Math 2551 Worksheet - Review from Single-Variable Calculus and Linear Algebra

This worksheet reviews some concepts and tools of calculation from linear algebra, differential calculus, and integral calculus that will be useful for this course.

1. Let $\mathbf{v} = \langle 1, 2, 1\rangle$ and $\mathbf{w} = \langle -1, 0, 2\rangle$. Compute $2\mathbf{v}$, $\mathbf{v} - \mathbf{w}$, and $\mathbf{v} \cdot \mathbf{w}$.

2. Let a, b, c be variables. Calculate

$$\det \begin{bmatrix} a & b & c \\ 1 & -1 & 2 \\ 4 & 3 & -7 \end{bmatrix}$$

by expanding the determinant along the first row.

3. Draw the vectors $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{v} = -\mathbf{i} + 2\mathbf{j}$ starting at the origin in $\mathbb{R}^2$ to show that these vectors are orthogonal. Compute $\mathbf{u} \cdot \mathbf{v}$. What is the result?

4. Let $\mathbf{v} = \langle 2, -4, \sqrt{5}\rangle$ and $\mathbf{u} = \langle -2, 4, -\sqrt{5}\rangle$. Compute the following:
   (a) $\mathbf{v} \cdot \mathbf{u}$
   (b) the cosine of the angle θ between $\mathbf{v}$ and $\mathbf{u}$. Recall that $\mathbf{v} \cdot \mathbf{u} = |\mathbf{v}||\mathbf{u}|\cos(\theta)$
   (c) $(3\mathbf{v}) \cdot (2\mathbf{u})$.

5. Compute the following integrals.
   (a) $\displaystyle\int_0^{\pi} (\sin(\theta))^5 \, d\theta$
   (b) $\displaystyle\int_0^{\pi} (\sin(\theta))^2 \, d\theta$
   (c) $\displaystyle\int_1^{2} t\ln(t) \, dt$

6. Find the maximum value of the function $f(t) = te^{t^2}$ on the interval from $t = 0$ to $t = 2$. Justify how you know this is the maximum.

7. Find the area enclosed by the curves $f(x) = x^2$ and $g(x) = 2 - x$.

Math 2551 Worksheet: $\mathbb{R}^3$

1. Find the components of the vector in 3-space of length 3 lying in the yz-plane pointing upward at an angle of $\pi/6$ measured from the positive y-axis.
2. Which is traveling faster, a car whose velocity vector is $\langle 28, 33\rangle$ km/h or a car whose velocity vector is $\langle 40, 0\rangle$ km/h? At what speed is the faster car traveling?
3. Find the equation of a sphere of radius 4 in $\mathbb{R}^3$ centered at the point $(1, -2, 3)$.

Math 2551 Worksheet: Cross Products

1. Let $P = (1, -1, 2)$, $Q = (2, 0, -1)$, and $R = (0, 2, 1)$.
   (a) Find the area of the triangle determined by the points P, Q, and R.
   (b) Find a unit vector normal to the plane containing P, Q, and R.

2. If the statement is *always* true, answer true. If the statement is *ever* false, answer false. Justify your answer.

   In each case, $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in $\mathbb{R}^3$.

   (a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
   (b) $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^{\mathbf{2}}$
   (c) $(\mathbf{u} \times \mathbf{u}) \cdot \mathbf{u} = \mathbf{0}$

3. Suppose $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in $\mathbb{R}^3$. Which of the following make sense, and which do not? For those that make sense, is the result a vector or a scalar?

   (a) $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$
   (b) $\mathbf{u} \times (\mathbf{v} \cdot \mathbf{w})$
   (c) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$
   (d) $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})$

4. Let $\mathbf{u} = \langle 2, 3 \rangle$. Find the maximum possible value for $\mathbf{u} \cdot \mathbf{v}$ if $\mathbf{v}$ is a unit vector, and find a $\mathbf{v}$ which gives this maximum. Then repeat the problem with "maximum" replaced by "minimum."

Math 2551 Worksheet: Lines and Planes

1. Find an equation for
   (a) the line through point $P = (1, 2, -1)$ and point $Q = (-1, 0, 1)$.
   (b) the line through $(0, -7, 0)$ perpendicular to the plane $x + 2y + 2z = 13$.
   (c) the line in which the planes $3x - 6y - 2z = 3$ and $2x + y - 2z = 2$ intersect.

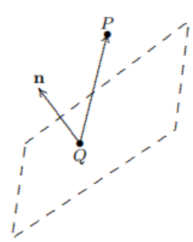
2. Find a vector in the direction of the line of intersection ℓ of the planes $2x + y - z = 3$ and $x + 2y + z = 2$. Find a plane which goes through $(2, 1, -1)$ and is perpendicular to ℓ (and thus both planes).

3. Find 2 planes that are not parallel that both contain the points $P(1, -1, 1)$, $Q(3, 2, 0)$, and $R(5, 5, -1)$. When will 3 distinct points NOT determine a unique plane?

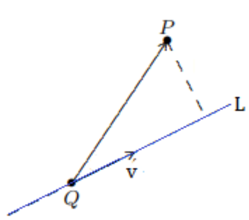
4. Find the point where the line $\mathbf{r}(t) = \langle 2, 3 + 2t, 1 + t\rangle$ intersects the plane $2x - y + 3z = 6$.

5. How can you tell when two planes $A_1x + B_1y + C_1z = D_1$ and $A_2x + B_2y + C_2z = D_2$ are parallel? Perpendicular? Justify your answer.

6. Recall from linear algebra that the **projection** of a vector $\mathbf{u}$ onto $\mathbf{v}$ is the component of $\mathbf{u}$ in the direction of $\mathbf{v}$: $\text{proj}_{\mathbf{v}}\,\mathbf{u} = \dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}$.
   (a) The distance from a point P to a plane is the shortest distance from P to any point on the plane. Use this and the above to compute the distance from the point $P = (1, 2, 3)$ to the plane $2x - y + 3z = 5$.



   (b) The distance from a point P to a line is the shortest distance from P to any point on the line. Use this and a well-chosen cross product to compute the distance from the point $P = (-1, 2, 1)$ to the line $\langle 1, 1, 1\rangle + t\langle 2, 3, -1\rangle$.



Math 2551 Worksheet: Quadric Surfaces

1. For the following, identify and describe (for example, which way is it oriented? what are the cross-sections?) the type of surface.

   (a) $x^2 + 4z^2 = 16$.

   (b) $9x^2 + z^2 + y^2 = 9$.

   (c) $y^2 = 3x^2 + 3z^2$.

   (d) $z = x^2 + y^2 + 4$.

   (e) $x^2 + y^2 = 16 - z^2$

Math 2551 Worksheet: Curves in Space and Their Tangents

1. Describe the graph of the curve $\mathbf{r}(t) = \langle t\cos(t), t\sin(t), t\rangle$, $t \in \mathbb{R}$.

2. Find a vector-valued function for the curve of intersection of the cylinder $x^2 + y^2 = 9$ and the plane $y + z = 2$.

   Hint: How could you parameterize the circle $x^2 + y^2 = 9$ in the plane?

3. What is the difference between the parameteric curves $\mathbf{f}(t) = \langle t, t, t^2\rangle$, $\mathbf{g}(t) = \langle t^2, t^2, t^4\rangle$, and $\mathbf{h}(t) = \langle \sin(t), \sin(t), \sin^2(t)\rangle$ as t runs over all real numbers?

4. With a parametric plot and a set of t values, we can associate a 'direction'. For example, the curve $\langle \cos(t), \sin(t)\rangle$, $t \in [0, 2\pi]$ is the unit circle traced counterclockwise. How can we change a set of given parametric equations and t values to get the same curve, only traced backwards?

5. The motion of a particle in the xy-plane at time t is described by the vector function

$$\mathbf{r}(t) = e^t\mathbf{i} + \frac{2}{9}e^{2t}\mathbf{j}$$

   (a) Find an equation in x and y whose graph is the path of the particle. Consider how $y(t)$ is related to $x(t)$ and what values $x(t)$ takes on.

