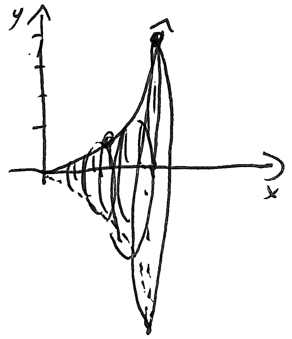
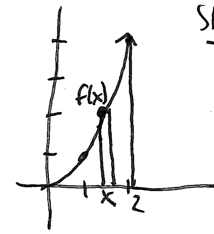
Volumes of Solids of Revolution I - Disk Method

Let's find the volume of the solid generated by rotating the region under $y=x^2$ about the x-axis for $0 \le x \le 2$.

Step 0:





Step 1: Cross-sections: We will look at rectangles perpendicular to axis of revolution.

when rotated, these form a disk of radius $f(x)$. So $A(x) = \pi f(x)^2$.

Step 2:

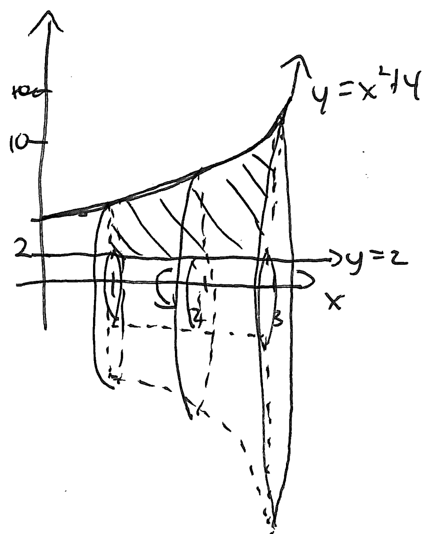
So $V = \int_0^2 \pi (x^2)\,dx = \frac{1}{3}\pi x^3 \Big|_0^2 = \boxed{\frac{8\pi}{3} \text{ units}^3}$

- This method works more broadly for solids of revolution:

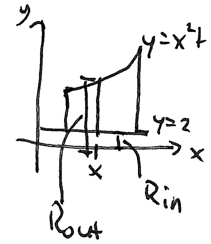
$$V = \int_a^b \pi R^2\,dh \quad \text{(think cylinders)} \quad \text{or} \quad V = \int_a^b \pi R_{outer}^2 - \pi R_{inner}^2\,dh$$
$$= \int_a^b \pi \left(R_{out}^2 - R_{in}^2\right) dh$$

ex: Find the volume V obtained by revolving the region between $y = x^2+4$ and $y=2$ about the x-axis for $1 \le x \le 3$.

Step 0:



Step 1: Find R_{out}, R_{in}:



So $R_{out} = x^2+4$
$R_{in} = 2$

Step 2: $V = \pi \int_1^3 (x^2+4)^2 - 2^2\,dx$

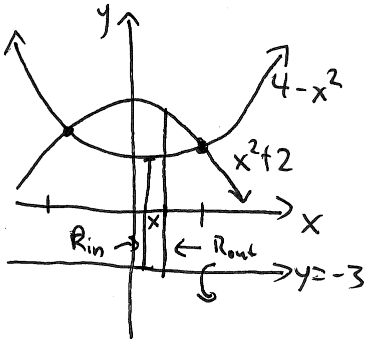
$= \pi \int_1^3 x^4 + 8x^2 + 16 - 4\,dx$

$= \pi \left(\frac{x^5}{5} + \frac{8}{3}x^3 + 12x\right)\Big|_1^3$

$= \pi \left(\frac{3^5}{5} + \frac{8}{3}3^3 + 12\cdot 3 - \frac{1}{5} - \frac{8}{3} - 12\right)$

$= \boxed{\frac{2126}{15}\pi}$

ex: Find the volume V of the "napkin ring" obtained by rotating the region between the graphs of $f(x)=x^2+2$ and $g(x)=4-x^2$ about $y=-3$.



First, we need to know the points of intersection btwn $f(x), g(x)$:

$$f(x)=g(x) \iff x^2+2=4-x^2 \iff 2x^2=2 \iff x^2=1 \iff x=\pm 1$$

So the graphs meet at $(1,3)$ and $(-1,3)$.

What are R_{out}, R_{in}?

$$R_{out}=(4-x^2)-(-3)=7-x^2 \qquad R_{in}=(x^2+2)-(-3)=x^2+5.$$

So $V=\pi\int_{-1}^{1}(7-x^2)^2-(x^2+5)^2\,dx = \pi\int_{-1}^{1} 49-14x^2+x^4-(x^4+10x^2+25)\,dx$

$$=\pi\int_{-1}^{1} 24-24x^2\,dx$$

$$=\pi(24x-8x^3)\Big|_{-1}^{1}$$

$$=\pi(24-8-(-24+8))$$

$$=\boxed{32\pi}$$

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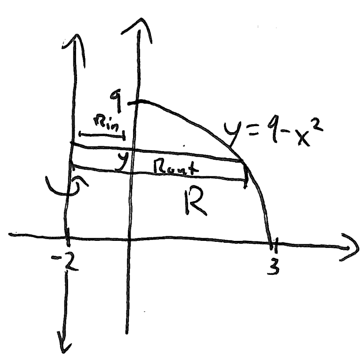
ex: Find the volume of the solid obtained by rotating the region R under the graph of $f(x)=9-x^2$ for $0\le x\le 3$ about $x=-2$.



Draw rectangles perpendicular to axis of rotation.

$R_{in}=0-(-2)=2$

R_{out}: Need to solve $y=9-x^2$ for x: $\quad x^2=9-y$, $\quad x=\sqrt{9-y}$

$$R_{out}=\sqrt{9-y}-(-2)=\sqrt{9-y}+2$$

So $V=\pi\int_0^9 R_{out}^2-R_{in}^2\,dy = \pi\int_0^9(\sqrt{9-y}+2)^2-2^2\,dy$

$$=\pi\int_0^9 (9-y)+4\sqrt{9-y}+4-4\,dy$$

$$=\pi\int_0^9 (9-y)+4\sqrt{9-y}\,dy$$

$$=\pi\left(9y-\frac{y^2}{2}-4\cdot\frac{2}{3}(9-y)^{3/2}\right)\Big|_0^9$$

$$=\pi\left(81-\frac{81}{2}-\frac{8}{3}(0)-0+0+\frac{8}{3}(9)^{\frac{3}{2}}\right)$$

$$=\boxed{\frac{225}{2}\pi}$$