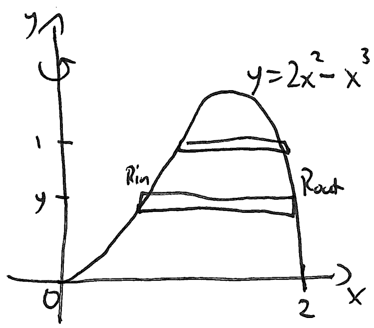
Cylindrical Shells

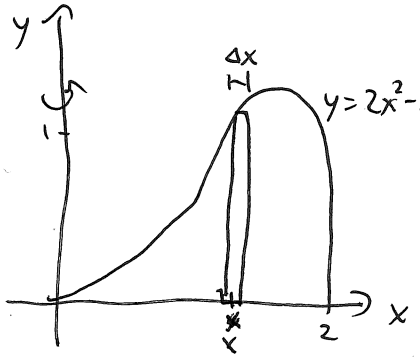
Consider finding the volume of the solid generated by rotating the area below $f(x)=2x^2-x^3$ on $0\le x\le 2$ about the y-axis.



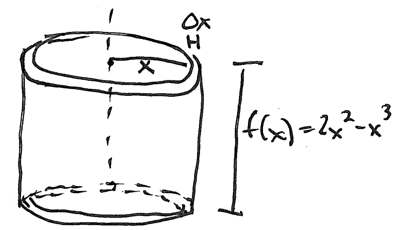
We need to find expressions for R_{in} & R_{out}, but this involves solving equations like

$$1 = 2x^2-x^3$$

for x. This is hard (annoying). Instead, let's try looking at rectangles drawn vertically (parallel to axis).



After rotating, this becomes



$$V_{scy} = (\text{circumference})\cdot\Delta x = (2\pi r h)\Delta x = 2\pi x(2x^2-x^3)\Delta x.$$

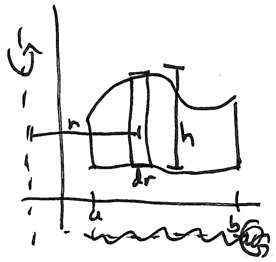
So, taking $\Delta x \to 0$, $V=\int_0^2 2\pi x(2x^2-x^3)\,dx = 2\pi\int_0^2 2x^3-x^4\,dx = 2\pi\left(\frac{x^4}{2}-\frac{x^5}{5}\right)\Big|_0^2$

$$= 2\pi\left(\frac{16}{2}-\frac{32}{5}\right)$$

$$= 3.2\pi$$

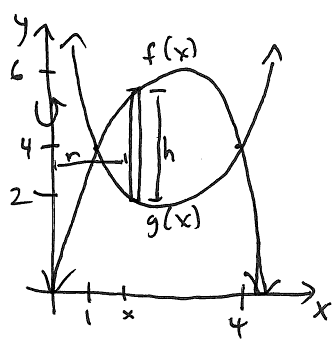
This is the method of cylindrical shells:

$$V=\int_a^b 2\pi r h\,dr$$



- draw rectangles parallel to axis of rotation

ex: Find the volume V obtained by rotating the region enclosed by $f(x)=5x-x^2$, $g(x)=8-5x+x^2$ around the y-axis.



First, we find points of intersection:
$5x-x^2=8-5x+x^2 \iff x^2-5x+4=0 \iff (x-4)(x-1)=0$
$x=1$ or $x=4$.

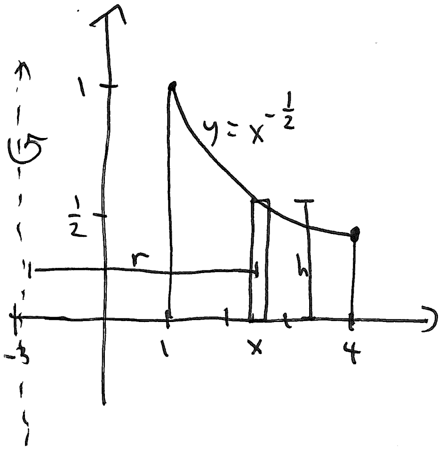
The "radius" (dist from rect to axis) is x.
The "height" is $f(x)-g(x)=10x-2x^2-8$ on $[1,4]$.

So, $V=2\pi\int_1^4 x(10x-2x^2-8)\,dx = 2\pi\int_1^4 10x^2-2x^3-8x\,dx$

$$= 2\pi\left(\frac{10x^3}{3}-\frac{x^4}{2}-4x^2\right)\Big|_1^4$$

$$= \boxed{45\pi}$$

ex: Find the volume V obtained by rotating the region under $f(x)=x^{-1/2}$ on $[1,4]$ about the axis $x=-3$.



radius $= r = x-(-3) = x+3$

height $= x^{-\frac{1}{2}} - 0 = x^{-\frac{1}{2}}$

$$V = 2\pi\int_a^b rh\,dr = 2\pi\int_1^4 (x+3)x^{-\frac{1}{2}}\,dx$$
$$= 2\pi\int_1^4 x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}\,dx$$
$$= 2\pi\left(\tfrac{2}{3}x^{\frac{3}{2}} + 6x^{\frac{1}{2}}\right)\Big|_1^4$$
$$= 2\pi\left(\tfrac{2}{3}(2)^3 + 6(2) - \tfrac{2}{3} - 6\right)$$
$$= \boxed{\frac{64\pi}{3}}$$

• note disks is awkward here b/c horiz rect. have different widths for different y: ($y<\frac{1}{2}$ vs $y>\frac{1}{2}$)

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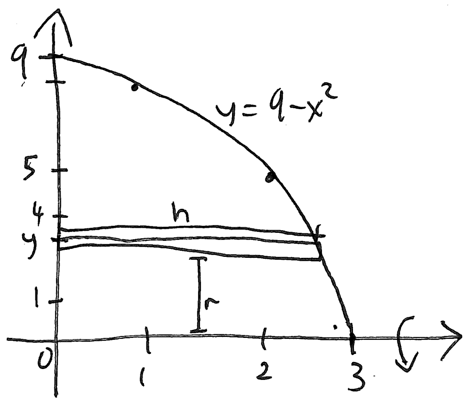
ex: Find the volume V obtained by rotating the region under $y=9-x^2$ and between $y=0$ and $x=0$ around the x-axis.

• Note: here we could easily use either method.



"r" $= y$

"h" $= \cancel{f(y)}\ x = f(y)$. : solve $9-x^2=y$ for x

$x=\sqrt{9-y}$

$= \sqrt{9-y}$

$$V = \int_0^9 2\pi y\sqrt{9-y}\,dy$$

Set $u=9-y$, $du=-dy$

$y=9-u$

$y=0$: $u=9$

$y=9$: $u=0$

$$= -\int_9^0 2\pi(9-u)\sqrt{u}\,du$$
$$= 2\pi\int_0^9 9u^{\frac{1}{2}} - u^{\frac{3}{2}}\,du$$
$$= 2\pi\left(9\cdot\tfrac{2}{3}u^{\frac{3}{2}} - \tfrac{2}{5}u^{\frac{5}{2}}\right)\Big|_0^9$$
$$= 2\pi\left(6(3)^3 - \tfrac{2}{5}(3)^5\right)$$
$$= \frac{648}{5}\pi$$