   (b) Find the particle's velocity and acceleration vectors at $t = \ln(3)$.

   (c) Sketch the path of the particle and include the particle's velocity and acceleration vectors at $t = \ln(3)$.

6. Find the parametric equations for the line that is tangent to the curve

$$\vec{r}(t) = \left\langle \ln t, \frac{t-1}{t+2}, t\ln t \right\rangle, \text{ at } t = 1.$$

7. Determine the point at which $\mathbf{f}(t) = \langle t, t^2, t^3\rangle$ and $\mathbf{g}(t) = \langle \cos(t), \cos(2t), t+1\rangle$ intersect, and find the angle between the curves at that point. (Hint: You'll need to set this up like the line intersection problems you've seen before, writing one in s and one in t).

   If these two functions were the trajectories of two bumblebees on the same scale of time, would the bees collide at their point of intersection? Explain.

8. Find the equation of the plane perpendicular to the curve $\langle \cos(t), \sin(t), \cos(6t)\rangle$ when $t = \pi/4$.

Math 2551 Worksheet: Integrals of Vector Valued Functions

1. Suppose that $\mathbf{r}(t)$ satisfies

$$\mathbf{r}''(t) = -\mathbf{i} - \mathbf{j} - \mathbf{k}, \quad t \geq 0, \qquad \mathbf{r}'(0) = 5\mathbf{i}, \qquad \mathbf{r}(0) = 10\mathbf{i} + 10\mathbf{j} + 10\mathbf{k}$$

   Find $\mathbf{r}(t)$.

2. A baseball is hit when it is 2.5 ft above the ground. It leaves the bat with an initial velocity of 140 ft/sec at a launch angle of 30°. At the instant the ball is hit, an instantaneous gust of wind blows against the ball, adding a component of $-14\hat{i}$ (ft/sec) to the ball's initial velocity. A 15 ft high fence lies 400 ft from the home plate in the direction of the flight. (Note that gravity, $\mathrm{g} = 32$ ft/sec^2)

   (a) Include an appropriate sketch.

   (b) Find a vector equation for the path of the baseball.

   (c) How high does the baseball go, and when does it reach maximum height?

   (d) Find the range and flight time of the baseball, assuming that the ball is not caught.

   (e) When is the baseball 20 ft high? How far (ground distance) is the baseball from home plate at that height?

   (f) Has the batter hit a home run? Explain.

Math 2551 Worksheet: Arc Length

1. Let $\mathbf{r}(t) = \langle 6\sin 2t, 6\cos 2t, 5t\rangle$. Find the unit tangent vector of $\mathbf{r}(t)$ and find the length of the portion of the graph of $\mathbf{r}(t)$ where $0 \le t \le \pi$.

2. Find the point on the curve
$$\mathbf{r}(t) = (5\sin t)\mathbf{i} + (5\cos t)\mathbf{j} + 12t\mathbf{k}$$
at a distance 26π units along the curve from the point $(0, 5, 0)$ in the direction of increasing arc length.

3. Suppose an object's position is given by $\mathbf{r}(t) = (2\ln(t+1))\mathbf{i} + (e^{2t} + t)\mathbf{j} + (\sin^2(t))\mathbf{k}$. Set up but do not evaluate the appropriate integral with limits to find the distance the object traveled from the point $A(0, 1, 0)$ to the point $B(\ln 4, e^2 + 1, \sin^2(1))$.

4. Find the length of the curve
$$\mathbf{r}(t) = \langle \sqrt{2}t, \sqrt{3}t, (1-t)\rangle$$
from $(0, 0, 1)$ to $(\sqrt{2}, \sqrt{3}, 0)$.

Math 2551 Worksheet: Curvature and Normals

1. Find the unit tangent vector, unit normal vector, and curvature of the curve $\mathbf{r}(t) = \langle \sqrt{2}t, e^t, e^{-t} \rangle$, $t \in \mathbb{R}$.

2. Find $\mathbf{T}, \mathbf{N}$ and κ for the space curve $\mathbf{r}(t) = (\cos(t) + t\sin(t))\mathbf{i} + (\sin(t) - t\cos(t))\mathbf{j} + 3\mathbf{k}$ with $t \geq 0$.

3. Compute $\mathbf{T}$ and $\mathbf{N}$ for the curve $\mathbf{r}(t) = \langle t, (1/3)t^3 \rangle, t \in \mathbb{R}$ for $t \neq 0$.

   Does $\mathbf{N}$ exist at $t = 0$? Graph the curve and explain what is happening to $\mathbf{N}$ as t passes from negative to positive values.

4. Before doing any computations, where do you think that the curvature of the parabola $y = x^2$ is greatest?

   Compute its curvature and find the point with greatest curvature.

Math 2551 Worksheet: Multivariable Functions

1. Find and sketch the domain for each function.
   (a) $f(x, y) = \sqrt{x - y - 1}$.
   (b) $f(x, y) = \sqrt{(x - 4)(y^2 - 1)}$.
   (c) $f(x, y) = \cos^{-1}(y - 4x^2)$.
   (d) $f(x, y) = \dfrac{1}{4 - x^2 - y^2}$.
   (e) $f(x, y) = \dfrac{1}{\ln(4 - x^2 - y^2)}$

2. Match the surfaces (a)-(g) with the written descriptions of the level curves (A)-(F). You may use each description once, multiple times, or not at all.
   (a) $z = 2x^2 + 3y^2$
   (b) $z = x^2 + y^2$
   (c) $z = \dfrac{1}{x - 1}$
   (d) $z = 2x + 3y$
   (e) $z = \sqrt{25 - x^2 - y^2}$
   (f) $z = \sqrt{x^2 + y^2}$
   (g) $z = xy$.

   (A) a collection of unequally spaced concentric circles
   (B) a collection of unequally spaced parallel lines
   (C) a collection of equally spaced concentric circles
   (D) two straight lines and a collection of hyperbolas
   (E) a collection of concentric ellipses
   (F) a collection of equally spaced parallel lines

3. Find an equation for the level curve of the function $F(x, y) = \dfrac{2y - x}{x + y + 1}$ passing through $(-1, 1)$.

4. Find the equation for the level surface of the function $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ passing through $(1, 1, 1)$.

5. Let $f(x, y) = (x - y)^2$. Determine the equations and shapes of the cross-sections when $x = 0$, $y = 0$, and $x = y$, and describe the level curves. Use this information to produce a sketch of the graph of the surface. Confirm your sketch using a 3d graphing utility.

Math 2551 Worksheet: Limits and Continuity

1. Let $f(x,y) = \left(\frac{1}{x} + \frac{1}{y}\right)^2$. Find $\lim_{(x,y)\to(2,-3)} f(x,y)$ or show it does not exist.

2. Let $f(x,y) = \frac{x-2y}{x^3-8y^3}$. Find $\lim_{(x,y)\to(2,1)} f(x,y)$ or show it does not exist.

3. Let $f(x,y) = \frac{\sqrt{2x-y}-2}{2x-y-4}$. Find $\lim_{(x,y)\to(2,0)} f(x,y)$ or show it does not exist.

4. Let $f(x,y) = \frac{y^2}{x^2+y^2}$. Find $\lim_{(x,y)\to(0,0)} f(x,y)$ or show it does not exist.

5. At what points (x,y) in the plane is $f(x,y) = \cos\left(\frac{1}{xy}\right)$ continuous?

6. At what points (x,y,z) is $h(x,y,z) = \frac{1}{1-\ln\left(x^2+y^2+z^2\right)}$ continuous?

Math 2551 Worksheet 8 - Review for Exam 1

1. Set up the integral to find the arc length of the curve $y = e^x$ from the point $(0, 1)$ to the point $(1, e)$. Focus on finding a parameterization, and on what values of t give these two points. Is this an integral you would want to compute? Why or why not?

