MATH 2551 L - Dr. Hunter Lehmann

- Log into Canvas & Piazza
- Talk to your neighbors

Values/Norms

- Mistakes are a learning opportunity
- Mathematics is collaborative
- Make sure everyone is included
- Criticize ideas, not people
- Be respectful of everyone
- Persevere

Big Ideas: Extend differential & integral calculus.

Q: Key ideas from these two courses?

Differential	Integral
- Limits	- Riemann sums
- Optimization	- Area/volume
- Rate of change	- Approximation
- parameterization	- Coordinate systems
	- Series

Def: A multivariable function is a rule that assigns one output to a set of inputs

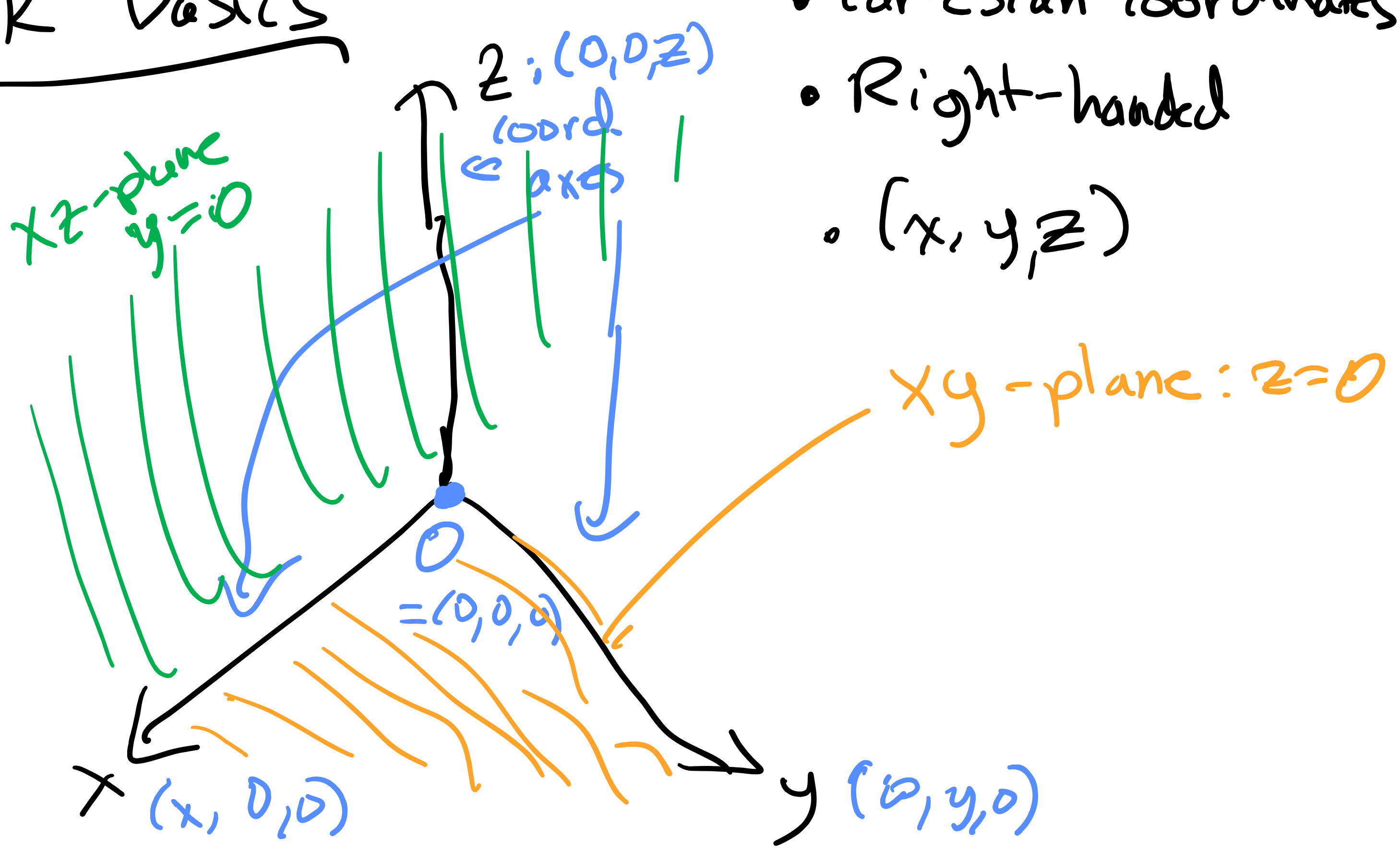
(x, y, z)

Examples: (first name, last name, bday) $\mapsto$ gtid

- (location coord) $\mapsto$ altitude, temp
- (size, position) $\mapsto$ appearance
- $c^2 = a^2 + b^2$
- recipe
- $F = ma$
- eqns. describing shapes

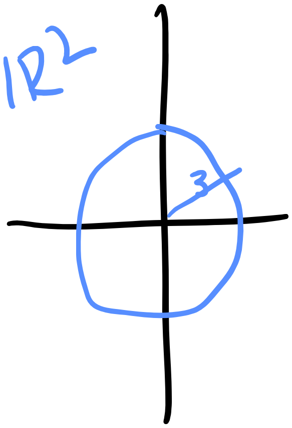
$\mathbb{R}^3$ Basis

- cartesian coordinates
- Right-handed
- (x, y, z)

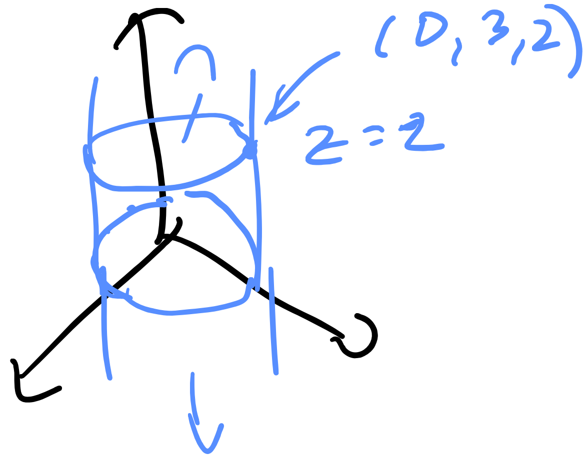


ex: What points satisfy $x^2+y^2=9$?

In $\mathbb{R}^2$



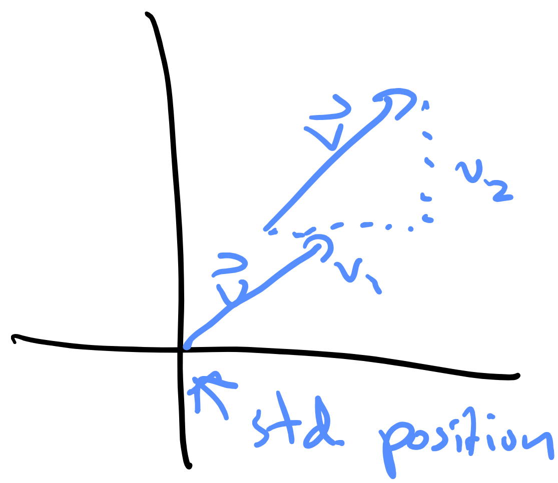
In $\mathbb{R}^3$: (x,y,z)



