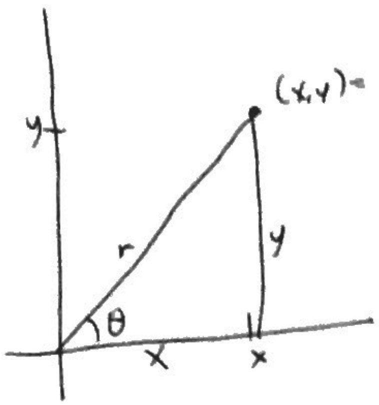
Polar Coordinates



We can describe points in the x-y plane by giving their x and y coords, but this is not the only way.

We can also specify the distance $\underline{r}$ from the origin along with the direction we travel to reach the point, given in radians from the positive x-axis $\underline{\theta}$.

These are <u>polar coords</u>.

How can we relate polar and Cartesian coordinates?

- Use trigonometry.

<u>Polar to Cartesian:</u>

From the triangle, $\cos\theta = \frac{x}{r}$, so $\underline{x = r\cos\theta}$. $\sin\theta = \frac{y}{r}$, so $\underline{y = r\sin\theta}$

<u>Cartesian to Polar:</u>

$r^2 = x^2 + y^2$, so $r = \sqrt{x^2+y^2}$

$\tan\theta = \frac{y}{x}$, so $\theta = \arctan\left(\frac{y}{x}\right)$.

<u>Ex</u>: $(3, 2) \to$ polar

$r = \sqrt{13}$

$\tan\theta = \frac{2}{3}$

$\theta = \tan^{-1}\left(\frac{2}{3}\right) \approx 0.588$

<u>Ex</u>: $\left(3, \frac{5\pi}{6}\right) \to$ xy

$x = 3\cos\left(\frac{5\pi}{6}\right) = -\frac{3\sqrt{3}}{2}$

$y = 3\sin\left(\frac{5\pi}{6}\right) = \frac{3}{2}$

<u>Ex</u>: $2x - 5x^3 = 1 + xy \Rightarrow$ polar

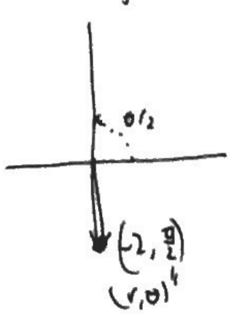
<u>Ex</u>: $r = -8\cos\theta$

<u>Ex</u>: $r\cos\theta = a$, $r\sin\theta = b$ $\Leftrightarrow$ $r = a\sec\theta$, $r = b\csc\theta$

- θ is not unique – $\left(4, \frac{\pi}{2}\right)$ and $\left(4, \frac{5\pi}{2}\right)$ are the same point for ex.
- The origin has polar coords $(0, \theta)$ for any angle θ
- r can be negative : rotate θ then go backwards $\underline{r}$.

<u>Ex</u>: $r = 1 - \cos\theta$ (cardioid)

$\left(-2, \frac{\pi}{2}\right)$:



<u>Polar Curves</u>

What do the graphs in the x-y plane of simple polar equations look like?

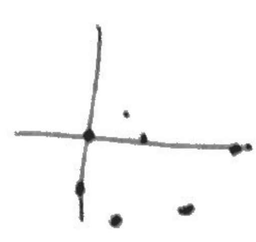
$r = 2$? $\theta = \frac{\pi}{4}$?

More complicated: $r = 2\cos\theta$

- convert to Cartesian sometimes makes it easier to see what a curve looks like?

: $r = 2\cos\theta - 1$

θ	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π
r	1	0.73	0	-1	-2	-2.73	-3



Tangents to Polar Curves

We will find tangents to polar curves ~~by~~ $r=f(\theta)$ by writing parametric equations for x & y in terms of ~~A~~ θ.

If, $r=f(\theta)$, then we have $x=r\cos\theta = f(\theta)\cos\theta$ & $y=r\sin\theta=f(\theta)\sin\theta$.

Then recall from last week that $\dfrac{dy}{dx}=\dfrac{dy/d\theta}{dx/d\theta}=\dfrac{f'(\theta)\sin\theta+f(\theta)\cos\theta}{f'(\theta)\cos\theta-f(\theta)\sin\theta}$

Ex: Find the slope of the tangent line to $r=\theta$ at $\theta=\frac{\pi}{2}$ and $\theta=\pi$.

