Parametric Curves & Calculus

Big Idea: Use the chain rule to adapt our techinques for parametric curves.

1) Tangents

Recall that if y is a function of x and x is a function of t (calc I), then

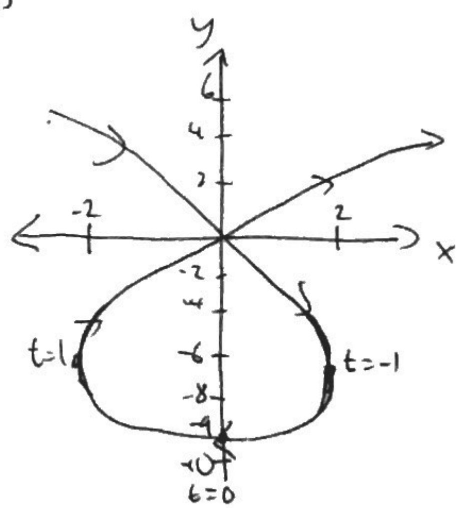
$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

So, $\boxed{\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}, \quad \text{if } dx/dt \neq 0}$

e.g. if $x(t) = t^2 - 2t$, $y(t) = t+1$, then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1}{2t-2}$ if $t \neq 1$.

ex: Find the tangent line(s) to the parametric curve given by $x = t^3 - 3t$, $y = 3t^2 - 9$ at the origin.

First, $\frac{dy}{dt} = 6t$, $\frac{dx}{dt} = 3t^2 - 3 = 3(t^2-1)$. So $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6t}{3(t^2-1)}$.



We want $\frac{dy}{dx}$ at $(0,0)$ so we need to know the appropriate values of t:

x: $0 = t^3 - 3t = t(t^2-3) \Rightarrow t = 0, \sqrt{3}, -\sqrt{3}$

y: $0 = 3t^2 - 9 = 3(t^2-3) \Rightarrow t = \sqrt{3}, -\sqrt{3}$.

So $t = \sqrt{3}$ or $t = -\sqrt{3}$ (thus there are 2 tangent lines)

$t = \sqrt{3}$: $\frac{dy}{dx} = \frac{6\sqrt{3}}{3(\sqrt{3}^2 - 1)} = \sqrt{3}$

So the tangent line here is $y - 0 = \sqrt{3}(x-0)$

$y = \sqrt{3}x$

$t = -\sqrt{3}$: $\frac{dy}{dx} = \frac{6(-\sqrt{3})}{3((-\sqrt{3})^2 - 1)} = -\sqrt{3}$

So the tangent line here is $y - 0 = -\sqrt{3}(x-0)$

$y = -\sqrt{3}x$

Horiz/Vert. Tangent Lines: We get a horizontal tangent when $\frac{dy}{dx} = 0$, so if $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$.

We get a vertical tangent when $\frac{dy}{dx}$ is undefined, so if $\frac{dx}{dt} = 0$, $\frac{dy}{dt} \neq 0$.

<u>ex</u>: Find the xy coordinates of the points where the previous parametric equations have horiz./vert. tangents.

~~Ho~~ We found $\frac{dy}{dt} = 6t$, $\frac{dx}{dt} = 3(t^2-1)$.

So we get horizontal tangent where $6t=0 \Rightarrow t=0$ (since $\frac{dx}{dt}(0) \neq 0$).

~~th~~ When $t=0$, $y = 3(0)^2 - 9 = -9$, $x = 0^3 - 3(0) = 0$, so there is a horiz. tangent at $(0,-9)$

There are vertical tangents where $3(t^2-1)=0 \Rightarrow t = \pm 1$.

When $t=1$, $x = 1^3 - 3 = -2$, $y = 3 \cdot 1^2 - 9 = -6$, while if $t=-1$, $x = (-1)^3 + 3 = 2$, $y = 3(-1)^2 - 9 = -6$

So the vertical tangents occur at $(-2,-6)$ and $(2,-6)$.