Vectors

- direction + magnitude
- coordinates

$\vec{v} = \langle v_1, v_2, v_3 \rangle$



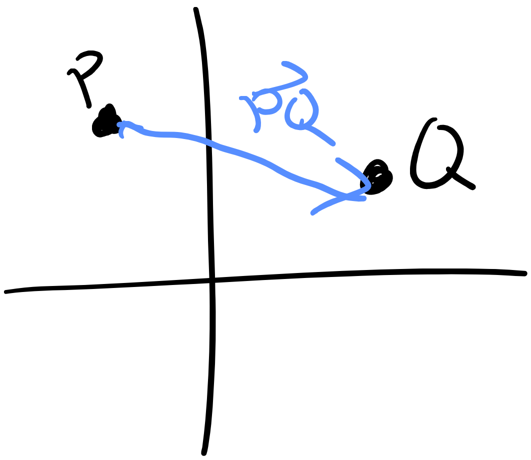
Add: $\vec{v} = \langle 1, 0 \rangle \quad \vec{w} = \langle 2, -3 \rangle$

$\vec{v} + \vec{w} = \langle 1+2, 0-3 \rangle = \langle 3, -3 \rangle$

· Product: $\vec{v} \cdot \vec{w} = 1 \cdot 2 + 0 \cdot (-3) = 2$

Scale: $2\vec{w} = \langle 2 \cdot 2, 2 \cdot (-3) \rangle = \langle 4, -6 \rangle$

length: $|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$



$P = (1,1) \quad Q = (-1,2)$

$\vec{PQ} = \langle -1-1, 2-1 \rangle$

$= \langle -2, 1 \rangle$

$\text{dist } \vec{PQ} = |\vec{PQ}| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$

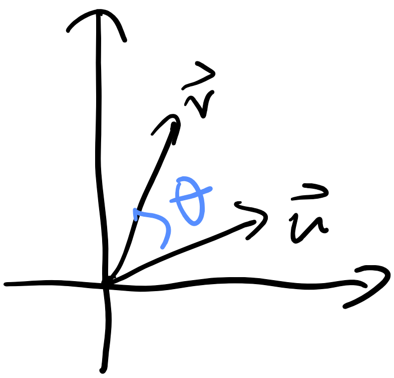
MATH 2551 L - 8/25

Today: Dot Product, Cross Product, Lines, Planes

PLUS sessions: M, Th 6-7pm in Clough

Last time: $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$ (dot product)

- dot product tells us about angle θ between $\vec{u}, \vec{v}$



$$\boxed{\vec{u} \cdot \vec{v} = |u||v| \cos\theta}$$

$\pi/2 < \theta < \pi$

$\vec{u} \cdot \vec{v} < 0 \Rightarrow \theta$ is obtuse

$\vec{u} \cdot \vec{v} = 0 \Rightarrow \vec{u} \perp \vec{v}$, $\theta = 90° = \pi/2$

$\vec{u} \cdot \vec{v} > 0 \Rightarrow \theta$ is acute
$0 \le \theta < \frac{\pi}{2}$

ex: $\vec{u} = \langle 1, 1 \rangle$, $\vec{v} = \langle 2, -1 \rangle$

angle θ is?

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \frac{2 + (-1)}{\sqrt{1+1}\,\sqrt{4+1}} = \frac{1}{\sqrt{10}}$$

θ is acute but close to $\pi/2$ b/c $\frac{1}{\sqrt{10}}$ is close to 0

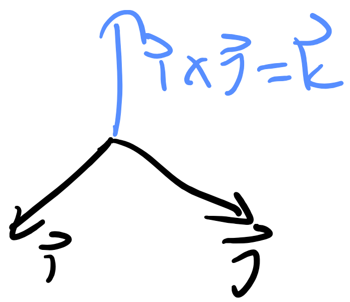
Cross Product (only in $\mathbb{R}^3$)

- Goal: produces a vector orthogonal to two given vectors (Right-handed!)

$\vec{i} = \langle 1,0,0\rangle$

$\vec{j} = \langle 0,1,0\rangle$

$\vec{k} = \langle 0,0,1\rangle$



$\vec{i}\times\vec{j}=\vec{k}$ $\qquad$ $\vec{j}\times\vec{i}=-\vec{k}$

$\vec{j}\times\vec{k}=\vec{i}$ $\qquad$ $\vec{k}\times\vec{j}=-\vec{i}$

$\vec{k}\times\vec{i}=\vec{j}$ $\qquad$ $\vec{i}\times\vec{k}=-\vec{j}$

- anti-commutative

Want: addition/scalar mult should distribute

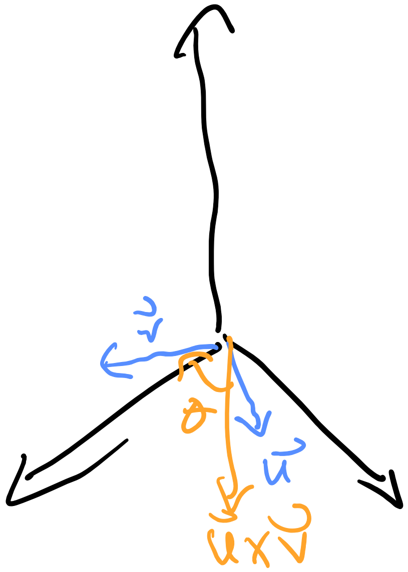
e.g. $(\vec{i}+\vec{j})\times\vec{k} = (\vec{i}\times\vec{k}) + (\vec{j}\times\vec{k})$

$(2\vec{i})\times\vec{k} = \vec{i}\times(2\vec{k}) = 2(\vec{i}\times\vec{k})$

if $\vec{u} = \langle u_1, u_2, u_3\rangle$, $\vec{v} = \langle v_1, v_2, v_3\rangle$

$$\boxed{\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}}$$

Ex A Find $\langle 1,2,0\rangle \times \langle 3,-1,0\rangle$.



$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ 3 & -1 & 0 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 2 & 0 \\ -1 & 0 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(-1-6)$$

$$= \boxed{-7\vec{k}} = \boxed{\langle 0,0,-7\rangle}$$

- $|\vec{u}\times\vec{v}| = |\vec{u}|\,|\vec{v}|\sin\theta$ = area

- $|(\vec{u}\times\vec{v})\cdot\vec{w}|$ = volume of the parallelipiped formed by $\vec{u},\vec{v},\vec{w}$

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

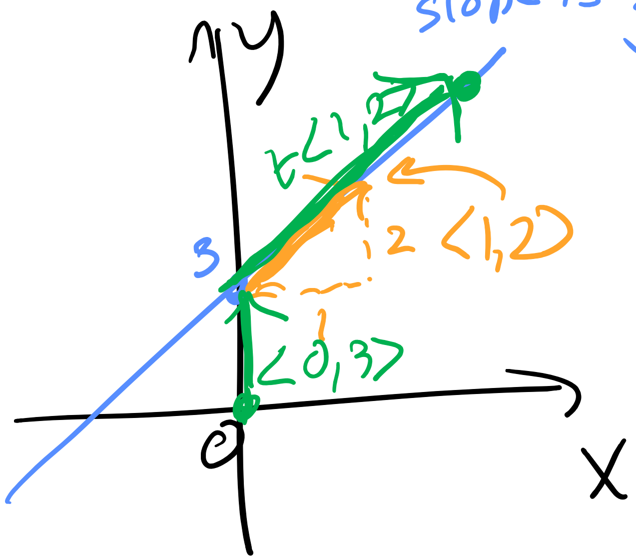
Lines:

In $\mathbb{R}^2$:

$y = 2x + 3$

slope is 2

$\rightarrow$ means if we move 1 unit $\rightarrow$ then we move 2 units $\uparrow$



$\vec{v} = \langle 1, 2\rangle$

$$\boxed{\vec{r}(t) = \langle 0, 3\rangle + t\langle 1, 2\rangle}$$

vector eqn of the line

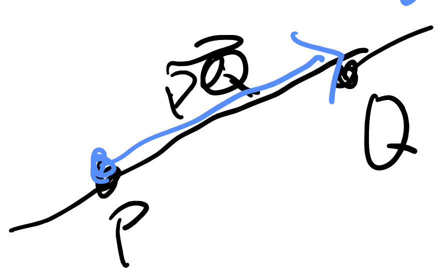
$$r(t) = \overrightarrow{OP} + t\,\vec{v}$$

$\leftarrow$ direction vector

P is a point on the line

ex: $P(1,2,-1)$, $Q(-2,1,-2)$, L is the line through both.