Ex: $r=3+8\sin\theta$ at $\theta=\frac{\pi}{6}$

$$\frac{dy}{dx}=\frac{\frac{d}{d\theta}(\theta\sin\theta)}{\frac{d}{d\theta}(\theta\cos\theta)}=\frac{\sin\theta+\theta\cos\theta}{\cos\theta-\theta\sin\theta}$$

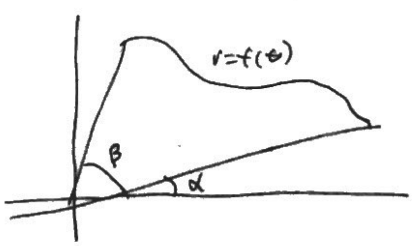
at $\theta=\frac{\pi}{2}$: $\dfrac{dy}{dx}=\dfrac{1+0}{0-\frac{\pi}{2}\cdot 1}=-\dfrac{2}{\pi}$

at $\theta=\pi$: $\dfrac{dy}{dx}=\dfrac{0+\pi\cdot(-1)}{(-1)-0}=\pi$

Area in Polar Coordinates

By approximation with circular arcs, each of area $A=\frac{1}{2}r^2\theta$ ,

we find the area of a region bounded by the polar curve $r=f(\theta)$ is



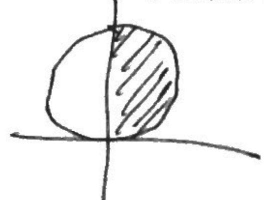
$$\boxed{A=\int_\alpha^\beta \frac{1}{2}\,[\underset{\uparrow\, r}{f(\theta)}]^2\,\underset{\uparrow\,\Delta\theta}{d\theta}}$$

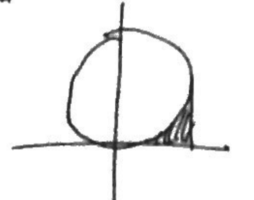
$$=\int_\alpha^\beta \frac{1}{2}r^2\,d\theta$$

Note, different area from before:

New: enclosed area

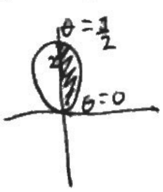
old: area below





$A=\int_\alpha^\beta \frac{1}{2}(f(\theta))^2\,d\theta$ $\qquad$ $A=\int_a^b f(x)\,dx$

Ex: Find the area of the right semicircle with equation $r=4\sin\theta$.



$$A=\int_0^{\pi/2}\frac{1}{2}\left(4\sin\theta\right)^2 d\theta$$

$$=\int_0^{\pi/2} 8\sin^2\theta\, d\theta$$

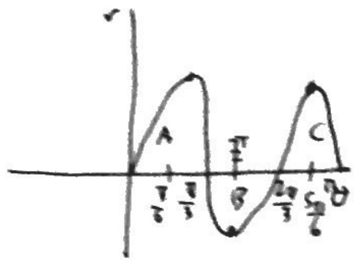
$$=\int_0^{\pi/2} 8\cdot\left(\tfrac{1}{2}-\tfrac{1}{2}\cos 2\theta\right)d\theta$$

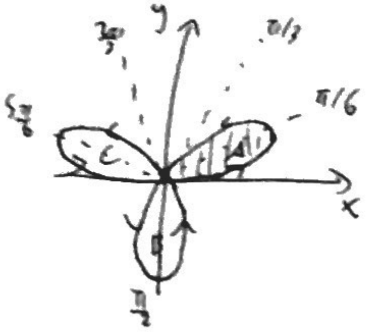
$$=\int_0^{\pi/2} 4-4\cos 2\theta\, d\theta$$

$$= 4\theta-2\sin 2\theta\Big|_0^{\pi/2}=2\pi$$

Check: $A=\frac{1}{2}\pi r^2$
$=\frac{1}{2}\pi(2)^2$
$=2\pi$ ✓

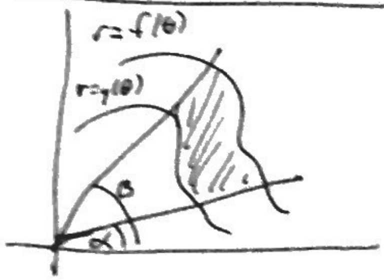
Ex. Sketch $r = \sin 3\theta$ and compute area bounded by one "petal"