2. Parameterize the line tangent to the curve
$$\mathbf{r}(t) = \langle \cos^2(t), \sin(t)\cos(t), \cos(t)\rangle$$
at the point where $t = \pi/2$.

3. Compute the unit tangent vector $\mathbf{T}(t)$ and the unit normal vector $\mathbf{N}(t)$ to the circle
$$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t)\rangle.$$
Before checking, should the normal vector be pointing into or out of the circle? Why?

4. We have seen that the curvature of a circle with radius a is $1/a$. Thinking about the geometry of a helix with radius a, do you think its curvature will be greater than or less than $1/a$? Why? Compute the curvature using the parameterization
$$\mathbf{r}(t) = \langle a\cos(t), t, a\sin(t)\rangle$$
to confirm or challenge your intuition.

5. The function $\ell(t)$ below describes a line. There is a particular plane that $\ell(t)$ is normal to at the point $t = 0$. Find an equation of this plane.
$$\ell(t) = \langle 3 - 3t, 2 + t, -2t\rangle.$$
Where does this line intersect the different plane $3x - y + 2z = -7$?

6. Find and sketch the domain of each of the following functions of two variables:
   (a) $\sqrt{9 - x^2} + \sqrt{y^2 - 4}$
   (b) $\arcsin(x^2 + y^2 - 2)$
   (c) $\sqrt{16 - x^2 - 4y^2}$

7. Solve the differential equation below, together with its given initial conditions. Remember that this means finding all functions $\mathbf{r}(t)$ which satisfy the given equations.
$$\mathbf{r}''(t) = 2\mathbf{i} + 6t\mathbf{j} + \frac{1}{2\sqrt{t}}\mathbf{k}, \quad \mathbf{r}'(1) = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}, \quad \mathbf{r}(1) = \mathbf{i} + \mathbf{j}$$

8. Let $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$ for $(x, y) \neq (0, 0)$. Is it possible to define $f(0, 0)$ in a way that makes f continuous at the origin? Why?

Math 2551 Worksheet: Partial Derivatives

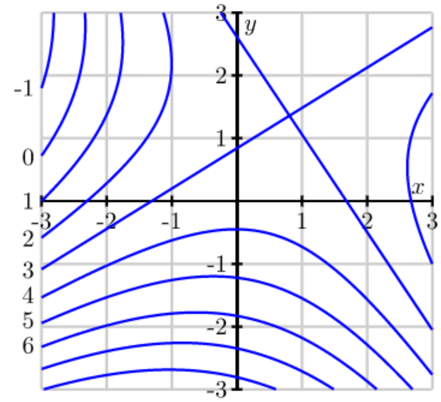
1. Below is a contour plot for a function $f(x, y)$, with values for some of the contours indicated on the left of the figure.

   (a) Estimate the partial derivative $f_x(-2, -1)$.

   (b) Estimate the partial derivative $f_y(-2, -1)$.

   (c) Locate, if possible, one point (x, y) where $f_x(x, y) = 0$.

   (d) Locate, if possible, one point (x, y) where $f_x(x, y) < 0$.



2. The speed of sound C traveling through ocean water is a functino of temperature, salinity, and depth. It may be modeled by the function

$$C(T, S, D) = 1450 + 4.5T - 0.05T^2 + 0.0003T^3 + (1.5 - 0.01T)(S - 35) + 0.015D,$$

   where C is the speed of sound in meters/second, T is the temprature in degrees Celsius, S is the salinity in grams/liter of water, and D is the depth below the ocean surface in meters.

   (a) State the units in which each of the partial derivatives C_T, C_S, and C_D are expressed and explain the physical meaning of each.

   (b) Find the partial derivatives C_T, C_S, and C_D.

   (c) Evaluate each of the three partial derivatives at the point where $T = 10, S = 35$, and $D = 100$. What does the sign of each partial derivative tell us about the behavior of the function C at the point $(10, 35, 100)$?

3. Find all the second partial derivatives for $f(x, y) = e^x + x\ln(y)$.

4. Find f_x, f_y, f_z, and f_{xzz} for the function $f(x, y, z) = x\sin(yz)$.

5. The fifth-order partial derivative $\partial^5 f/\partial x^2 \partial y^3$ is zero for each of the following functions. To show this as quickly as possible, which variable would you differentiate with respect to first: x or y?

   Try to answer without writing anything down. Why did you make the choice you did?

   (a) $f(x, y) = y^2x^4e^x + 2$

   (b) $f(x, y) = y^2 + y(\sin(x) - x^4)$

   (c) $f(x, y) = x^2 + 5xy + \sin(x) + 7e^x$

   (d) $f(x, y) = xe^{y/2}$

Math 2551 Worksheet: Chain Rule

1. An object travels along a path on a surface. The exact path and surface are not known, but at time $t = t_0$ it is known that

$$\frac{\partial z}{\partial x} = 5, \quad \frac{\partial z}{\partial y} = -2, \quad \frac{dx}{dt} = 3, \quad \frac{dy}{dt} = 7.$$

   Use this information to determine the rate of change of the height z of the object with respect to time t at $t = t_0$.

2. Find the values of t where $\dfrac{dz}{dt} = 0$ if $z = 3x + 4y$, $x = t^2$, and $y = 2t$.

3. Let $w = xy + yz + zx$, where $x = r\cos\theta, \quad y = r\sin\theta, \quad z = r\theta$. Find $\dfrac{\partial w}{\partial r}$ and $\dfrac{\partial w}{\partial \theta}$ when $r = 2$ and $\theta = \dfrac{\pi}{2}$.

4. Suppose we have a differentiable function $w = g(x, y)$ and x and y are differentiable functions of t and we know the following information.

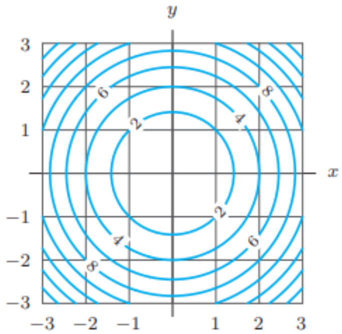
$$g(1,0) = 1,\ g_x(1,0) = -2,\ g_y(1,0) = 2,\ g(-1,2) = 3,\ g_x(-1,2) = 1,\ g_y(-1,2) = -2,$$

$$x(2) = 1,\ y(2) = 0,\ x(1) = 1,\ y(1) = 3,\ x'(2) = 4,\ y'(2) = -1,\ x'(1) = 0,\ y'(1) = 2$$

   If possible, find $\dfrac{dw}{dt}(1)$ and $\dfrac{dw}{dt}(2)$ or explain why the given information is not enough to do so. Which of these pieces of information would you not use at all to compute either value?

Math 2551 Worksheet: Gradient and Directional Derivatives

1. Use the contour diagram of the differentiable function f given below to decide if the specified directional derivative is positive, negative, or approximately zero.



   (a) At the point $(-2, 2)$ in the direction $\mathbf{i}$

   (b) At the point $(0, -2)$ in the direction $\mathbf{j}$

   (c) At the point $(-1, 1)$ in the direction $\mathbf{i} + \mathbf{j}$

   (d) At the point $(-1, 1)$ in the direction $-\mathbf{i} + \mathbf{j}$

   (e) At the point $(0, -2)$ in the direction $\mathbf{i} - 2\mathbf{j}$

2. Let $f(x, y) = xy$. Sketch the curve $f(x, y) = -4$ together with $\nabla f(2, -2)$ and the tangent line at $(2, -2)$. Then, find an equation for the tangent line. What do you notice?

3. Find the derivative of $g(x, y) = \dfrac{x - y}{xy + 2}$ at $(1, -1)$ in the direction of $\langle 12, 5 \rangle$.

4. Suppose you are climbing a hill whose shape is given by the equation

$$z = 1000 - 0.005x^2 - 0.01y^2,$$

   where x, y, and z are measured in meters, and you are standing at a point with coordinates $(60, 40, 966)$. The positive x-axis points east and the positive y-axis points north.