Find vector eqn.



$\vec{v} = \overrightarrow{PQ} = \langle -3, -1, -1\rangle$

$r(t) = \langle 1, 2, -1\rangle + t\langle -3, -1, -1\rangle$

$t = 0 \Rightarrow$ get P

$t = 1 \Rightarrow$ get Q

MATH 2551 L Lecture 3 - 8/30/22

Today: Lines cont., planes, quadric surfaces

- 12.1-12.4 HW due tonight
- Quiz 1 on 12.1-12.4 in studio tomorrow
- CalcPlot3d

Lines • A line is all terminal pts of vectors emanating from a given point P parallel to a fixed vector $\vec{v}$.

$\rightarrow P=(1,2,-1)$ $\vec{v}=\langle -3,-1,-1\rangle$

vector eqn: $\vec{r}(t)=\overrightarrow{OP}+\vec{v}\cdot t$

e.g. $\vec{r}(t)=\langle 1-3t, 2-t, -1-t\rangle$

Parametric eqns : $\vec{r}=\langle x(t), y(t), z(t)\rangle$

e.g.
$$x(t)=1-3t$$
$$y(t)=2-t$$
$$z(t)=-1-t$$

Planes: A plane is all terminal points of vectors emanating from a given point P_0 $\perp$ to a fixed vector $\vec{n}$.

- $P_0 = (x_0, y_0, z_0)$
- $P = (x, y, z)$
- $\vec{n} = \langle a, b, c \rangle$ (normal vector)

- $\overrightarrow{P_0P} \perp \vec{n}$

$\Rightarrow \overrightarrow{P_0P} \cdot \vec{n} = 0$

scalar eqn of a plane: $\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle a, b, c \rangle = 0$

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

$$ax + by + cz = d, \quad d = \overrightarrow{OP_0} \cdot \vec{n}$$

ex: $P = (1, 2, -1)$, $Q = (1, 0, -1)$, $R = (0, 1, 3)$

Find the plane containing all 3 points.

- Need $\vec{n}$
- Use $\overrightarrow{PQ} = \langle 0, -2, 0 \rangle$

$\overrightarrow{PR} = \langle -1, -1, 4 \rangle$

$$\vec{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -2 & 0 \\ -1 & -1 & 4 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} -2 & 0 \\ -1 & 4 \end{vmatrix} - \vec{j} \begin{vmatrix} 0 & 0 \\ -1 & 4 \end{vmatrix} + \vec{k} \begin{vmatrix} 0 & -2 \\ -1 & -1 \end{vmatrix}$$

$$= \langle -8-0,\ 0,\ 0-2 \rangle = \langle -8, 0, -2 \rangle$$

plane: $-8(x-1)+0(y-2)-2(z+1)=0$

$P_0 = P$ $\quad -8(x-0)+0(y-1)-2(z-3)=0$

$$8x+2z=6$$

$$4x+z=3$$

$$\vec{n} = \langle 4, 0, 1 \rangle$$

Quadric surfaces

- next nicest objects

$$Ax^2 + By^2 + Cz^2 + Dz = E$$

- **spheres**: $A = B = C$ equal, no D

$$x^2 + y^2 + z^2 = 4$$

- sphere of radius 2 center 0

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$$

center (x_0, y_0, z_0)
radius r

complete the square

$$x^2 - 2x + y^2 + z^2 = 0$$
$$(x - 1)^2 + y^2 + z^2 = 1$$

- **cylinders**: one variable missing

e.g. $y^2 + z^2 = 2$

MATH 2551 L - 9/1

- Today: vector-valued functions and their calculus

Def: A vector-valued function is a function whose input is a real parameter t and whose output is a vector.
The graph of a v.-v. function is the set of all terminal pts of its output vectors with initial pt $\vec{0}$.

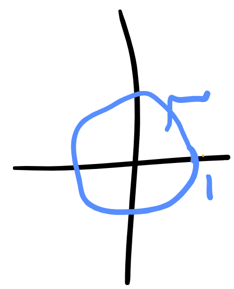
$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

ex: Lines $\vec{r}(t) = \langle 1-t, -1+t, 2+2t \rangle$
has the graph which is a line through $(1,-1,2)$ in the direction $\langle -1, 1, 2 \rangle$

ex: Parametric curves in $\mathbb{R}^2$

$x(t) = \cos(t)$
$y(t) = \sin(t)$

$$\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$$



ex: $\vec{r}_1(t) = \langle \cos(t), \sin(t), t \rangle$
$\vec{r}_2(t) = \langle \cos(2t), \sin(2t), 2t \rangle$
$\vec{r}_3(t) = \langle \cos(t+\pi), \sin(t+\pi), t+\pi \rangle$

Q: Are these functions equal?
Do they have the same graph?

No, $\vec{r}_1(1) \neq \vec{r}_2(1) \neq \vec{r}_3(1)$

How are they similar or different?

Calculus of V-V functions

Theme: Work componentwise

Limits: The limit $\lim_{t\to t_0} \vec{r}(t)$ is

$$\left\langle \lim_{t\to t_0} x(t),\ \lim_{t\to t_0} y(t),\ \lim_{t\to t_0} z(t) \right\rangle$$

ex: $\lim_{t\to 1} \langle \ln(t), t+1, t^2 \rangle$

$$= \left\langle \lim_{t\to 1} \ln(t),\ \lim_{t\to 1} t+1,\ \lim_{t\to 1} t^2 \right\rangle$$

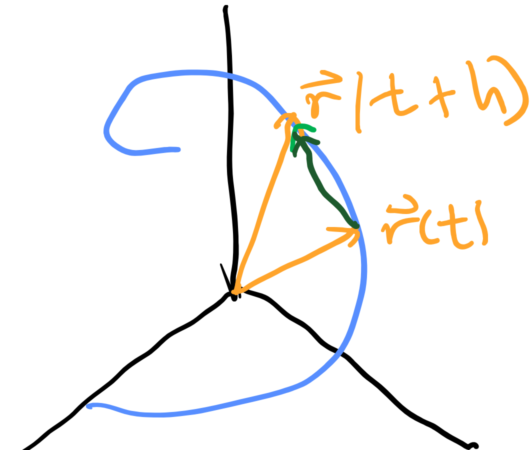
$$= \langle 0, 2, 1 \rangle$$

Continuity: $\vec{r}(t)$ is continuous at t_0 if

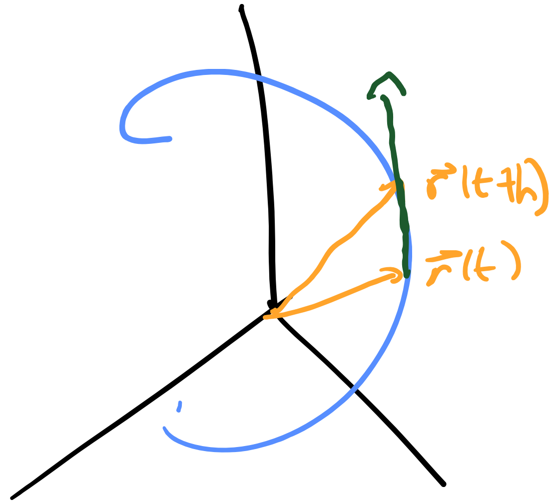
$$\vec{r}(t_0) = \lim_{t\to t_0} \vec{r}(t)$$

- equivalently, $\vec{r}(t)$ is cts at $t = t_0$ if $x(t), y(t), z(t)$ are cts at $t = t_0$.

