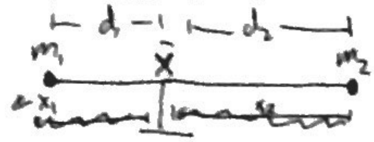
Center of Mass

Goal: Find balance point or center of mass or centroid

Simple example: 1-D



This balances if its moments are equal:

· moment about a particular axis measures tendency to rotate and is mass · signed distance to axis

$$m_1 d_1 = m_2 d_2$$

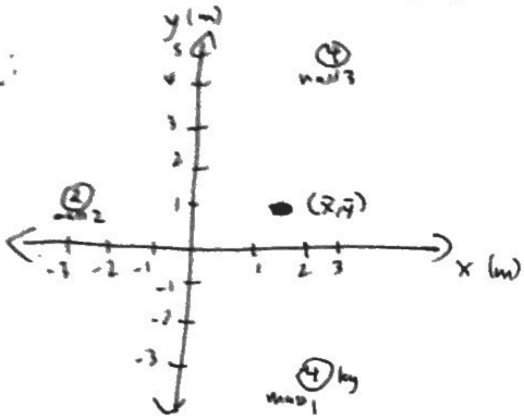
$$m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$$

$$m_1\bar{x} + m_2\bar{x} = m_1x_1 + m_2x_2$$

$$\bar{x} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} \quad \begin{matrix}\leftarrow \text{moments} \\ \leftarrow \text{masses}\end{matrix}$$

In 2-D, we find $\bar{x} = \frac{\sum m_i x_i}{\sum m_i} := \frac{M_y}{M}$, $\bar{y} = \frac{\sum m_i y_i}{\sum m_i} = \frac{M_x}{M}$ where M is the total mass of our system, M_x is its moment about the x-axis, M_y is its moment about y-axis

Ex:



$$M_y = m_1x_1 + m_2x_2 + m_3x_3$$
$$= 4 \cdot 3 + 2 \cdot (-3) + 4 \cdot 3$$
$$= 18 \text{ m·kg}$$

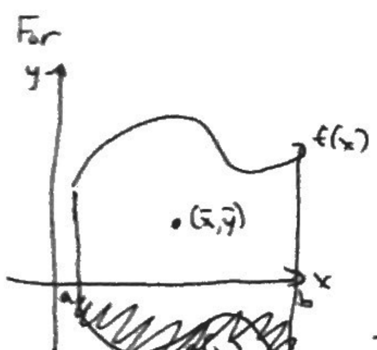
$$M_x = m_1y_1 + m_2y_2 + m_3y_3$$
$$= 4 \cdot (-3) + 2 \cdot 1 + 4 \cdot 5$$
$$= 10 \text{ m·kg}$$

$$M = 4 + 2 + 4 = 10 \text{ kg}$$

So, $\bar{x} = \frac{M_y}{M} = \frac{18}{10} = 1.8\text{ m}$, $\bar{y} = \frac{M_x}{M} = \frac{10 \text{ m·kg}}{10 \text{ kg}} = 1\text{ m}$

Flat Plates or Laminas:

For



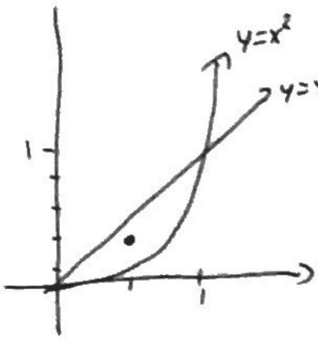
We have $M = \int_a^b \underbrace{\rho}_{\text{density}} \underbrace{f(x)}_{\text{area}}\, dx$ We will assume ρ is a constant for simplicity.

$$M_x = \int_a^b \underbrace{\rho}_{\text{density}} \underbrace{f(x)}_{\text{area}} \underbrace{\frac{f(x)}{2}}_{\substack{\text{distance to x-axis} \\ \text{on avg at } x}} dx = \int_a^b \frac{1}{2}\rho f(x)^2\, dx$$

$$M_y = \int_a^b \underbrace{\rho}_{\text{density}} \underbrace{f(x)}_{\text{area}} \underbrace{x}_{\text{distance to y-axis}} dx$$

