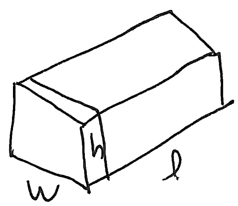
Volumes

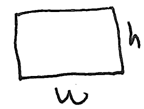
We will find volumes as $V = (\text{cross-sectional area}) \cdot (\text{height/width/thickness})$.
length

For familiar shapes, this gives the formulas we know from geometry

ex:



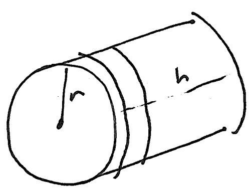
Cross-sections are rectangles



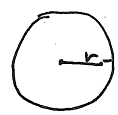
and the length is l.

So $V = (wh) \cdot l = lwh$.

ex:



Cross-sections are circles



and the height is h.

So $V = A_{\circ} \cdot h = \pi r^2 h$.

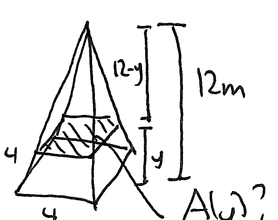
We will use integration to deal with more complicated shapes.

$$V = \int_a^b \underbrace{A(x)}_{\text{cross-sectional area at point } x} dx \quad \text{or} \quad \int_a^b A(y)\, dy$$

(Intuition: we are taking very small cross-sectional cylinders/cubes)

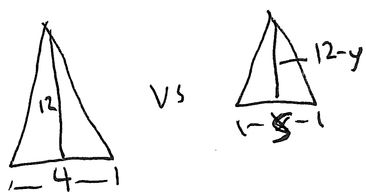
ex: Volume of a Pyramid. Find the volume of a pyramid of height 12m whose base is a square of side 4m.

Step 0: Draw the shape!



Step 1: Find $A(y)$. We see that $A(y)$ is a square. What is its side length s?

Use similar triangles



$$\frac{s}{4} = \frac{12-y}{12}$$

$$12s = 48 - 4y$$

$$s = 4 - \frac{1}{3}y$$

So $A(y) = s^2 = (4 - \frac{1}{3}y)^2$

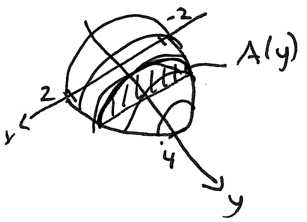
Step 3: Compute V using an integral.

$$V = \int_0^{12} A(y)\, dy = \int_0^{12} (4 - \tfrac{1}{3}y)^2 dy = \frac{-3}{3}(4 - \tfrac{1}{3}y)^3 \Big|_0^{12} = -(4-4)^3 + (4-0)^3 = \boxed{64 \text{ m}^3}$$

Check: $V_{\text{pyramid}} = \frac{1}{3}Bh = \frac{1}{3}(4 \cdot 4) \cdot 12 = 64 \text{ m}^3$. ✓

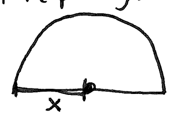
ex: Find the volume of the solid whose base is the region enclosed by the parabola $y=4-x^2$ and the x-axis and whose cross-sections perpendicular to the y-axis are semicircles.

Step 0: Draw the shape.



Step 1: Identify cross-sections $A(y)$.

At the point y:



where $y=4-x^2$ so $x=\sqrt{4-y}$

so $A(y)=\frac{1}{2}\pi r^2=\frac{1}{2}\pi x^2$

$=\frac{1}{2}\pi(4-y)$

Step 2: Find the volume by integrating.

$$V=\int_0^4 A(y)\,dy=\int_0^4\frac{1}{2}\pi(4-y)\,dy=-\frac{1}{2}\pi\frac{(4-y)^2}{2}\Big|_0^4=-\frac{1}{4}\pi(4-4)^2+\frac{1}{4}\pi(4-0)^2=\boxed{4\pi}\text{ units}^3$$

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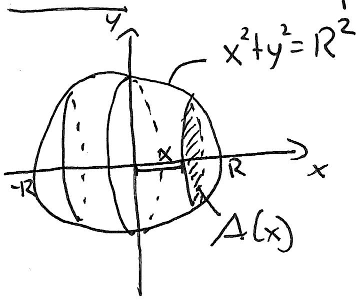
ex: Find the volume of a sphere of radius R centered at the origin.

Step 0: Draw the shape.



Step 1: Find $A(x)$.

The CSA is a circle of radius y where $x^2+y^2=R^2$

So $y=\sqrt{R^2-x^2}$. So $A(x)=\pi y^2=\pi(R^2-x^2)$

Step 2: Find the volume by integrating.

$$V=\int_{-R}^{R}A(x)\,dx=\pi\int_{-R}^{R}(R^2-x^2)\,dx=\pi\left(R^2x-\frac{1}{3}x^3\right)\Big|_{-R}^{R}$$

$$=\pi\left(R^3-\frac{1}{3}R^3\right)-\pi\left(R^2(-R)-\frac{1}{3}(-R)^3\right)$$

$$=\frac{2}{3}\pi R^3-\left(-\frac{2}{3}\pi R^3\right)$$

$$=\frac{4}{3}\pi R^3.$$