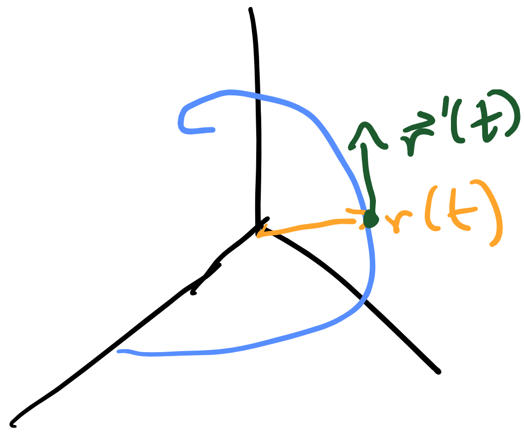
Derivatives



$$\frac{\vec r(t+h)-\vec r(t)}{h}$$



$$\lim_{h\to 0}\frac{\vec r(t+h)-\vec r(t)}{h}=\vec r\,'(t)$$



- $\vec r\,'(t_0)$ is tangent to the graph of $\vec r(t)$ if we place its source at $\vec r(t_0)$
- $\vec r\,'(t)=\langle x'(t), y'(t), z'(t)\rangle$

Interpretation: If $\vec r(t)$ is position, t is time

then $\vec r\,'(t)$ is velocity $\vec v(t)$, speed $|\vec r\,'(t)|=|\vec v(t)|$

$\vec r\,''(t)=\vec v\,'(t)=\vec a(t)$ is acceleration

ex: $\vec r(t)=\langle 2t-\frac{1}{2}t^2+1, t-1\rangle$

then $\vec v(t)=\langle 2-t,\ 1\rangle$

$\vec a(t)=\langle -1, 0\rangle$

ex: Find tangent line to $\vec{r}(t) = \langle \cos(t), -\sin(t), t \rangle$ at $t = \pi$.

- $\vec{r}(\pi) = \langle -1, 0, \pi \rangle$ so $(-1, 0, \pi)$ is on tangent line
- $\vec{r}\,'(t) = \langle -\sin(t), -\cos(t), 1 \rangle$

  $\vec{r}\,'(\pi) = \langle 0, 1, 1 \rangle$ is tangent to $r(t)$ at $t = \pi$

so the tangent line is

$$L(s) = \vec{r}(\pi) + s\,\vec{r}\,'(\pi)$$

$$= \langle -1, 0, \pi \rangle + s\langle 0, 1, 1 \rangle$$

MATH 2551 L 9/6 - 13.2 & 13.3

Today: • Integrals of v.v. functions
• Initial value problems
• Arc length
• Arc length parameterization

$\boxed{\text{-Do warm-up poll}}$

Differentiation Rules for Vector Functions

Let **u** and **v** be differentiable vector functions of t, **C** a constant vector, c any scalar, and f any differentiable scalar function.

1. *Constant Function Rule:* $\frac{d}{dt}\mathbf{C} = \mathbf{0}$
2. *Scalar Multiple Rules:* $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

   $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
3. *Sum Rule:* $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
4. *Difference Rule:* $\frac{d}{dt}[\mathbf{u}(t) - \mathbf{v}(t)] = \mathbf{u}'(t) - \mathbf{v}'(t)$
5. *Dot Product Rule:* $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$
6. *Cross Product Rule:* $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$
7. *Chain Rule:* $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

standard function

Integrals:

v.v. function

$\langle c_1, c_2, c_3 \rangle$

- indefinite $\int \vec{r}(t)\,dt = \langle \int x(t)\,dt, \int y(t)\,dt, \int z(t)\,dt \rangle + \vec{C}$

  $+c_1$ $+c_2$ $+c_3$

- definite: $\int_a^b \vec{r}(t)\,dt = R(b) - R(a)$

  $R(t) = \int \vec{r}(t)\,dt$

  vector

  $= \langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt, \int_a^b z(t)\,dt \rangle$

ex: Find $\int_0^1 \langle t, e^{2t}, \sec^2(t)\rangle dt$

$$\int \langle t, e^{2t}, \sec^2(t)\rangle dt = \langle \int t\,dt, \int e^{2t}\,dt, \int \sec^2(t)\,dt\rangle$$

$$= \langle \tfrac{1}{2}t^2, \quad u=2t,\ du=2dt, \quad \tan(t)\rangle$$

$\int e^u \cdot \frac{1}{2} du$

$\frac{1}{2}e^u = \frac{1}{2}e^{2t}$

$$= \langle \tfrac{1}{2}t^2, \tfrac{1}{2}e^{2t}, \tan(t)\rangle \Big|_0^1$$

$$= \langle \tfrac{1}{2}, \tfrac{1}{2}e^2, \tan(1)\rangle - \langle 0, \tfrac{1}{2}, 0\rangle$$

$$= \langle \tfrac{1}{2}, \tfrac{1}{2}e^2 - \tfrac{1}{2}, \tan(1)\rangle$$

ex: Initial Value Problems

If $\vec{r}\,'(t) = \vec{v}(t) = \langle -2\sin(2t), 2\cos(t), 1 - \frac{1}{1+t}\rangle$
and $\vec{r}(0) = \langle 1.5, -1, 0\rangle$, find $\vec{r}(t)$.

1) Take antiderivative

$u = 2t$

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \langle \int -2\sin(2t)\,dt, \int 2\cos(t)\,dt, \int 1 - \tfrac{1}{1+t}\,dt\rangle$$

$u = 1+t$

$$= \langle \cos(2t), 2\sin(t), t - \ln|1+t|\rangle + \vec{c}$$

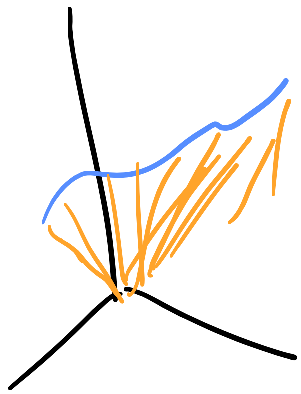
2) Apply initial condition

$\cos(0)\quad 2\sin(0)\quad 0 - \ln(1+0)$

$$\vec{r}(0) = \langle 1.5, -1, 0\rangle = \langle 1, 0, 0\rangle + \vec{c}$$

$$\vec{c} = \langle 1.5, -1, 0\rangle - \langle 1, 0, 0\rangle = \langle 0.5, -1, 0\rangle$$

$\vec{r}(t) = \langle \cos(2t) + 0.5,\ 2\sin(t) - 1,\ t - \ln|1+t| \rangle$



Arc Length

The length of a smooth curve $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ from $t=a$ to $t=b$ that is traced out exactly once is

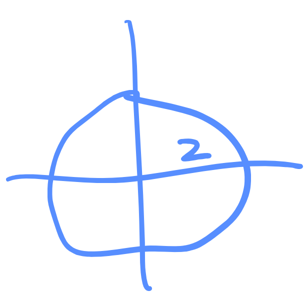
$$L = \int_a^b |\vec{r}\,'(t)|\, dt$$

↑ speed · time = dist trav.

$$= \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt$$

ex: Find the length of the curve $\vec{r}(t) = \langle 2\cos(t), 2\sin(t) \rangle$

