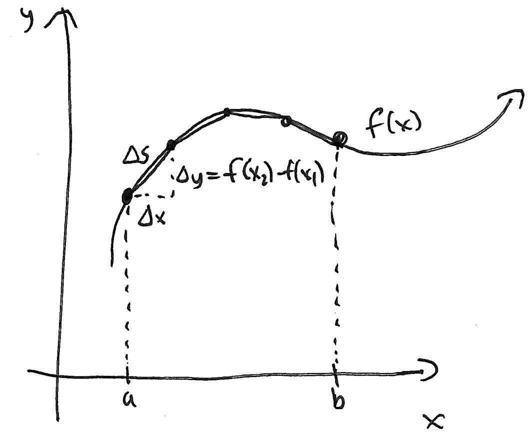
Arc Length

Consider a curve $y=f(x)$. How long is the segment between $x=a$ and $x=b$?



Let's approximate with straight lines:

How long is the segment Δs?

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2 \quad \text{by Pythagorean Theorem}$$

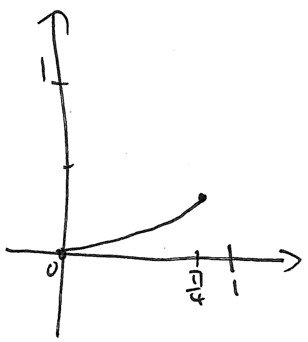
If we take $\Delta x \to 0$, this becomes.

$$(ds)^2 = (dx)^2 + (dy)^2. \qquad (1)$$

The arc length of the curve is $s = \int_a^b ds$. (add up these approx. lines)

Let's see how to use this w/ an example.

ex: Find the length of $y=\ln(\sec(x))$ between $x=0$ and $x=\frac{\pi}{4}$



We want an expression for ds: since we have $y=f(x)$, lets solve for an integral in terms of x

$$(ds)^2 = (dx)^2\left(1+\left(\frac{dy}{dx}\right)^2\right)$$

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

So, $s = \int_0^{\pi/4} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$ and $\frac{dy}{dx} = \frac{d}{dx}\left(\ln(\sec(x))\right) = \frac{\sec(x)\tan(x)}{\sec(x)} = \tan(x)$.

$$= \int_0^{\pi/4} \sqrt{1+\tan^2(x)}\, dx$$

$$= \int_0^{\pi/4} \sqrt{\sec^2(x)}\, dx = \int_0^{\pi/4} \sec(x)\, dx = \ln|\sec(x)+\tan(x)|\Big|_0^{\pi/4}$$

$$= \ln|\sec(\tfrac{\pi}{4})+\tan(\tfrac{\pi}{4})| + \ln|\sec(0)+\tan(0)|$$

$$= \ln|\sqrt{2}+1| + \ln|1+0|$$

$$= \boxed{\ln(\sqrt{2}+1)}$$

ex] Find the length of $x=\frac{2}{3}(y-1)^{\frac{3}{2}}$, $1\le y\le 4$.

We have $x=f(y)$, so let's rewrite $(ds)^2=(dx)^2+(dy)^2$ to get an integral in terms of y.

$$(ds)^2=(dy)^2\left(\left(\frac{dx}{dy}\right)^2+1\right) \Rightarrow ds=\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$s=\int_1^4\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy \qquad \frac{dx}{dy}=\frac{d}{dy}\left(\frac{2}{3}(y-1)^{\frac{3}{2}}\right)=(y-1)^{\frac{1}{2}}$$

$$=\int_1^4\sqrt{1+(y-1)}\,dy$$

$$=\int_1^4\sqrt{y}\,dy$$

$$=\frac{2}{3}y^{3/2}\Big|_1^4$$

$$=\frac{2}{3}(8-1)=\boxed{\frac{14}{3}}$$

ex) We could also have solved for y in the previous problem.

$$x=\frac{2}{3}(y-1)^{\frac{3}{2}} \Rightarrow \frac{3}{2}x=(y-1)^{\frac{3}{2}} \Rightarrow \left(\frac{3}{2}x\right)^{\frac{2}{3}}=y-1 \Rightarrow y=1+\left(\frac{3}{2}x\right)^{\frac{2}{3}},\quad 0\le x\le\frac{2}{3}(3)^{\frac{3}{2}}$$

$$\frac{dy}{dx}=\frac{2}{3}\left(\frac{3}{2}x\right)^{-\frac{1}{3}}\cdot\frac{3}{2}=\left(\frac{3}{2}x\right)^{-\frac{1}{3}}$$

So, $$s=\int_0^{\frac{2}{3}(3)^{\frac{3}{2}}}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_0^{\frac{2}{3}(3)^{\frac{3}{2}}}\sqrt{1+\left(\frac{3}{2}x\right)^{-\frac{2}{3}}}\,dx$$

$$=\int_0^{\frac{2}{3}(3)^{\frac{3}{2}}}\sqrt{\frac{\left(\frac{3}{2}x\right)^{\frac{2}{3}}+1}{\left(\frac{3}{2}x\right)^{\frac{2}{3}}}}\,dx$$

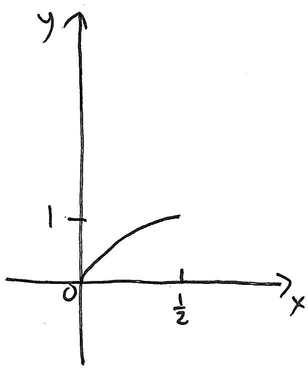
$$=\int_0^{\frac{2}{3}(3)^{\frac{3}{2}}}\left(\frac{3}{2}x\right)^{-\frac{1}{3}}\sqrt{\left(\frac{3}{2}x\right)^{\frac{2}{3}}+1}\,dx \qquad u=\left(\frac{3}{2}x\right)^{\frac{2}{3}}+1 \quad 0\Rightarrow 1$$

$$du=\left(\frac{3}{2}x\right)^{-\frac{1}{3}}dx \quad \frac{2}{3}(3)^{\frac{3}{2}}\Rightarrow 4$$

$$=\int_1^4\sqrt{u}\,du \quad \text{(look familiar?)}$$

$$=\boxed{\frac{14}{3}}$$

ex: Determine the length of $x=\frac{1}{2}y^2$, $0\le x\le\frac{1}{2}$, $y\ge 0$.



Use the eq (1) solved for an integral in terms of y:

$$ds=\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

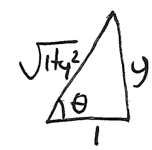
$$\frac{dx}{dy}=y.$$

$$s=\int_0^1\sqrt{1+(y)^2}\,dy$$

Let $y=\tan\theta$, $dy=\sec^2\theta\,d\theta$, $y=0\Rightarrow\theta=0$

$y=1\Rightarrow\theta=\frac{\pi}{4}$

$\sqrt{1+y^2}=\sec\theta$



So, $s=\int_0^{\pi/4}\sec\theta\cdot\sec^2\theta\,d\theta$

$$=\int_0^{\pi/4}\sec^3\theta\,d\theta$$

$$=\frac{1}{2}\left(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\right)\Big|_0^{\pi/4}$$

$$=\frac{1}{2}\left(\sec\frac{\pi}{4}\tan\frac{\pi}{4}+\ln\left|\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right|-\sec(0)\tan(0)-\ln|\sec(0)+\tan(0)|\right)$$

$$=\frac{1}{2}\left(\sqrt{2}\cdot 1+\ln|\sqrt{2}+1|-0-\ln|1+0|\right)$$

$$=\boxed{\frac{1}{2}\left(\sqrt{2}+\ln(\sqrt{2}+1)\right)}$$