$$A = \int_0^{\pi/3} \frac{1}{2}(\sin 3\theta)^2\, d\theta$$
$$= \frac{1}{2}\int_0^{\pi/3} \frac{1}{2}(1 - \cos 6\theta)\, d\theta$$
$$= \frac{1}{4}\left[\theta - \frac{1}{6}\sin 6\theta\right]_0^{\pi/3}$$
$$= \frac{1}{4}\cdot\frac{\pi}{3} = \boxed{\frac{\pi}{12}}$$

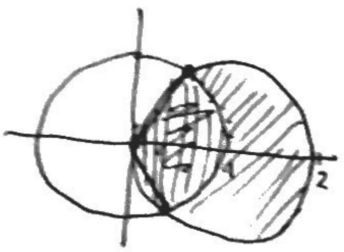
Area between two polar curves



Subtract area enclosed by $r = g(\theta)$ from area enclosed by $r = f(\theta)$

$$A = \frac{1}{2}\int_\alpha^\beta [f(\theta)]^2\, d\theta - \frac{1}{2}\int_\alpha^\beta [g(\theta)]^2\, d\theta$$
$$= \frac{1}{2}\int_\alpha^\beta \left([f(\theta)]^2 - [g(\theta)]^2\right) d\theta$$

Ex: Find the area inside circle $r = 2\cos\theta$ but outside circle $r = 1$



$\cancel{A = \frac{1}{2}}$ Find intersection: need $(r, 2\cos\theta) = (r, 1)$

i.e. $2\cos\theta = 1$ so $\cos\theta = \frac{1}{2}$ so $\theta = \pm \pi/3$.

Then $$A = \frac{1}{2}\int_{-\pi/3}^{\pi/3} (2\cos\theta)^2\, d\theta - \frac{1}{2}\int_{-\pi/3}^{\pi/3} 1^2\, d\theta$$
$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} 4\cos^2\theta - 1\, d\theta = \frac{1}{2}\int_{-\pi/3}^{\pi/3} 2\cos 2\theta + 2 - 1\, d\theta$$
$$= \frac{1}{2}(\sin 2\theta + \theta)\Big|_{-\pi/3}^{\pi/3} = \frac{1}{2}\left(\sin 2\pi/3 + \pi/3 - \sin(-2\pi/3) + \pi/3\right)$$
$$= \frac{1}{2}\left(\frac{\sqrt{3}}{2} + \frac{\pi}{3} + \frac{\sqrt{3}}{2} + \frac{\pi}{3}\right)$$
$$= \boxed{\frac{\sqrt{3}}{2} + \frac{\pi}{3}}$$

Arc Length : Use parametrization again.

$x = f(\theta)\cos\theta \Rightarrow x'(\theta) = -f(\theta)\sin\theta + f'(\theta)\cos\theta$

$y = f(\theta)\sin\theta \Rightarrow y'(\theta) = f(\theta)\cos\theta + f'(\theta)\sin\theta$

$$s = \int_\alpha^\beta \sqrt{(x'(\theta))^2 + (y'(\theta))^2}\, d\theta = \int_\alpha^\beta \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$$

(some algebra + Pythag. Thm)

Ex: Find the length of $r = \theta^2$ for $0 \le \theta \le \pi$.

$u = \theta^2 + 4 \quad du = 2\theta\, d\theta$

$$s = \int_0^\pi \sqrt{(\theta^2)^2 + (2\theta)^2}\, d\theta = \int_0^\pi \sqrt{\theta^4 + 4\theta^2}\, d\theta = \int_0^\pi \theta\sqrt{\theta^2 + 4}\, d\theta = \int \frac{1}{2}\sqrt{u}\, du = \frac{1}{3}u^{3/2} = \frac{1}{3}(\theta^2 + 4)^{3/2}\Big|_0^\pi$$
$$= \frac{1}{3}\left[(\pi^2 + 4)^{3/2} - 4^{3/2}\right]$$