$0 \le t \le 2\pi$



Ans is circumference $= 4\pi$

$$\vec{r}\,'(t) = \langle -2\sin(t), 2\cos(t) \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{4\sin^2(t) + 4\cos^2(t)}$$

$$= \sqrt{4(\underbrace{\sin^2(t) + \cos^2(t)}_{1})}$$

$$= 2$$

$$L = \int_0^{2\pi} 2\,dt = 2t\Big|_0^{2\pi} = 4\pi$$

ex: Find the length $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$ from $t=0$ to $t=2\pi$.

$$\vec{r}'(t) = \langle -\sin(t), \cos(t), 1 \rangle$$

$$|\vec{r}'(t)| = \sqrt{\underbrace{\sin^2(t) + \cos^2(t)}_{1} + 1}$$

$$= \sqrt{2}$$

$$L = \int_0^{2\pi} \sqrt{2}\,dt = \sqrt{2}\,t\Big|_0^{2\pi} = \boxed{2\sqrt{2}\,\pi}$$

- arc length function

$$s(t) = \int_{t_0}^{t} |\vec{r}'(T)|\,dT$$

- arc-length parameterization

a) route given by $\vec{r}(t)$ parameterized by time
- different depending on speed, traffic

b) route given by $\vec{r}(s)$ parametrized by distance
- like mile markers

Q: How?

ex: Circle of radius 4 in $\mathbb{R}^2$ about origin

$\vec{r}(t) = \langle 4\cos(t), 4\sin(t)\rangle$, $0 \le t \le 2\pi$

1) Find $s(t)$.

$\vec{r}\,'(t) = \langle -4\sin(t), 4\cos(t)\rangle$

$|\vec{r}\,'(t)| = \sqrt{16\sin^2(t) + 16\cos^2(t)}$

$= 4$

$$s(t) = \int_{t_0}^{t} |\vec{r}\,'(T)|\, dT$$

$$= \int_0^t 4\, dT$$

$$= 4T\Big|_0^t$$

$$= 4t$$

Q: How long does it take to travel $\sqrt{3}$ units?

$s = \sqrt{3} = 4t \Rightarrow t = \sqrt{3}/4$

Solve for t: $s = 4t$

$t = s/4$

arc-length parametization is

$\vec{r}(t) = \vec{r}\left(\frac{s}{4}\right) \Rightarrow$

$\vec{r}(s) = \langle 4\cos\left(\frac{s}{4}\right), 4\sin\left(\frac{s}{4}\right)\rangle$

- Always possible but usually hard

MATH 2551 L -9/8 - 13.3, 13.4

Today: - Review arc length

- Curvature
- Unit tangent & normal vectors

Next week is exam 1 on W - see Canvas

ex: Find arc length parametrization for the curve

$\vec{r}(t) = \langle t^2, \frac{8}{3}t^{3/2}, 4t \rangle$ for $t \geq 0$.

1) Find $s(t) = \int_0^t |\vec{r}\,'(T)| \, dT$

$\vec{r}\,'(t) = \langle 2t, 4t^{1/2}, 4 \rangle$

$$s(t) = \int_0^t \sqrt{(2T)^2 + (4T^{1/2})^2 + 4^2} \, dT$$

$$= \int_0^t \sqrt{4T^2 + 16T + 16} \, dT$$

$$= \int_0^t \sqrt{4} \sqrt{T^2 + 4T + 4} \, dT$$

$$= \int_0^t \sqrt{4} \sqrt{(T+2)^2} \, dT$$

$$= \int_0^t 2(T+2) \, dT$$

$$= T^2 + 4T \Big|_0^t = t^2 + 4t$$

$s = t^2 + 4t$

2) Invert and find $t = f(s)$ (solve for t)

$t \geq 0$

$$t^2 + 4t - s = 0$$

$$t = \frac{-4 \pm \sqrt{16 + 4s}}{2} = -2 + \sqrt{4+s}$$

e.g. $s = 5$ $\quad t = -2 + \sqrt{9}$
$= 1$

3) Plug in for t

$$\vec{r}(s) = \left\langle \left(-2+\sqrt{4+s}\right)^2, \frac{8}{3}\left(-2+\sqrt{4+s}\right)^{3/2}, 4\left(-2+\sqrt{4+s}\right) \right\rangle$$

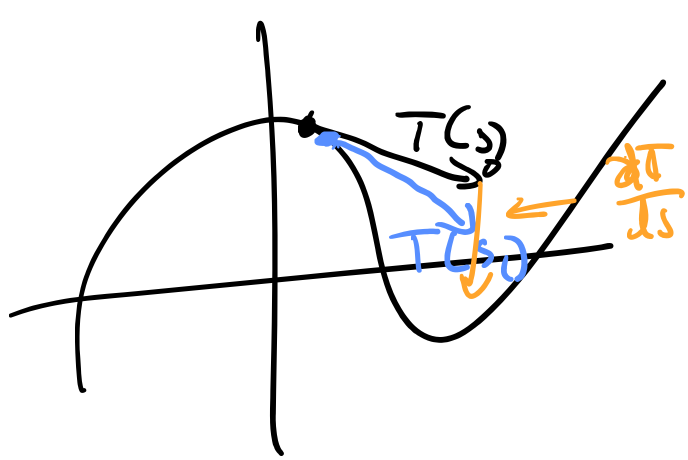
Curvature: measure the rate at which the curve is turning

- **unit tangent vector**: $\vec{T}(s) = \vec{r}\,'(s)$

$$\left(\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}\right)$$

curvature: $$\left|\frac{d\vec{T}}{ds}\right| = \kappa$$

$$= |\vec{r}\,''(s)|$$



ex: - circle of radius 4 centered at $\vec{O}$ in $\mathbb{R}^2$

$$\vec{r}(s) = \langle 4\cos(\tfrac{s}{4}), 4\sin(\tfrac{s}{4})\rangle, \quad 0 \le s \le 8\pi$$

Find $\vec{T}$, κ.

$$\vec{T}(s) = \vec{r}\,'(s) = \langle -4\sin(\tfrac{s}{4})\cdot\tfrac{1}{4}, 4\cos(\tfrac{s}{4})\cdot\tfrac{1}{4}\rangle$$
$$= \langle -\sin(\tfrac{s}{4}), \cos(\tfrac{s}{4})\rangle$$

Check: $|\vec{T}(s)| = \sqrt{\sin^2(\tfrac{s}{4}) + \cos^2(\tfrac{s}{4})}$
$$= \sqrt{1} = 1$$

$$\kappa = \left|\frac{dT}{ds}\right| = \left|\langle -\tfrac{1}{4}\cos(\tfrac{s}{4}), -\tfrac{1}{4}\sin(\tfrac{s}{4})\rangle\right|$$
$$= \left|-\tfrac{1}{4}\right| \left|\langle\cos(\tfrac{s}{4}), \sin(\tfrac{s}{4})\rangle\right|$$
$$= \frac{1}{4}$$

For a circle: $\kappa = \dfrac{1}{r}$

Q: which direction is $\vec{T}$ changing in?

$$\vec{N}(s) = \frac{dT}{ds}\cdot\frac{1}{\kappa}$$

principal unit normal

1) $\vec{N} \cdot \vec{T} = 0$ 2) $\vec{N}$ points in direction of turn

- Arclength parameterization is hard
- If we have $\vec{r}(t)$ <u>any</u> parameterization

1) $\vec{T}(t) = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$ 2) $\vec{N}(t) = \dfrac{\vec{T}'(t)}{|T'(t)|}$

