New Power Series from old:

Given a power series, we can substitute expressions for x to get a new power series, possibly changing the radius of convergence.

Ex: Find a power series for $\frac{1}{1-3x}$ that converges on $|x|<\frac{1}{3}$.

We know that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x|<1$, so plug in $3x$ for x:

$$\frac{1}{1-3x} = \sum_{n=0}^{\infty} (3x)^n = \sum_{n=0}^{\infty} 3^n x^n \text{ for } |3x|<1, \text{ i.e. } |x|<\tfrac{1}{3}.$$

Ex: Find a power series for $\frac{x}{1+x^2}$ and determine its radius of convergence.

Use the same technique as above:

$$\frac{x}{1+x^2} = x \cdot \frac{1}{1+x^2} = x \cdot \frac{1}{1-(-x^2)}.$$

So $\frac{x}{1+x^2} = x \cdot \sum_{n=0}^{\infty} (-x^2)^n = x \sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=0}^{\infty} (-1)^n x^{2n+1}$ for $|-x^2|<1$, i.e. $|x|<1$.

We can also integrate and differentiate a power series term-by-term: This does not change the radius of convergence

Ex: Find a power series for $\frac{1}{(1-x)^2}$ on $|x|<1$.

We know $\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right)$, so $\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=0}^{\infty} \frac{d}{dx}(x^n) = \sum_{n=0}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} (n+1)x^n$

$$= 1 + 2x + 3x^2 + 4x^3 + \dots$$

Ex: Find a power series for $-\ln|1-x|$ on $|x|<1$

$$-\ln|1-x| = \int \frac{1}{1-x}dx = \int \sum_{n=0}^{\infty} x^n \, dx = \sum_{n=0}^{\infty} \int x^n dx = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots$$