<u>Second Derivatives</u>: $\frac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$

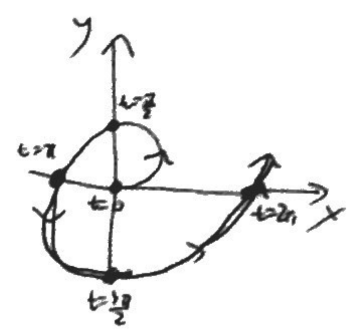
<u>ex</u>: Find $\frac{d^2y}{dx^2}$ for the parametric curve $x = e^t$, $y = te^{-t}$

$$\frac{dy}{dt} = e^{-t} - te^{-t} = e^{-t}(1-t) \qquad \frac{dx}{dt} = e^t$$

So $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{e^{-t}(1-t)}{e^t} = e^{-2t}(1-t)$. So $\frac{d}{dt}\left(\frac{dy}{dx}\right) = -2e^{-2t}(1-t) + e^{-2t}(-1)$
$= e^{-2t}(-3+2t)$

Thus $\frac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \dfrac{e^{-2t}(-3+2t)}{e^t} = \underline{e^{-3t}(-3+2t)}$

<u>ex</u>: Find a tangent line to $x = t\cos(t)$, $y = t\sin(t)$, at $t=\pi$.



$$\frac{dy}{dt} = \sin(t) + t\cos(t) \qquad \frac{dx}{dt} = \cos(t) - t\sin(t)$$

$\frac{dy}{dx} = \frac{\sin(t) + t\cos(t)}{\cos(t) - t\sin(t)}$. At $t=\pi$, $\frac{dy}{dx} = \frac{0 + \pi(-1)}{-1-0} = \pi$.

$x(\pi) = -\pi$, $y(\pi) = 0$, So $y - 0 = \pi(x+\pi)$ is the tangent line.

$\boxed{y = \pi x + \pi^2}$

Arc Length for Parametric Curves

Remember that for a curve $y=f(x)$ or $x=g(y)$, we had

$$s=\int_a^b ds \quad \text{where} \quad ds^2=dx^2+dy^2.$$

For parametric curves $x(t), y(t)$, $\alpha \le t \le \beta$, this becomes

$$s=\int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\, dt$$

$$ds^2 = dt^2\left(\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right)$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\, dt$$

Ex: Find the length of the curve $x=3t+1$, $y=9-4t$, $0\le t\le 2$

$$s=\int_0^2 \sqrt{3^2+4^2}\, dt = \int_0^2 5\, dt = 10.$$

Ex: Find the length of the curve $x=t^3+1$, $y=t^2-3$, $0\le t\le 1$.

$$\frac{dx}{dt}=3t^2 \qquad \frac{dy}{dt}=2t$$

$$s=\int_0^1 \sqrt{(3t^2)^2+(2t)^2}\, dt$$

$$=\int_0^1 \sqrt{9t^4+4t^2}\, dt$$

$$=\int_0^1 t\sqrt{9t^2+4}\, dt \qquad u=9t^2+4,\ du=18t\, dt \qquad t=0: u=2,\ t=1: u=\sqrt{13}$$

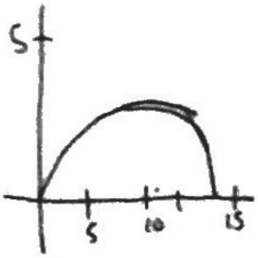
$$=\int_2^{\sqrt{13}} \frac{1}{18}\sqrt{u}\, du$$

$$=\frac{1}{18}\cdot\frac{2}{3}\, u^{3/2}\Big|_2^{\sqrt{13}}$$

$$=\frac{1}{27}\left(13^{3/2}-2^{3/2}\right)$$

$$\approx 1.631$$

Ex: Find the length of one arch of the cycloid $x=2(\theta-\sin\theta)$, $y=2(1-\cos\theta)$. $(0\le\theta\le 2\pi)$



$$\frac{dx}{d\theta}=2(1-\cos\theta) \qquad \frac{dy}{d\theta}=2\sin\theta$$

$$s=\int_0^{2\pi}\sqrt{(2(1-\cos\theta))^2+(2\sin\theta)^2}\, d\theta$$

$$=2\int_0^{2\pi}\sqrt{1-2\cos\theta+\cos^2\theta+\sin^2\theta}\, d\theta$$

$$=2\int_0^{2\pi}\sqrt{2-2\cos\theta}\, d\theta$$

Use $\cos(2u)=1-2\sin^2 u \iff 1-\cos\theta=2\sin^2\left(\frac{\theta}{2}\right)$

$$=2\int_0^{2\pi}\sqrt{2\left(2\sin^2\left(\tfrac{\theta}{2}\right)\right)}\, d\theta$$

$$=4\int_0^{2\pi}\left|\sin\left(\tfrac{\theta}{2}\right)\right| d\theta$$

$$=-4\cos\left(\tfrac{\theta}{2}\right)\cdot 2\Big|_0^{2\pi}$$

$$=-8\cos(\pi)+8\cos(0)=\boxed{16}$$