3) $\kappa(t) = \dfrac{|T'(t)|}{|\vec{r}'(t)|}$ by Chain Rule / FTC

$$= \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$$

<u>ex</u>: Find $\vec{T}, \vec{N}, \kappa$ for the helix $\vec{r}(t)$

$\vec{r}(t) = \langle 2\cos(t), 2\sin(t), t-1 \rangle$

$\vec{r}'(t) = \langle -2\sin(t), 2\cos(t), 1 \rangle$

$|\vec{r}'(t)| = \sqrt{4\sin^2(t) + 4\cos^2(t) + 1} = \sqrt{4+1} = \sqrt{5}$

$\vec{T}(t) = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|} = \left\langle -\dfrac{2}{\sqrt{5}}\sin(t), \dfrac{2}{\sqrt{5}}\cos(t), \dfrac{1}{\sqrt{5}} \right\rangle$

$\vec{T}'(t) = \left\langle -\dfrac{2}{\sqrt{5}}\cos(t), -\dfrac{2}{\sqrt{5}}\sin(t), 0 \right\rangle$

$$|\vec{T}'(t)| = \left|\frac{-2}{\sqrt{5}}\right| \, |\langle \cos(t), \sin(t), 0\rangle| = \frac{2}{\sqrt{5}}$$

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = \langle -\cos(t), -\sin(t), 0\rangle$$

$$\kappa = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{2/\sqrt{5}}{\sqrt{5}} = \frac{2}{5}$$

MATH 2551 L - 9/13 - Review for Exam 1

Topics: 12.1–12.6, 13.1–13.4

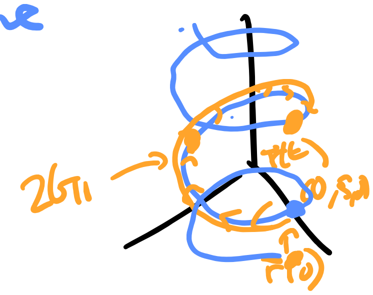
Exam 1:
- Tomorrow, W 9/14 in studio
- bring pencils/Buzzcard
- 50 minutes, 5 problems

5. Let $\mathbf{r}(t) = \langle 6\sin 2t, 6\cos 2t, 5t\rangle$. Find the unit tangent vector of $\mathbf{r}(t)$ and find the length of the portion of the graph of $\mathbf{r}(t)$ where $0 \le t \le \pi$.

1) Find $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$ ← a) Find $\vec{r}\,'(t)$ b) Divide by $|\vec{r}\,'(t)|$

2) Find the length of the portion of the curve

$$L = \int_0^{\pi} |\vec{r}\,'(t)|\, dt$$



6. Find the point on the curve

$$\mathbf{r}(t) = (5\sin t)\mathbf{i} + (5\cos t)\mathbf{j} + 12t\mathbf{k}$$

at a distance 26π units along the curve from the point $(0, 5, 0)$ in the direction of increasing arc length.

1) Use $s(t) = \int_{t_0}^{t} |\vec{r}\,'(\tau)|\, d\tau$ ← unknown point

t_0 ← reference pt (given)

a) Find t_0: $\vec{r}(t_0) = \langle 0,5,0\rangle$

$\langle 5\sin t_0, 5\cos t_0, 12t_0\rangle = \langle 0,5,0\rangle$

$12t_0 = 0 \Rightarrow \underline{t_0 = 0}$

b) Find $|\vec{r}\,'(t)|$

$\vec{r}'(t) = \langle 5\cos(t), -5\sin(t), 12 \rangle$

$|\vec{r}'(t)| = \sqrt{(5\cos(t))^2 + (-5\sin(t))^2 + 12^2}$

$= \sqrt{25(\underbrace{\cancel{\cos^2(t) + \sin^2(t)}}_{1}) + 144}$
$= 13$

c) Do the integral

$s = \int_0^t 13\, dT = 13T \Big|_0^t = 13t - 0 = 13t$

d₁) Solve for t: $t = \frac{s}{13}$

$\vec{r}(s) = \langle 5\sin(\frac{s}{13}), 5\cos(\frac{s}{13}), \frac{12}{13}s \rangle$

Plug in $s = 26\pi$

$\hookrightarrow \vec{r}(26\pi) = \langle 5\sin(2\pi), 5\cos(2\pi), 24\pi \rangle$

$= \langle 0, 5, 24\pi \rangle$

d₂) Plug in $s = 26\pi$:

$26\pi = 13t \Rightarrow t = 2\pi$

Plug in!

7. Suppose an object's position is given by $\mathbf{r}(t) = (2\ln(t+1))\mathbf{i} + (e^{2t} + t)\mathbf{j} + (\sin^2(t))\mathbf{k}$. Set up but do not evaluate the appropriate integral with limits to find the distance the object traveled from the point $A(0, 1, 0)$ to the point $B(\ln 4, e^2 + 1, \sin^2(1))$.

find t values: A: $(0,1,0) = \langle 2\ln(t+1), e^{2t} + t, \sin^2(t) \rangle$

$0 = 2\ln(t+1)$ $\qquad e^0 + 0 = 1 \quad 0^2 = 0$

$0 = \ln(t+1) \Rightarrow t+1 = 1$

$t = 0$

B: $(\ln 4,\ e^2+1,\ \sin^2(1))$

$t=1 \Rightarrow \sin^2(1) = \sin^2(1)$

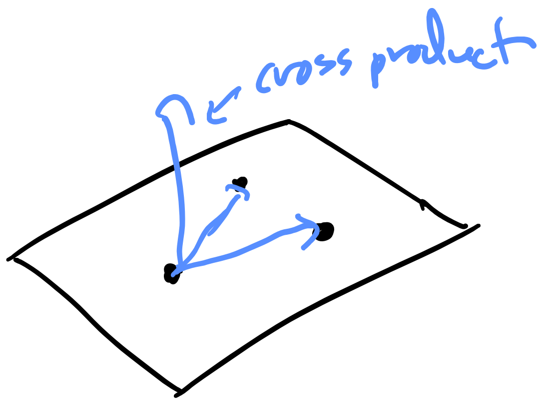
$e^{2\cdot 1}+1 = e^2+1$

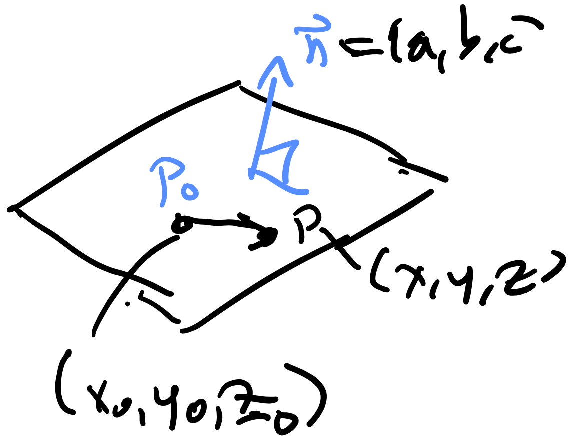
$2\ln(2) \overset{?}{=} \ln 4$

$\qquad \parallel$

$\ln(2^2) =$

Planes





$$\vec{n} \cdot \overrightarrow{P_0P} = 0$$

$$\langle a, b, c\rangle \cdot \langle x-x_0, y-y_0, z-z_0\rangle = 0$$

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

$$ax + by + cz = d$$

$$d = \vec{n} \cdot \overrightarrow{OP_0}$$

27. ~~Find the point of intersection of the lines~~ $x = 2t + 1$, $y = 3t + 2$, $z = 4t + 3$, and $x = s + 2, y = 2s + 4, z = -4s - 1$, and then find the plane determined by these lines.

