# Taylor Polynomials & Taylor Series

Recall from calculus I that we can approximate a diff'able function $f(x)$ near $x=a$ with the tangent line at $x=a$,

$$T_1(x) = f'(a)(x-a) + f(a).$$

What if we want to approximate $f$ with a polynomial where the $2^{nd}$ derivative also agrees with $f$?

Well, let's try a quadratic:

$$T_2(x) = c(x-a)^2 + f'(a)(x-a) + f(a).$$

$$T_2(a) = f(a)$$

$$T_2'(x) = 2c(x-a) + f'(a)$$

$$T_2'(a) = f'(a)$$

$$T_2''(x) = 2c$$

$$T_2''(a) = 2c = f''(a), \text{ so } c = \frac{f''(a)}{2}$$

$$\boxed{T_2(x) = \frac{f''(a)}{2}(x-a)^2 + f'(a)(x-a) + f(a).}$$

We can keep going! For the third derivative to match, we need a cubic and we get $T_3(x) = d(x-a)^3 + \frac{f''(a)}{2}(x-a)^2 + f'(a)(x-a) + f(a)$.

$$T_3'''(x) = 3\cdot 2\cdot d \qquad \text{so if } T_3'''(a) = f'''(a), \text{ then } d = \frac{f'''(a)}{3\cdot 2} = \frac{f'''(a)}{3!}$$

In general, the <u>$n^{th}$ Taylor polynomial of $f(x)$ at $x=a$</u> is

$$T_n(x) = \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + \dots + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f'(a)}{1!}(x-a) + f(a)$$

The <u>Taylor series</u> of $f$ at $x=a$ is what we get if we take $n\to\infty$, if the resulting series converges.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

<u>ex</u>: Find the Taylor series for $f(x)=e^x$ at $x=0$ and determine its radius of convergence.

We need to figure out what $f^n(0)$ is for $f(x)=e^x$:

$$f(0) = e^0 = 1$$

$$f'(0) = e^0 = 1$$

$$\vdots$$

$$f^{(n)}(x) = e^x, \text{ so } f^{(n)}(0) = 1 \text{ for all } n!$$

So $e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$ whenever the series converges:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right| = \lim_{n\to\infty}\frac{|x|}{n+1} = 0 \text{ for all } x\in\mathbb{R}.$$

Since $0<1$, the series converges to $e^x$ for all $x\in\mathbb{R}$, i.e. it has infinite radius of convergence.

## More examples:

ex: Find the ~~fourth~~ sixth-order Taylor polynomial for $f(x) = xe^{3x^4}$ at $x=0$.

Let's use the Taylor series we already know: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all $x$.

$$\text{So } f(x) = xe^{3x^4} = x \cdot \sum_{n=0}^{\infty} \frac{(3x^4)^n}{n!} = \sum_{n=0}^{\infty} \frac{3^n x^{4n+1}}{n!} = x + \frac{3}{1!}x^5 + \frac{9}{2!}x^9 + \frac{27}{3!}x^{13} + \dots$$

The sixth order Taylor polynomial is the partial sum of this series with terms up to $x^6$:

$\boxed{T_6(x) = x + 3x^5}$ (a lot of coefficients are 0 in this power series). (since $e^x$ has a Taylor series w/ $R=\infty$, so does $f$)

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ex: Find a Taylor series for $f(x) = \sin(x)$ at $x=0$.

$f(x) = \sin(x)$ $\quad f(0) = 0$

$f'(x) = \cos(x)$ $\quad f'(0) = 1$

$f''(x) = -\sin(x)$ $\quad f''(0) = 0$

$f'''(x) = -\cos(x)$ $\quad f'''(0) = -1$

$f^{(4)}(x) = \sin(x)$ $\quad f^{(4)}(0) = 0$

$$\text{So } \sin(x) = 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 - \frac{1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \dots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \text{ if this series converges.}$$

Test for convergence:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{x^{2(n+1)+1}}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x^{2n+3}}{x^{2n+1}} \cdot \frac{(2n+1)!}{(2n+3)!} \right| = \lim_{n\to\infty} \frac{|x|^2}{(2n+3)(2n+2)} = 0 \text{ for all } x.$$

Since $0<1$, the series converges for all $x$ in $(-\infty, \infty)$.

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ex Find the Taylor series for $f(x) = x^{-3}$ at $c=1$

$f'(x) = -3x^{-4}$ $\quad f'(1) = -3$

$f''(x) = (-3)(-4)x^{-5}$ $\quad f''(1) = (-3)(-4)$

$f'''(x) = (-3)(-4)(-5)x^{-6}$ $\quad f'''(1) = (-3)(-4)(-5)$

$f^{(n)}(1) = (-1)^n(3)(4)(5)\cdots(n+1)(n+2)$
$= (-1)^n \frac{1}{2}(n+2)!$

$$\text{So } T(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \frac{1}{2}(n+2)!}{n!}(x-1)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(n+2)(n+1)}{2}(x-1)^n$$

which converges on $(-1,1)$ by Ratio Test.

ex: Approximate $\int_0^1 e^{-x^2}dx$ using a fourth-order Taylor polynomial.

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \dots$$

$$T_4(x) = 1 - x^2 + \frac{x^4}{2} \approx e^{-x^2}$$

So $\int_0^1 e^{-x^2}dx \approx \int_0^1 (1 - x^2 + \frac{x^4}{2})\,dx = x - \frac{x^3}{3} + \frac{x^5}{10}\Big|_0^1 = 1 - \frac{1}{3} + \frac{1}{10} \approx \frac{30-10+3}{30} = \boxed{\frac{23}{30}}$

Taylor's Remainder Thm:

If $f^{(n+1)}(x)$ is continuous around $a$, then $\exists c$ between $a$ and $x$ so that

$$f(x) = \sum T_n(x) + R_n(x)$$

where $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$

So, $|f(x) - T_n(x)| = |R_n(x)|$.

ex: How many terms to approximate $e$ to 4 decimal places w/ $e = \sum_{n=0}^{\infty} \frac{1}{n!}$?

Want $R_n^{(1)} = \frac{e^c}{(n+1)!}(1-0)^{n+1} = \frac{e^c}{(n+1)!}$ for $0 \le c \le 1$.

Use $e^c \le e^1 \le 3$. $|R_n| < \frac{3}{(n+1)!} < 0.00005$. Some computation gives $n=8$. $(n+1=9)$.

$1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} \approx 2.7182788$, $e \approx 2.7182818$