   (a) If you walk due south, will you start to ascend or descend? At what rate?

   (b) If you walk northwest, will you start to ascend or descend? At what rate?

   (c) In which direction is the slope largest? What is the rate of ascent in that direction?

5. Let $f(x, y) = -x^2y + xy^2 + xy$ and $P = (2, 1)$.

   (a) Find the direction of maximal increase of f at P.

   (b) What is the maximum rate of change of f at P?

   (c) Find the direction of maximal decrease of f at P.

   (d) Find a direction $\mathbf{u}$ such that $D_{\mathbf{u}}f(P) = 0$ (note this forces $\mathbf{u}$ to be a unit vector!).

Math 2551 Worksheet: Linearization and Tangent Planes

1. Find the linearization of $f(x, y) = e^{2y-x}$ at $(1, 2)$.
2. Find the linearization of $f(x, y, z) = \tan^{-1}(xyz)$ at $(1, 1, 0)$.
3. Use the linearization to approximate $f(2.95, 7.1)$ for the function $f(x, y) = \sqrt{x^2 + y}$, knowing that $f(3, 7) = 4$.
4. Graph the function $z = x^{1/3}y^{1/3}$. Examine the graph at $(0, 0)$ - does it look like the function can be well-approximated by a tangent plane there? Based on your conclusion, is this function differentiable at $(0, 0)$?
5. Find the equation for the plane tangent to the graph of $f(x, y) = \sqrt{y - x}$ when $x = 1$ and $y = 2$.

Math 2551 Worksheet: Optimization I

1. Find all the local maxima, local minima, and saddle points of $f(x, y) = e^y(x^2 - y^2)$.
2. Find and classify all critical points for the function $x^3 + 3xy + y^3$.
3. Can you conclude anything about $f(a, b)$, if f and its first and second partial derivatives are continuous around the critical point (a, b) and $f_{xx}(a, b)$ and $f_{yy}(a, b)$ differ in sign? Justify your answer.
4. In each case, the origin is a critical point of f and $f_{xx}f_{yy} - (f_{xy})^2 = 0$ at the origin, so the Second Derivative Test fails at the origin. Use some other method to determine whether the function f has a maximum, a minimum, or neither at the origin.
   (a) $f(x, y) = x^2y^2$
   (b) $f(x, y) = 1 - x^2y^2$
   (c) $f(x, y) = xy^2$
   (d) $f(x, y) = x^3y^2$
   (e) $f(x, y) = x^3y^3$
   (f) $f(x, y) = x^4y^4$

Math 2551 Worksheet: Optimization II

1. Find the absolute maxima and minima of the function $f(x, y) = x^2 - xy + y^2 + 1$ on the closed triangular plate bounded by lines $x = 0$, $y = 4$, $y = x$ in the first quadrant.

2. Among all rectangular boxes of volume 27 cm^3, what are the dimensions of the box with the smallest surface area? What is the smallest possible surface area? (assume this occurs at a local min of the surface area function)

Math 2551 Worksheet: Lagrange Multipliers

1. A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

2. Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

3. Find the extreme values of the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $x^4 + y^4 + z^4 = 1$.

4. The plane $x + y + 2z = 2$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the points on this ellipse that are nearest to and farthest from the origin. Hint: It may be helpful algebraically to work with the square of the distance to the origin.

5. Find the maximum volume of a rectangular box that is inscribed in a sphere of radius r.

Math 2551 Worksheet: Double Integrals on Rectangles

1. Without using an iterated integral, evaluate the double integral $\iint_R (4-2y)\,dA$, $R = [0,1]\times[0,1]$, by identifying it as the volume of a solid.
2. The integral $\iint_R \sqrt{9-y^2}$, where $R = [0,4]\times[0,2]$, represents the volume of a solid. Sketch the solid.
3. Find $\displaystyle\iint_R \frac{xy^2}{x^2+1}\,dA, \qquad R\colon\ 0\le x\le 1,\ -3\le y\le 3$
4. Find the volume of the region bounded above by the elliptical paraboloid $z = 16-x^2-y^2$ and below by the square $R: 0\le x\le 2, 0\le y\le 2$.
5. Use Fubini's Theorem to evaluate the integral
$$\int_0^1\int_0^3 xe^{xy}\,dx\,dy.$$
6. **Challenge**: Use a trig substitution $y = x\tan(\theta)$ or $x = y\tan(\theta)$ to show that the iterated integrals
$$\int_0^1\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dy\,dx \qquad \text{and} \qquad \int_0^1\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dx\,dy$$
are not equal. Why does this not violate Fubini's Theorem?

Math 2551 Worksheet: Double Integrals on General Regions

1. Compute the integrated integral

$$\int_0^{\pi}\int_0^{x} x\sin(y)\,dy\,dx.$$

2. Decide, without calculation, if each of the integrals below are positive, negative, or zero. Let D be the region inside the unit circle centered at the origin. Let T, B, R, and L denote the regions enclosed by the top half, the bottom half, the right half, and the left half of unit circle, respectively.

   (a) $\iint_B (y^3+y^5)\,dA$
   (b) $\iint_T (y^3+y^5)\,dA$
   (c) $\iint_D (y^3+y^5)\,dA$
   (d) $\iint_L (y^3+y^5)\,dA$
   (e) $\iint_R (y^3+y^5)\,dA$

3. Write an iterated integral for $\iint_R 1\,dA$ over the region R using vertical cross-sections and horizontal cross-sections.

   (a) Bounded by $y=e^{-x}$, $y=1$, and $x=\ln 3$.
   (b) Bounded by $y=x^2$ and $y=x+2$

4. Sketch the region of integration, reverse the order of integration, and evaluate the integral.

$$\int_0^{\sqrt{\pi}}\int_y^{\sqrt{\pi}} \cos(x^2)\,dx\,dy.$$

5. Sketch the region of integration and evaluate the integral

$$\iint_R xy^2\,dA,$$

   where R is enclosed by $x=0$ and $x=\sqrt{1-y^2}$.

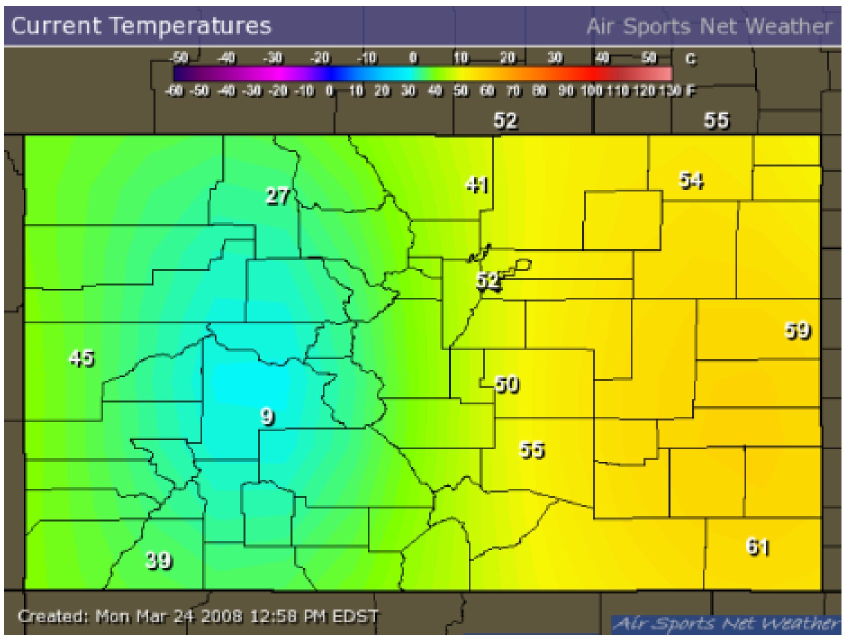
6. Find the volume of the solid bounded by the cylinder $y^2+z^2=4$ and the planes $x=2y$, $x=0$, $z=0$ in the first octant.