1) Find intersection point:

$x: \ 2t+1 = s+2$

$y: \ 3t+2 = 2s+4$

$z: \ 4t+3 = -4s-1$

$s = -1$

$s = 2t-1$

$3t+2 = 2(2t-1)+4$

$3t+2 = 4t+2$

$t = 0$

$1 = 1$ ✓

$2 = 2$ ✓

$3 = 3$ ✓

2) Find $\vec{n}$

$$\vec{n} = \langle 2,3,4\rangle \times \langle 1,2,-4\rangle = \begin{vmatrix} i & j & k \\ 2 & 3 & 4 \\ 1 & 2 & -4 \end{vmatrix} = \langle -20, 12, 1\rangle$$

$-20(x-1) + 12(y-2) + 1(z-3) = 0$

$-20(x-2) \ 12(y-4) + 1(z+1) = 0$

$-20x + 12y + z = d$

MATH 2551 L - 9/15 - Section 14.1

Topics: Functions of Multiple Variables

- examples
- domains
- level curves/contours
- graphs
- traces

Def: $w = f(x_1, x_2, x_3, \dots, x_n)$ is a function of multiple variables

Usually: $z = f(x,y)$ $\quad w = f(x,y,z)$

$z = 2x + 3y$ $\quad w = \sqrt{x^2+y^2+z^2}$ $\quad V = \pi r^2 h$

$z = \sqrt{x + 3\cos(\ln y)}$

Domain: set of all points $(x_1, x_2, \dots, x_n)$ we can input to f
- exclude 1/0, complex #s

Range: set of all outputs of f
- in $\mathbb{R}$

ex: Find domain & range

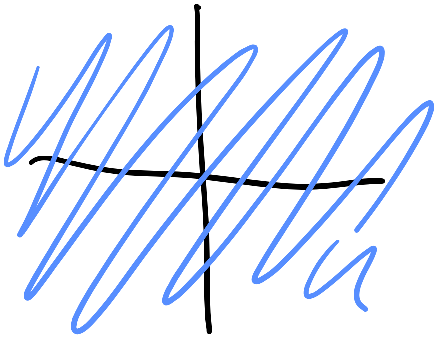
$f(x,y) = 2x + y$

Domain: All of $\mathbb{R}^2$

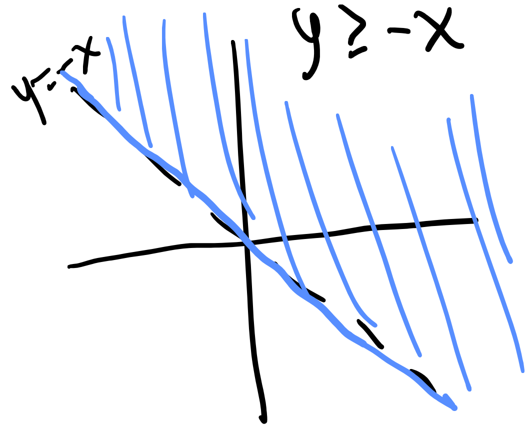
$f(x,y) = \sqrt{x+y}$

Domain: $x+y \geq 0$

$\{(x,y) \mid x+y \geq 0\}$



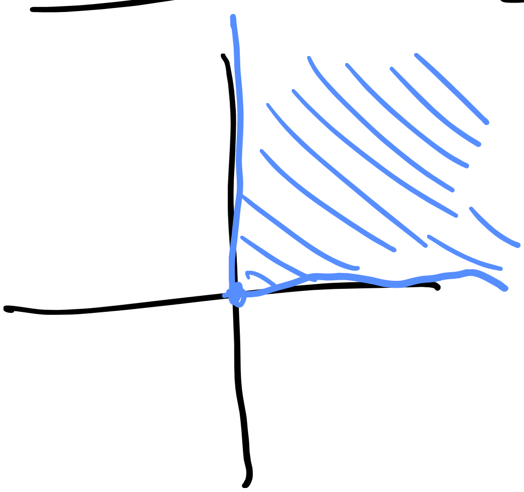
<u>Range</u>: $\mathbb{R}$ or $(-\infty,\infty)$



<u>Range</u>: $[0,\infty)$

$$f(x,y)=\sqrt{x}+\sqrt{y}$$

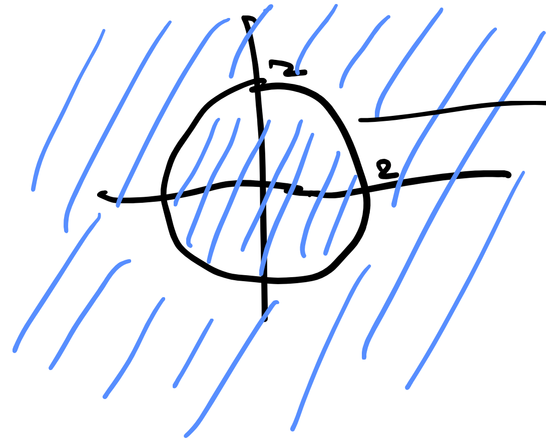
<u>Domain</u>: $x\geq 0$ <u>and</u> $y\geq 0$



<u>Range</u>: $[0,\infty)$

$$f(x,y)=\frac{1}{4-x^2-y^2}$$

<u>Domain</u>: All (x,y) except $4-x^2-y^2=0$



exclude circle $x^2+y^2=4$

<u>Range</u>: $(-\infty,0)\cup[\frac{1}{4},\infty)$

when is $z=-2$

$$-2=\frac{1}{4-x^2-y^2}$$

$$-8+2x^2+2y^2=1$$

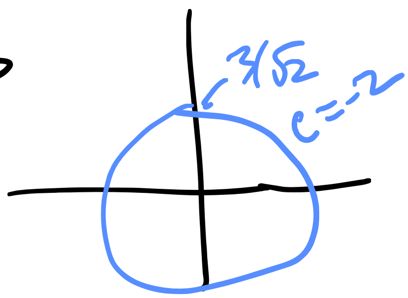
$$x^2+y^2=9/2$$

<u>Level Curves/Contours</u>

A <u>level curve</u> of $f(x,y)$ is the set of points in $\mathbb{R}^2$

where $f(x,y)=c$.

<u>ex</u>: Work to the right show that $f(x,y)=-2$ for $f(x,y)=\frac{1}{4-x^2-y^2}$ is



when is $z=-c$ $(c>0)$

$$-c=\frac{1}{4-x^2-y^2}$$

$$-4c+cx^2+cy^2=1$$

$$x^2+y^2=\frac{1}{c}+4$$

when is $z=\frac{1}{8}$

There is no contour $c=\frac{1}{8}$

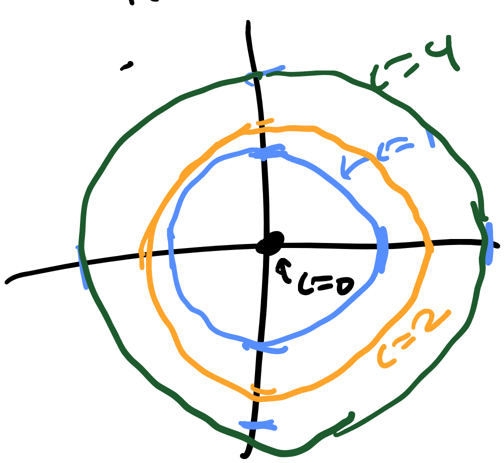
$$\frac{1}{8}=\frac{1}{4-x^2-y^2}$$

$$8=4-x^2-y^2$$

$$4=-x^2-y^2 \quad ???$$

<u>ex</u>: $z=f(x,y)=x^2+y^2$

level curves



contour map

$c=0$: $0=x^2+y^2$

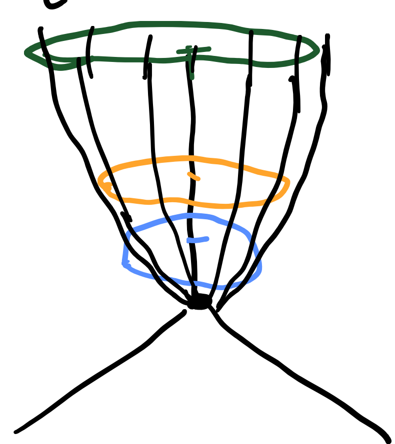
$x=y=0$

$c=1$ $1=x^2+y^2$

$c=2$: $2=x^2+y^2$

$c=4$: $4=x^2+y^2$

contours



A <u>contour</u> is the intersection of $z=f(x,y)$ and the plane $z=c$ in $\mathbb{R}^3$