Then $\bar{x} = \frac{M_y}{M} = \frac{\rho\int_a^b f(x)\, x\, dx}{\rho\int_a^b f(x)\, dx} = \frac{\int_a^b x f(x)\, dx}{\int_a^b f(x)\, dx}$, $\bar{y} = \frac{M_x}{M} = \frac{\int_a^b \frac{1}{2}(f(x))^2\, dx}{\int_a^b f(x)\, dx}$

ex: Find the centroid of the region bounded by $y=x, y=x^2$



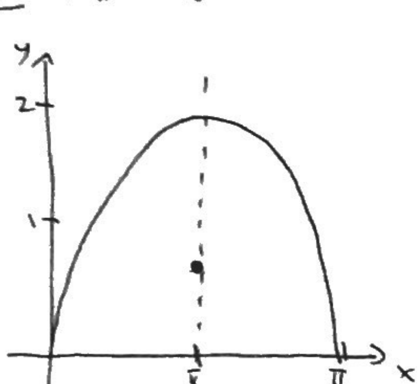
$$M = \int_0^1 x - x^2\, dx$$
$$= \tfrac{1}{2}x - \tfrac{1}{3}x^3 \Big|_0^1$$
$$= \frac{1}{6}$$

$$M_y = \int_0^1 x(x - x^2)\, dx$$
$$= \int_0^1 x^2 - x^3\, dx$$
$$= \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 \Big|_0^1$$
$$= \frac{1}{12}$$

$$M_x = \int_0^1 \tfrac{1}{2}\left((x)^2 - (x^2)^2\right) dx$$
$$= \frac{1}{2}\int_0^1 x^2 - x^4\, dx$$
$$= \frac{1}{2}\left(\tfrac{1}{3}x^3 - \tfrac{1}{5}x^5\right)\Big|_0^1$$
$$= \frac{1}{15}$$

So $(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$
$$= \left(\frac{\frac{1}{12}}{\frac{1}{6}}, \frac{\frac{1}{15}}{\frac{1}{6}}\right)$$
$$= \left(\frac{1}{2}, \frac{2}{5}\right)$$

ex: Find the centroid of the region bounded by $y = 2\sin(2x), y=0$ on $[0, \frac{\pi}{2}]$



Notice by symmetry $\bar{x} = \frac{\pi}{4}$.

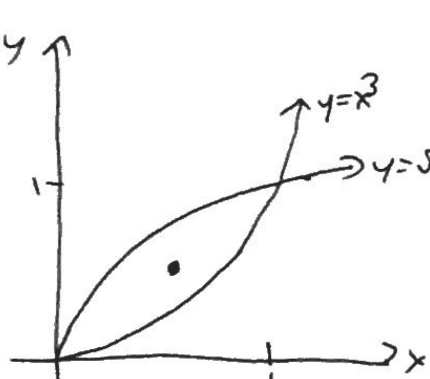
$$M = \int_0^{\pi/2} 2\sin(2x)\, dx$$
$$= -\cos(2x)\Big|_0^{\pi/2}$$
$$= 2$$

$$M_y = \int_0^{\pi/2} x \cdot (2\sin(2x))\, dx$$
(by parts) $= -x\cos(2x) + \int_0^{\pi/2} \cos(2x)\, dx$
$$= -x\cos(2x) + \tfrac{1}{2}\sin(2x)\Big|_0^{\pi/2}$$
$$= \frac{\pi}{2}$$

$$M_x = \int_0^{\pi/2} \frac{1}{2}(2\sin(2x))^2\, dx$$
$$= \int_0^{\pi/2} 2\sin^2(2x)\, dx$$
$$= \int_0^{\pi/2} 1 - \cos(4x)\, dx$$
$$= x - \tfrac{1}{4}\sin(4x)\Big|_0^{\pi/2}$$
$$= \frac{\pi}{2}.$$

So $(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$
$$= \left(\frac{\pi}{4}, \frac{\pi}{4}\right)$$

ex: Find the center of mass for the region bounded by $y = x^3$ and $y = \sqrt{x}$



$$M = \int_0^1 \sqrt{x} - x^3\, dx$$
$$= \tfrac{2}{3}x^{3/2} - \tfrac{1}{4}x^4 \Big|_0^1$$
$$= \frac{2}{3} - \frac{1}{4}$$
$$= \frac{5}{12}$$

$$M_y = \int_0^1 x(\sqrt{x} - x^3)\, dx$$
$$= \int_0^1 x^{3/2} - x^4\, dx$$
$$= \tfrac{2}{5}x^{5/2} - \tfrac{1}{5}x^5 \Big|_0^1$$
$$= \frac{1}{5}$$

$$M_x = \int_0^1 \tfrac{1}{2}\left((\sqrt{x})^2 - (x^3)^2\right) dx$$
$$= \frac{1}{2}\int_0^1 x - x^6\, dx$$
$$= \frac{1}{2}\left(\tfrac{1}{2}x^2 - \tfrac{1}{7}x^7\right)\Big|_0^1$$
$$= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{7}\right)$$
$$= \frac{5}{28}$$

So $(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$
$$= \left(\frac{\frac{1}{5}}{\frac{5}{12}}, \frac{\frac{5}{28}}{\frac{5}{12}}\right)$$
$$= \left(\frac{12}{25}, \frac{12}{28}\right)$$