7. The integral expression below gives the area of a region in the xy-plane. Sketch the region, labeling the bounding curves with their equations, and giving the coordinates of points where the curves intersect. Then find the area of the region.

$$\int_0^2\int_{x^2-4}^{0} dy\,dx + \int_0^4\int_0^{\sqrt{x}} dy\,dx$$

Math 2551 Worksheet: Applications of Double Integrals

1. The figure below shows a temperature map of Colorado. Use the data to estimate the average temperature in the state using 4, 16, and 25 subdivisions. Give both an upper and lower estimate for each number of subdivisions. Why do we like Colorado for this problem? What other state(s) or countries might we like?



2. Which do you think will be larger, the average value of $f(x, y) = xy$ over the square $0 \leq x \leq 1, 0 \leq y \leq 1$, or the average value of f over the quarter circle $x^2 + y^2 \leq 1$ in the first quadrant? Calculate them to find out.

3. If $f(x, y) = 100(y + 1)$ represents the population density in people per square mile of a planar region on Earth, where x and y are measured in miles, find the number of people in the region bounded by the curves $x = y^2$ and $x = 2y - y^2$.

Math 2551 Worksheet: Polar Double Integrals

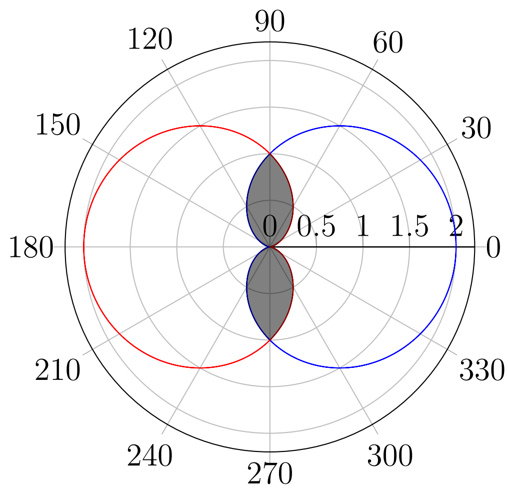
1. Evaluate $\iint_D y^2 + 3x \, dA$ where D is the region in the 3rd quadrant between $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$.

2. Change the Cartesian integral

$$\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} e^{-x^2-y^2} \, dx \, dy$$

into an equivalent polar integral and evaluate the integral.

3. Find the area of the region common to the interiors of the cardioids $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$.



4. Use a double integral to determine the volume of the solid that is inside the cylinder $x^2 + y^2 = 16$, below $z = 2x^2 + 2y^2$, and above the xy-plane.

5. **Challenge:** An integral of great importance in statistics is the Gaussian integral $I = \int_0^\infty e^{-x^2} \, dx$. The function $f(x) = e^{-x^2}$ has no elementary antiderivative, so this integral was hard to compute with the methods of single-variable calculus.

   Let $I^2 = \left(\int_0^\infty e^{-x^2} \, dx\right)\left(\int_0^\infty e^{-y^2} \, dy\right)$.

   (a) Express I^2 as the limit of a double integral in polar coordinates with an appropriately chosen domain. (Hint: As R goes to infinity, what happens to a disk of radius R centered at the origin?)

   (b) Evaluate your double interal to compute the value of I^2. Use this to find the value of the original Gaussian integral I.

You can find some history of this integral here.

Math 2551 Worksheet: Exam 2 Review

1. Which of the following statements are true if $f(x, y)$ is differentiable at (x_0, y_0)? Give reasons for your answers.
   (a) If $\mathbf{u}$ is a unit vector, the derivative of f at (x_0, y_0) in the direction of $\mathbf{u}$ is $(f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j}) \cdot \mathbf{u}$.
   (b) The derivative of f at (x_0, y_0) in the direction of $\mathbf{u}$ is a vector.
   (c) The directional derivative of f at (x_0, y_0) has its greatest value in the direction of ∇f.
   (d) At (x_0, y_0), the vector ∇f is normal to the curve $f(x, y) = f(x_0, y_0)$.
2. Find dw/dt at $t = 0$ if $w = \sin(xy + \pi), x = e^t$, and $y = \ln(t + 1)$.
3. Find the extreme values of $f(x, y) = x^3 + y^2$ on the circle $x^2 + y^2 = 1$.
4. Test the function $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2$ for local maxima and minima and saddle points and find the function's value at these points.
5. Find the points on the surface $xy + yz + zx - x - z^2 = 0$ where the tangent plane is parallel to the xy-plane.
6. Evaluate the integral $\int_0^1 \int_{2y}^2 4\cos(x^2)\, dx\, dy$. Describe why you made any choices you did in the course of evaluating this integral.
7. If $f(x, y) \geq 2$ for all (x, y), is it possible that the average value of $f(x, y)$ on a unit disk centered at the origin is $\frac{2}{\pi}$?
8. A swimming pool is circular with a 40 foot diameter. The depth is constant along east-west lines and increases linearly from 2 feet at the south end to 7 feet at the north end. Find the volume of water in the pool.

Math 2551 Worksheet 18: Triple Integrals

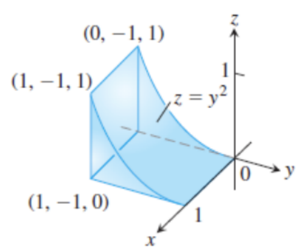
1. Evaluate the triple iterated integral
$$\int_0^2 \int_{-1}^4 \int_0^1 z^3 - 4x^2 y \; dz \; dy \; dx.$$
When you evaluate the innermost integral, you should treat both x and y as constants and take the antiderivative with respect to z.

   What is the region of integration for this integral in $\mathbb{R}^3$?

2. Evaluate $\iiint_E z dV$, where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$. It may be helpful to make a sketch of the solid.

3. Evaluate the integral
$$\int_0^{\pi/2} \int_0^y \int_0^x \cos(x + y + z) \; dz \; dx \; dy$$

4. Set up integrals that would calculate the volume of the region below, using the specified orders of integration.



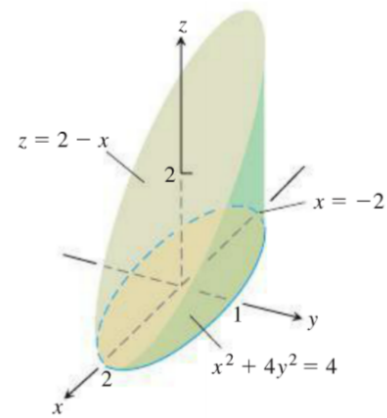
(a) $dy\ dz\ dx$ (b) $dy\ dx\ dz$ (c) $dx\ dy\ dz$ (d) $dx\ dz\ dy$ (e) $dz\ dx\ dy$

5. Let D be the region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 2y$. Write triple iterated integrals in the order $dz\ dx\ dy$ and $dz\ dy\ dx$ that give the volume of D.

Math 2551 Worksheet: Mass and Moments

1. Find the center of mass and the moment of inertia about the y-axis of a thin plate bounded by the line $y = 1$ and the parabola $y = x^2$ if the density is $\delta(x, y) = y + 1$.

2. A solid of constant density $\delta(x, y) = 1$ is bounded below by the plane $z = 0$, on the sides by the elliptical cylinder $x^2 + 4y^2 = 4$, and above by the plane $z = 2 - x$. Find $\bar{x}$ and $\bar{y}$.



Math 2551 Worksheet: Triple Integrals in Cylindrical & Spherical Coordinates

1. Convert the integral $\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{0}^{x}(x^2+y^2)\ dz\ dx\ dy$ into an integral in cylindrical coordinates, and evaluate the integral.

