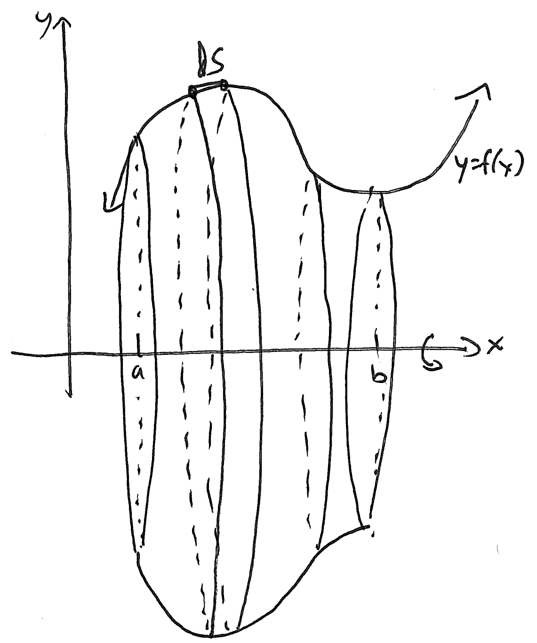
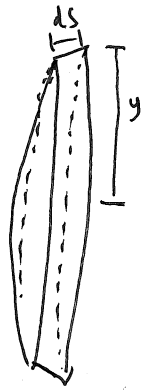
Surface Area



We will find the surface area of the solid generated by rotating the area under $y=f(x)$ from $x=a$ to $x=b$ around the x-axis.

Rotating the strip $\underline{ds}$ gives a frustrum



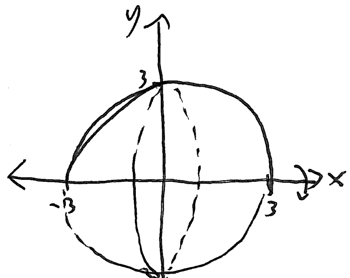
with area $dA = \underbrace{2\pi y}_{\text{circumference}} \overbrace{ds}^{\text{width}}$

So $SA = \int_a^b 2\pi y \, ds$

or, if we rotate around y-axis

$$SA = \int_a^b 2\pi x \, ds.$$

Ex: Find the surface area of the solid obtained by rotating $y=\sqrt{9-x^2}$, $-3 \le x \le 3$ about x-axis.



$$V = \int_{-3}^{3} 2\pi y \, ds$$

Since $y=f(x)$, we use $ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ then we substitute $\sqrt{9-x^2}$ for y and $\frac{-2x}{2\sqrt{9-x^2}}$ for $\frac{dy}{dx}$

$$= \int_{-3}^{3} 2\pi \sqrt{9-x^2}\sqrt{1+\left(\frac{-x}{\sqrt{9-x^2}}\right)^2}\,dx$$

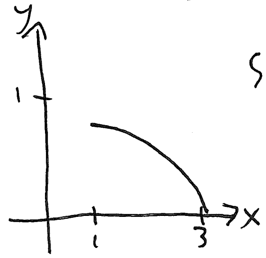
$$= 2\pi \int_{-3}^{3} \sqrt{9-x^2+x^2}\,dx$$

$$= 2\pi \int_{-3}^{3} 3\,dx$$

$$= 36\pi$$

(Notice this solid is a sphere of radius 3, so it should have $SA = 4\pi r^2 = 4\pi \cdot 9 = 36\pi$)

Ex: Find the surface area of the surface obtained by rotating $y = x^{1/2} - \frac{1}{3}x^{3/2}$ about the x-axis for $1 \le x \le 3$.



$$SA = \int_1^3 2\pi y \, ds$$

Again $y=f(x)$, so write $ds = \sqrt{1+(f'(x))^2}\,dx$

$$f'(x) = \frac{1}{2x^{1/2}} - \frac{1}{2}x^{1/2}$$

$$1+f'(x)^2 = 1 + \left(\frac{x^{-1/2}-x^{1/2}}{2}\right)^2$$

$$= 1 + \frac{x^{-1}-2+x}{4}$$

$$= \frac{4+x^{-1}-2+x}{4}$$

$$= \frac{x^{-1}+2+x}{4} = \left(\frac{x^{-1/2}+x^{1/2}}{2}\right)^2$$

$$= \int_1^3 2\pi y \sqrt{1+f'(x)^2}\,dx$$

$$= \int_1^3 2\pi \left(x^{1/2} - \frac{1}{3}x^{3/2}\right)\sqrt{\left(\frac{x^{-1/2}+x^{1/2}}{2}\right)^2}\,dx$$

$$= \int_1^3 2\pi \left(x^{1/2} - \frac{1}{3}x^{3/2}\right)\frac{1}{2}\left(x^{-1/2}+x^{1/2}\right)dx$$

$$= \pi \int_1^3 1 + x - \frac{1}{3}x - \frac{1}{3}x^2 \, dx$$

$$= \pi \left(x + \frac{1}{3}x^2 - \frac{1}{9}x^3\right)\Big|_1^3 = \frac{16\pi}{9}$$

Ex] Find the surface area of the surface obtained by rotating $y = x^{1/3}$, $1 \le y \le 2$ about y-axis. Use both forms of ds to compute the surface area.

a) Since we have $y = f(x)$, let's start with $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$.

$1 = x^{1/3} \Rightarrow x = 1$

$2 = x^{1/3} \Rightarrow x = 8$

$f'(x) = \frac{1}{3} x^{-2/3}$, so our integral is

$$SA = 2\pi \int_1^8 x \, ds = 2\pi \int_1^8 x \sqrt{1 + \left(\tfrac{1}{3} x^{-2/3}\right)^2}\, dx$$

distance to y-axis

$$= 2\pi \int_1^8 x \sqrt{1 + \frac{1}{9x^{4/3}}}\, dx$$

$$= 2\pi \int_1^8 x \sqrt{\frac{9x^{4/3}+1}{9x^{4/3}}}\, dx$$

$$= 2\pi \int_1^8 x \cdot \frac{1}{3x^{2/3}} \sqrt{9x^{4/3}+1}\, dx$$

$$= \frac{2\pi}{3} \int_1^8 x^{1/3} \sqrt{9x^{4/3}+1}\, dx$$

Let $u = 9x^{4/3} + 1$, $du = 12x^{1/3}\,dx$

$x = 1: u = 10$

$x = 8: u = 9 \cdot 16 + 1 = 145$

$$= \frac{2\pi}{3} \cdot \frac{1}{12} \int_{10}^{145} \sqrt{u}\, du$$

$$= \frac{\pi}{18} \cdot \frac{2}{3} u^{3/2} \Big|_{10}^{145}$$

$$= \boxed{\frac{\pi}{27}\left(145^{3/2} - 10^{3/2}\right)}$$

$$\approx \boxed{199.48}$$

b) Alternately, solve $y = x^{1/3}$ for x and use $ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$:

$x = y^3$, $\frac{dx}{dy} = 3y^2$, so we get

$$SA = 2\pi \int_1^2 x \sqrt{1 + \left(\tfrac{dx}{dy}\right)^2}\, dy = 2\pi \int_1^2 y^3 \sqrt{1 + (3y^2)^2}\, dy$$

$$= 2\pi \int_1^2 y^3 \sqrt{1 + 9y^4}\, dy$$

$u = 1 + 9y^4$, $du = 36y^3\,dy$

$y = 1: u = 10$

$y = 2: u = 145$

$$= \frac{2\pi}{36} \int_{10}^{145} \sqrt{u}\, du$$

(see above).