# Power Series

Start lecture with discussion of wksheet12a

Remember the geometric series $1+r+r^2+r^3+\cdots$

We said that this series converges to $\frac{1}{1-r}$ if $|r|<1$. In other words, $1+x+x^2+x^3+\cdots = \frac{1}{1-x}$ on the interval $(-1,1)$.

The left-hand expression $\sum_{n=0}^{\infty} x^n$ is an example of a power series in $x$.

We will investigate:

1) when does a power series converge
2) what does it converge to?
3) can we make new powerseries from old ones
4) Which functions have power series representations? (like $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $x$ in $(-1,1)$).

A <u>power series</u> is a series $\sum_{n=0}^{\infty} c_n (x-a)^n$. We say the power series is <u>centered at a</u>, since the series will definitely converge at $x=a$: we get $c_0 + c_1(a-a) + c_2(a-a)^2 + \cdots = c_0$.

<u>Ex</u>: $1 + 2(x-2) + 3(x-2)^2 + 4(x-2)^3 + 5(x-2)^4 + \cdots = \sum_{n=0}^{\infty} (n+1)(x-2)^n$

if $x=\frac{3}{2}$, this becomes the convergent series $1+1+\frac{3}{4}+\frac{4}{8}+\frac{5}{16}+\cdots = \sum_{n=0}^{\infty} \frac{n+1}{2^n}$ (Ratio Test)

$x=3$, this is the divergent series $1+2+3+4+5+\cdots$

<u>Ex</u>: (Do this first)
$1-x^2+3x^5+x^7$ is a powerseries.
Where does it converge?
$\mathbb{R} = (-\infty,\infty)$

For which x-values does it converge?

~~1+2(x-2)~~ Let's use the Ratio Test:

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+2)(x-2)^{n+1}}{(n+1)(x-2)^n}\right| = \lim_{n\to\infty} \left|\frac{(n+2)}{(n+1)}(x-2)\right| = |x-2|.$$

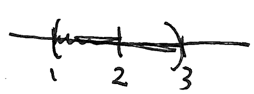
The series converges if $L<1$, so let's set $L=|x-2|<1$: $|x-2|<1$ means $-1<x-2<1$

$\boxed{1<x<3}$

What about $L=1$, when $x=1$ or $x=3$?

We saw $x=3$ diverges earlier. If $x=1$, we get $1-2+3-4+5+\cdots$, which diverges by Div. Test.

So $\sum_{n=0}^{\infty}(n+1)(x-2)^n$ converges on $1<x<3$.



Notice that in our examples the series either converged on an interval that was symmetric around the center or converged everywhere.

In general, a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has a <u>radius of convergence</u>, $R$.

The series converges absolutely on the interval $(a-R, a+R)$. The endpoints need to be checked manually and may or may not converge. If $R=0$, the series only converges at $x=a$. If $R=\infty$, the series converges everywhere. The set of all values of $x$ where the series converges is the <u>interval of convergence</u> (usually $(a-R, a+R)$, $[a-R, a+R)$, $(a-R, a+R]$, or $[a-R, a+R]$).

Ex: $R=1$ for $\sum x^n = \frac{1}{1-x}$, $R=1$ for $\sum (n+1)(x-2)^n$

Ex: Find the radius & interval of convergence of $1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Again, we use the Ratio Test:

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \overset{\lim_{n\to\infty}}{=} \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \lim_{n\to\infty} |x| \cdot \frac{\cancel{n!}}{(n+1)\cancel{n!}} = 0, \text{ for all } x.$$

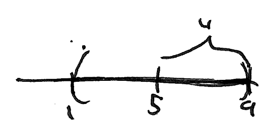
So the series converges for all $x$, hence $R=\infty$ and the interval of convergence is $(-\infty, \infty)$.

Ex: Find the radius & interval of convergence of $\sum_{n=1}^{\infty} \frac{(-1)^n}{4^n n}(x-5)^n$.

Use Ratio Test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-5)^{n+1}}{4^{n+1}(n+1)} \cdot \frac{n\,4^n}{(x-5)^n}\right| = \lim_{n\to\infty} \frac{1}{4}|x-5| \cdot \frac{n}{n+1} = \frac{1}{4}|x-5|$.

Want $\frac{1}{4}|x-5| < 1$, so $|x-5| < 4$, i.e. $\boxed{R=4}$

The series conv. abs. on $-4 < x-5 < 4$, so $1 < x < 9$



Endpoints: if $x=1$, we get $\sum_{n=1}^{\infty} \frac{(-1)^n(-4)^n}{4^n n} = \sum_{n=1}^{\infty} \frac{1}{n}$, which diverges.

if $x=9$, we get $\sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{4^n n} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$, which converges.