2. Let D be the right circular cylinder whose base is the circle $r = 2\sin\theta$ in the xy-plane and whose top lies in plane $z = 4 - y$. Recall that $r = 2\sin\theta$ describes a circle centered at $(0, 1)$ with radius 1 in the xy-plane. Using cylindrical coordinates,
   (a) find the volume of the region D.
   (b) find the $\bar{x}$ component of the centroid of the region (hint: use symmetry).

3. Find the volume of the solid that is between the spheres $\rho = \sqrt{2}$ and $\rho = 2$, but outside of the circular cylinder $x^2 + y^2 = 1$.

4. Suppose $a \geq 0$. Find the volume of the region cut from the solid sphere $\rho \leq a$ by the half-planes $\theta = 0$ and $\theta = \pi/6$ in the first octant.

5. Find the volume of the solid that lies within the sphere $x^2 + y^2 + z^2 = 4$, above the xy-plane, and below the cone $z = \sqrt{x^2 + y^2}$.

6. Find the volume of the region bounded above by the paraboloid $z = 9 - x^2 - y^2$, below by the xy-plane, and lying *outside* the cylinder $x^2 + y^2 = 1$.

Math 2551 Worksheet: Change of Variables

1. Find the Jacobian of the transformation $x = e^{-r}\sin(\theta), y = e^{r}\cos(\theta)$.

2. Find the image of the set S which is the disk given by $u^2+v^2 \leq 1$ under the transformation $x = au, y = bv$ for some constants $a, b > 0$.

3. Find equations for a transformation $\mathbf{T}$ that maps a rectangular region S in the uv-plane whose sides are parallel to the u- and v-axes onto the region R bounded by the hyperbolas $y = 1/x, y = 3/x$ and the lines $y = x, y = 3x$ in the first quadrant.

4. Solve the system
$$u = 2x - 3y, v = -x + y$$
for x and y in terms of u and v. Then find the value of the Jacobian and find the image of the parallelogram R in the xy-plane with boundaries $x = -3, x = 0, y = x$, and $y = x + 1$ under this transformation. Sketch the transformed region in the uv-plane. Use your results to rewrite the integral
$$\iint_R 2(x-y)\, dx\, dy$$
as an integral in uv-coordinates.

5. Use a change of variables to compute
$$\iint_R xy\, dA,$$
where R is the region in the first quadrant bounded by the lines $y = x$ and $y = 3x$ and the hyperbolas $xy = 1, xy = 3$.

   Hint: Think about problem 3 above.

6. Compute $\iint_R e^{x+y}\, dA$, where R is the region given by the inequality $|x| + |y| \leq 1$.

Math 2551 Worksheet: Scalar Line Integrals

1. For each curve, find a parameterization of the curve with the specified orientation.
   (a) The line segment in $\mathbb{R}^3$ from $(0, 1, -2)$ to $(3, -1, 2)$.
   (b) The line segment in $\mathbb{R}^3$ from $(3, -1, 2)$ to $(0, 1, -2)$.
   (c) The circle of radius 3 in $\mathbb{R}^2$ centered at the origin, beginning at the point $(0, -3)$ and proceeding clockwise around the circle.
   (d) In $\mathbb{R}^2$, the portion of the parabola $y^2 = x$ from the point $(4, 2)$ to the point $(1, -1)$.

2. Find the line integral of $f(x, y, z) := \sqrt{x^2 + y^2}$ over the curve $\vec{r}(t) = (-4\sin t)\mathbf{i} + (4\cos t)\mathbf{j} + 3t\mathbf{k}$, $t \in [0, 2\pi]$.

3. Find the line integral of $f(x, y) = \sqrt{4x + 1}$ over C where C is the part of the curve $x = y^2$ from the point $(4, -2)$ to $(1, 1)$.

4. Let C be the curve with parameterization
$$\vec{r}(t) = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j} + e^t\mathbf{k}, \quad t \in [0, \pi].$$
   Find the mass of C if the density of a wire along C is $\delta(x, y, z) = z^{-1}$.

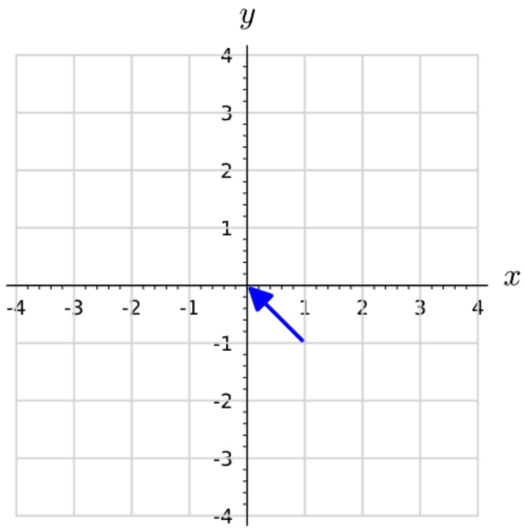
   Preview Problems: In Chapter 13, we talked about vector-valued functions: functions whose input is a single real number and whose output is a vector in $\mathbb{R}^2$ or $\mathbb{R}^3$. A very important related type of function in this unit is a **vector field**, which is a function that takes a point (x, y) in $\mathbb{R}^2$ and outputs a vector in $\mathbb{R}^2$ or a point (x, y, z) in $\mathbb{R}^3$ and outputs a vector in $\mathbb{R}^3$.

5. If we want to make a decision based on what the wind is doing, then we need to keep track of not just its strength at any point, but also its direction! So a vector field is a good model here: at each point (x, y), we can record the velocity vector for the wind.

   Suppose that given the point (x, y) in the plane, we know that the wind velocity at that point is given by the vector $\mathbf{F}(x, y) = \langle y, x\rangle$. For example, at the point $(1, -1)$, the wind velocity is $\mathbf{F}(1, -1) = \langle -1, 1\rangle$. Fill in the table below with the wind velocity vectors for the given points.

(x, y)	$(-2, 0)$	$(-1, 2)$	$(0, -2)$	$(1, 1)$	$(2, 3)$	$(3, 2)$	$(-1, 0)$	$(1, 3)$
$\mathbf{F}(x, y)$								

6. Another useful way of recording this information is to plot the velocity vectors! At each point (x, y) we will draw the vector $\mathbf{F}(x, y)$ starting with the tail of the vector at (x, y). This has been done for you with the point $(1, -1)$ below. Fill in the plot with the other vectors you found in the last problem.



CalcPlot3d can graph vector fields too! Use it to check your work and see the full detail of this vector field.

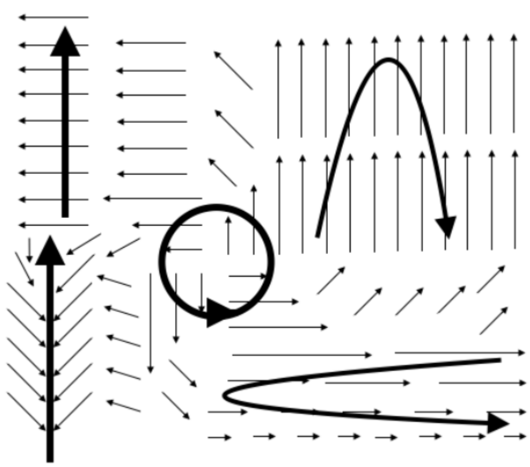
7. Let $f(x, y) = \frac{1}{2}(x - y)^2$. Find the gradient vector field, ∇f, of f and sketch it.

Math 2551 Worksheet: Vector Line Integrals

1. Consider the vector field $\mathbf{F}$ (thin arrows) and and let $\mathbf{T}$ denote the unit tangent vector to the directed curves shown below (denoted with thick arrows). Determine whether

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

is positive, negative, or zero for each directed curve C. In other words, determine whether the *work* done by the vector field on each curve is positive, negative, or zero.



2. Evaluate $\int_C (2x - y)\, dx$ where C is parameterized by $\mathbf{r}(t) = (t^2)\mathbf{i} + (3t-2)\mathbf{j}$, $t \in [0, 1]$.

3. Find the work done by the force $\mathbf{F} = xy\mathbf{i} + (y-x)\mathbf{j}$ over the straight line from $(1, 1)$ to $(2, 3)$.

4. Consider the closed curve C consisting of a semicircle and a straight line segment as follow:

$$\mathbf{r}_1(t) = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j},\ t \in [0, \pi], \qquad \mathbf{r}_2(t) = t\mathbf{i},\ t \in [-2, 2]$$

Let the vector field $\mathbf{F}$ be given by

$$\mathbf{F}(x, y) = -y^2\mathbf{i} + x^2\mathbf{j}.$$

Find the circulation of $\mathbf{F}$ around C and the flux of $\mathbf{F}$ across C.

5. Give an example of a non-trivial force field $\mathbf{F}$ (not the zero vector at all points) and a non-trivial path $\mathbf{r}(t)$ (not the stationary path at a point P) for which the total work done moving along the path is zero.

Math 2551 Worksheet: Potentials and Conservative Vector Fields

1. Show that the vector field $\mathbf{F} = \dfrac{y}{1+x^2y^2}\mathbf{i} + \dfrac{x}{1+x^2y^2}\mathbf{j}$ is conservative using the mixed partials test, then find a potential function f such that $\mathbf{F} = \nabla f$.

2. Find a potential function f for

$$\mathbf{F}(x, y, z) = 2xy\mathbf{i} + (x^2 - z^2)\mathbf{j} - 2yz\mathbf{k}.$$

   Evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

   where C is any path from $(0, 0, 0)$ to $(1, 2, 3)$.

3. Let a, b, c, d, e be real numbers and

$$\begin{aligned} P(x, y, z) &= 3x + 7y + 2z; \\ Q(x, y, z) &= ax + by + 4z; \\ R(x, y, z) &= cx + dy + ez. \end{aligned}$$

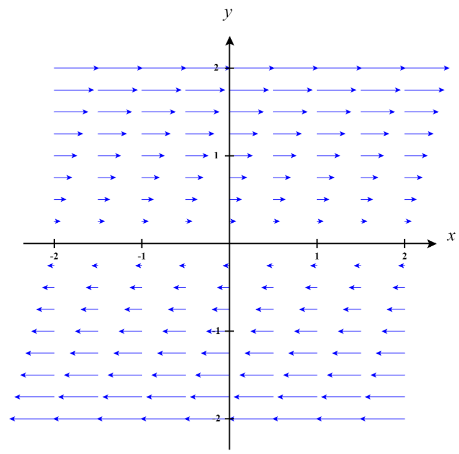
   For which values of the constants a, b, c, d, e is $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ a conservative vector field?

4. Find a potential function f for $\mathbf{F}(x, y, z) = (y^2z + 2xz^2)\mathbf{i} + 2xyz\mathbf{j} + (xy^2 + 2x^2z)\mathbf{k}$ and use it to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the curve $C : \mathbf{r}(t) = \langle \sqrt{t}, t + 1, t^2 \rangle, 0 \le t \le 1$.

5. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ for the vector field $\mathbf{F} = \left\langle \dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \right\rangle$ where the curve C is the unit circle oriented counterclockwise.

Math 2551 Worksheet: Curl, Divergence, Green's Theorem

1. Below is a plot of a vector field $\mathbf{F}(x, y)$. Use this to decide whether the values of $\operatorname{curl} \mathbf{F} \cdot \mathbf{k}$ and $\operatorname{div} \mathbf{F}$ in each quadrant are positive, negative, or zero.



2. Compute $\operatorname{div} \mathbf{F}$ and $\operatorname{curl} \mathbf{F} \cdot \mathbf{k}$ for the vector field $\mathbf{F}(x, y) = \langle \frac{y}{4}, 0 \rangle$, which was plotted above.

3. Let C be the ellipse
$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1.$$
   (a) Parametrize this ellipse to give it a positive orientation.

   (b) Let $\mathbf{F}(x, y) = 2x\mathbf{i} + 2y\mathbf{j}$. Use Green's theorem to find the circulation of $\mathbf{F}$ around C and its flux across C.

4. Let R be the region in the xy-plane bounded above by the curve $y = 3 - x^2$ and below by the curve $y = x^4 + 1$. Orient this boundary positively. Let
$$\mathbf{F}(x, y) = (y + e^x \ln y)\mathbf{i} + (e^x/y)\mathbf{j}.$$
Use Green's theorem to find the circulation of $\mathbf{F}$ around C. What happens when you try to use Green's theorem to evaluate the flux of $\mathbf{F}$ across C? Should you use Green's theorem to evaluate the flux integral?

5. Use Green's Theorem to find the work done by the force $\mathbf{F}(x, y) = \langle x(x + y), xy^2 \rangle$ in moving a particle from the origin along the x-axis to $(1, 0)$, then along the line segment to $(0, 1)$, and then back to the origin along the y-axis.

6. **Looking ahead:** Find a parameterization (a function $\mathbf{r}(s, t) = \langle x(s, t), y(s, t), z(s, t) \rangle$) of the plane through the origin that contains the vectors $\mathbf{i} - \mathbf{j}$ and $\mathbf{j} - \mathbf{k}$. Linear algebra ideas may be useful.

Math 2551 Worksheet: Surfaces

1. Consider the surface cut from the parabolic cylinder $y = 4 - x^2$ by the planes $z = 0$, $z = 2$, and $y = 0$. Sketch S and find a parameterization of S.

2. Consider the surface left in the hemisphere $x^2 + y^2 + z^2 = 4,\ z \geq 0$ after cutting off the lower part between the planes $z = 0$ and $z = \sqrt{3}$ from the hemisphere. (In short, S is the surface which is the part of $x^2 + y^2 + z^2 = 4$ above $z = \sqrt{3}$). Sketch S, parametrize S and find the surface area of S.

3. The tangent plane at a point $P_0 = (f(u_0, v_0), g(u_0, v_0), h(u_0, v_0))$ on a parameterized surface $\mathbf{r}(u, v) = \langle f(u, v), g(u, v), h(u, v)\rangle$ is the plane through P_0 normal to the vector $\mathbf{r}_u(u_0, v_0) \times \mathbf{r}_v(u_0, v_0)$.

   Use this to find an equation to the tangent plane of the surface parameterized by $\mathbf{r}(r, \theta) = (r\cos(\theta))\mathbf{i} + (r\sin(\theta))\mathbf{j} + r\mathbf{k}, r \geq 0, 0 \leq \theta \leq 2\pi$ at the point where $(r, \theta) = (2, \pi/4)$.

   What is a Cartesian equation for this surface? Sketch it and the tangent plane.

4. Find the area of the part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$.

5. Find the area of the surface cut from the "nose" of the paraboloid $x = 1 - y^2 - z^2$ by the yz-plane.

Math 2551 Worksheet: Surface Integrals

1. Integrate $f(x, y, z) = yz$ over the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the cone $z = \sqrt{x^2 + y^2}$.

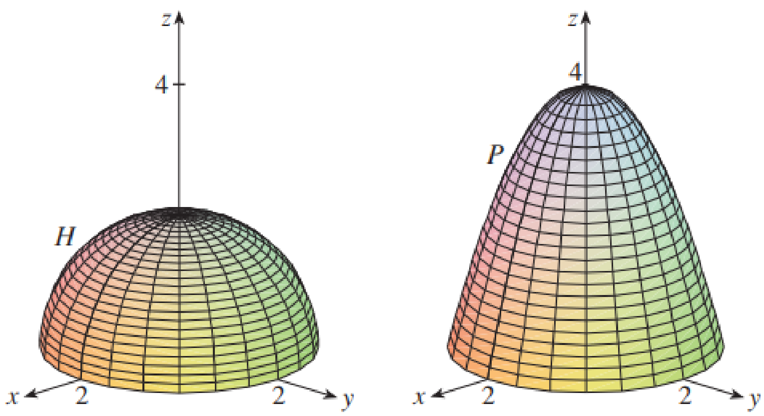
2. Find the flux of the field $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ across the surface S which is the boundary of the solid half-cylinder $0 \leq z \leq \sqrt{1 - y^2}, 0 \leq x \leq 2$, with the outward orientation.

3. A fluid has density 870 kg/m^3 and flows with velocity $\mathbf{v} = \langle z, y^2, x^2 \rangle$, where x, y, z are measured in meters and the components of $\mathbf{v}$ in meters per second. Find the rate of flow outward through the cylinder $x^2 + y^2 = 4, 0 \leq z \leq 1$.

Math 2551 Worksheet: Stokes' Theorem

1. Let H be the hemisphere and P be the portion of a paraboloid shown below. Use Stokes' Theorem to explain why, if $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ whose components have continuous partial derivatives, we must have

$$\iint_H (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \iint_P (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma.$$



2. Use Stokes' Theorem to evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma$, where $\mathbf{F} = \langle x^2z^2, y^2z^2, xyz\rangle$ and S is the part of the paraboloid $z = x^2 + y^2$ that lies inside the cylinder $x^2 + y^2 = 4$, oriented upward.

3. A particle moves along line segments from the origin to the points $(1, 0, 0)$, $(1, 2, 1)$, $(0, 2, 1)$, and back to the origin under the influence of the force field

$$\mathbf{F}(x, y, z) = z^2\mathbf{i} + 2xy\mathbf{j} + 2y^2\mathbf{k}.$$

Find the work done by the field on the particle.

Math 2551 Worksheet: Review for Exam 3

1. Set up an iterated integral in spherical coordinates for $\iiint_E z^2\,dV$ where E is the region between the spheres $x^2+y^2+z^2=4$ and $x^2+y^2+z^2=25$ and inside $z=-\sqrt{\frac{1}{3}(x^2+y^2)}$.

2. Set up an integral that computes the volume of the solid which is bounded above by the cylinder $z=4-x^2$, on the sides by the cylinder $x^2+y^2=4$, and below by the xy-plane using

   (a) Cartesian coordinates

   (b) cylindrical coordinates

   Which integral would you rather evaluate and why?

3. Find an integral that computes the mass of the wire which lies along the curve $y^2=x^3$ from $(0,0)$ to $(1,-1)$ and has density function $\rho(x,y)=2xy^2$.

4. Show that the field $\mathbf{F}=2x\mathbf{i}-y^2\mathbf{j}-\frac{4}{1+z^2}\mathbf{k}$ is conservative, find a potential function, and use it to compute the integral

   $$\int_C 2x\ dx - y^2\ dy - \frac{4}{1+z^2}\ dz$$

   where C is any path from $(0,0,0)$ to $(3,3,1)$.

5. Compute $\int_C (6y+x)\ dx+(y+2x)\ dy$ using any method, where C is the circle $(x-2)^2+(y-3)^2=4$.

6. Find the flux of the field $\mathbf{F}=y\mathbf{i}-x\mathbf{j}+\mathbf{k}$ through the portion of the sphere $x^2+y^2+z^2=a^2$ in the first octant in the direction away from the origin.

7. Use Stokes' theorem to show that the circulation of the field $\mathbf{F}=\langle 2x,2y,2z\rangle$ around the boundary curve C of **any** smooth orientable surface S in $\mathbb{R}^3$ is 0.

8. Find the outward flux of $\mathbf{F}=(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})/\sqrt{x^2+y^2+z^2}$ through the boundary S of the "thick sphere" D given by the points satisfying $1\le x^2+y^2+z^2\le 4$.

Math 2551 Worksheet: Review for Final

1. Find the equation of the plane through $(1, -1, 3)$ parallel to the plane $3x + y + z = 7$. Is there a unique plane through $(1, -1, 3)$ which is perpendicular to the plane $3x+y+z = 7$. Explain why or why not.

2. Find the point on the curve
$$\mathbf{r}(t) = (5\sin(t))\mathbf{i} + (5\cos(t))\mathbf{j} + 12t\mathbf{k}$$
at a distance 26π units along the curve from the point $(0, 5, 0)$ in the direction of increasing parameter t.

3. Find the domain and range of $f(x, y) = \sqrt{x^2 - y}$ and identify its level curves.

4. Compute $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y}{x^2 - y}$ or show this limit does not exist.

5. Let $f(x, y, z) = xy + 2yz - 3xz$. Find the tangent plane to the surface $f(x, y, z) = 1$ at $(1, 1, 0)$ and the linearization $L(x, y, z)$ at $(1, 1, 0)$.

6. At the point $(1, 2)$, the function $f(x, y)$ has a derivative of 2 in the direction toward $(2, 2)$ and a derivative of -2 in the direction toward $(1, 1)$. Find $\nabla f(1, 2)$ and the derivative of f at $(1, 2)$ in the direction toward the point $(4, 6)$.

7. Find the value of the derivative of $f(x, y, z) = xy+yz+xz$ with respect to t on the curve $\mathbf{r}(t) = \langle\cos(t), \sin(t), \cos(2t)\rangle$ at $t = 1$.

8. Find the local minima, local maxima, and saddle points of the function $f(x, y) = x^4 - 8x^2 + 3y^2 - 6y$.

9. Find the extreme values of $f(x, y) = 4xy - x^4 - y^4 + 16$ on the triangular region bounded below by the line $y = -2$, above by the line $y = x$, and on the right by the line $x = 2$.

10. Find the extreme values of $f(x, y) = xy$ on the circle $x^2 + y^2 = 1$.

11. Sketch the region of integration and reverse the order of integration for the integral
$$\int_0^{3/2} \int_{-\sqrt{9-4y^2}}^{\sqrt{9-4y^2}} y \, dx \, dy.$$

12. Evaluate the integral
$$\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{2 \, dy \, dx}{(1 + x^2 + y^2)^2}$$
by changing to polar coordinates.

13. Find the centroid of the region bounded by the lines $x = 2, y = 2$, and the hyperbola $xy = 2$ in the xy-plane.

14. Find the volume of the region bounded above by the sphere $x^2 + y^2 + z^2 = 2$ and below by the paraboloid $z = x^2 + y^2$.

15. Use the transformation $u = 3x + 2y, v = x + 4y$ to evaluate the integral

$$\iint_R (3x^2 + 14xy + 8y^2)\ dx\ dy$$

where R is the region in the first quadrant bounded by the lines $y = (-3/2)x + 1, y = (-3/2)x + 3, y = -(1/4)x$, and $y = -(1/4)x + 1)$.

16. Evaluate the integral $\int_C y^2\ dx + x^2\ dy$ where C is the circle $x^2 + y^2 = 4$.

17. Find the outward flux of $\mathbf{F} = 2xy\mathbf{i} + 2yz\mathbf{j} + 2xz\mathbf{k}$ across the boundary of the cube cut from the first octant by the planes $x = 1, y = 1, z = 1$.

18. Find the work done by $\mathbf{F} = \dfrac{x\mathbf{i} + y\mathbf{j}}{(x^2 + y^2)^{3/2}}$ over the plane curve $\mathbf{r}(t) = \langle e^t \cos(t), e^t \sin(t)\rangle$ from the point $(1, 0)$ to the point $(e^{2\pi}, 0)$.

19. Find the flux of the field $\mathbf{F} = \langle 2xy + x, xy - y\rangle$ outward across the boundary of the square bounded by $x = 0, x = 1, y = 0, x = 1$.

20. Find the flux of $\mathbf{F} = xz\mathbf{i} + yz\mathbf{j} + \mathbf{k}$ across the upper cap cut from the sphere $x^2 + y^2 + z^2 = 25$ by the plane $z = 3$, oriented away from the xy-plane.