Level surfaces: points in $\mathbb{R}^3$ where $f(x,y,z)=c$

ex: level surfaces of $f(x,y,z)=\sqrt{x^2+y^2+z^2}$ are spheres of radius c b/c $c=\sqrt{x^2+y^2+z^2}$

Trace: A trace of $f(x,y)$ is the intersection of $x=c$ or $y=c$ with $z=f(x,y)$ in $\mathbb{R}^3$

ex: $f(x,y)=x^2+y^2=z$

The traces $x=c$ are: $z=c^2+y^2$ (parabola in $x=c$ plane)

The traces $y=c$ are: $z=x^2+c^2$ (parabola in $y=c$ plane)

Poll 2

$$2=e^{\sin(x^2+y^2)}$$

$$\ln 2=\sin(x^2+y^2)$$

$$\arcsin(\ln 2)+2\pi k=x^2+y^2 \quad \Big\}\ \text{circle}$$

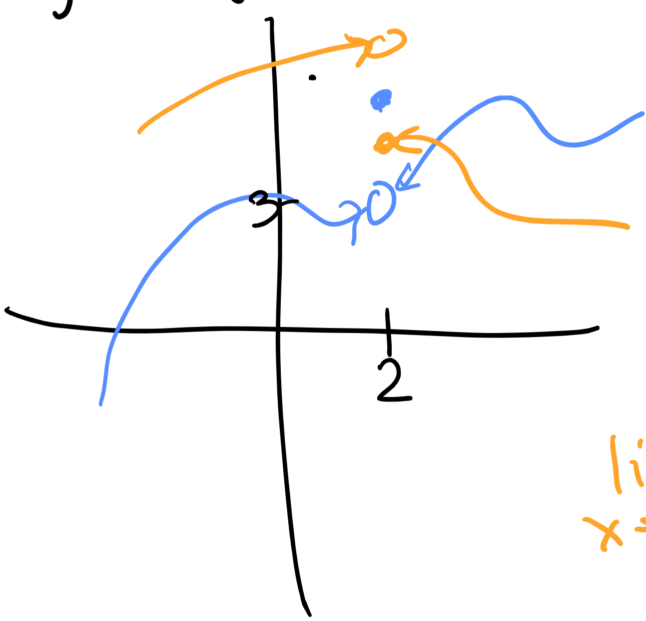
!!

MATH 2551 L - 9/20 - 14.2

Today: Limits & continuity for functions of multiple variables

Warmup Poll: $z = f(x,y) \Rightarrow 0 = \underbrace{f(x,y) - z}_{0 = g(x,y,z)}$

Recall: $\lim_{x \to a} f(x) = L$ if we can make $f(x)$ as close as we like to L by ~~~~ taking x close to a



$\lim_{x \to 2} f(x) = 3$

f continuous at $x=2$?

No, b/c $f(2) \neq 3$

$\lim_{x \to 2} g(x)$ DNE b/c left and right limits don't agree

Def: $f(x,y)$ is continuous at (a,b) if

1) $f(a,b)$ exists
2) $\lim_{(x,y) \to (a,b)} f(x,y)$ exists
3) $f(a,b) = \lim_{(x,y) \to (a,b)} f(x,y)$

- f is continuous if it is continuous everywhere in its domain

- $f(x,y) = x$ is cts
  $g(x,y) = 5x + 2y$

$h(x,y) = \frac{1}{x}$ is cts away from $x=0$

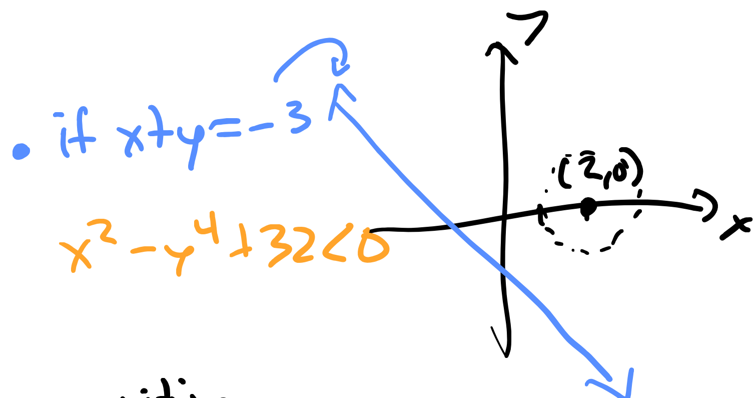
e^{x+y} is cts

- Add, subtract, multiply, divide, compose two cts functions $\to$ cts function

$\underline{Q}$: Is $f(x,y) = \cos(3x) + \sin(3y)$ cts?

Limits of $f(x,y)$

$\underline{ex}$: $\displaystyle\lim_{(x,y)\to(2,0)} \frac{\sqrt{x^2-y^4+32}}{x+y+3}$

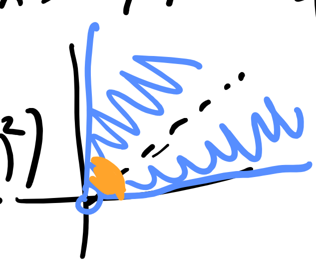


$(2,0)$ is in the domain & this is a composition of cts functions away from $x+y=-3$, so the limit is

$$f(2,0) = \frac{\sqrt{2^2-0^4+32}}{2+0+3} = \frac{6}{5}$$

$\underline{ex}$: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}}$ $\quad$ $\underline{\text{Domain}}$: $x \geq 0$, $y \geq 0$ & $x \neq y$

$$\frac{x^2-xy}{\sqrt{x}-\sqrt{y}} = \frac{x(x-y)}{\sqrt{x}-\sqrt{y}} = \frac{x((\sqrt{x})^2-(\sqrt{y})^2)}{\sqrt{x}-\sqrt{y}}$$



$\dfrac{x^2-xy}{\sqrt{x}-\sqrt{y}}$ is not cts at $(0,0)$

or multiply by $\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}}$

$$= \frac{x(\sqrt{x}+\sqrt{y})\cancel{(\sqrt{x}-\sqrt{y})}}{\cancel{(\sqrt{x}-\sqrt{y})}}$$

$$= x(\sqrt{x}+\sqrt{y}) \leftarrow \text{cts at } (0,0)$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} = \lim_{(x,y)\to(0,0)} x(\sqrt{x}+\sqrt{y})$$

$$= 0(\sqrt{0}+\sqrt{0}) = 0$$

but

$$f(x,y) = \begin{cases} \dfrac{x^2-xy}{\sqrt{x}-\sqrt{y}} & \text{if } (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is cts at $(0,0)$

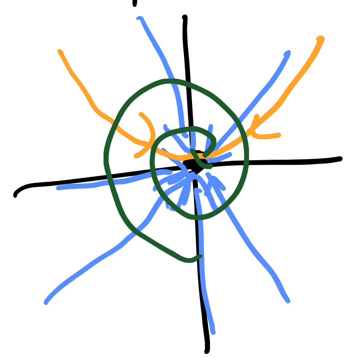
ex: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{-2x}{\sqrt{x^2+y^2}}$ Domain: $\mathbb{R}^2$ except $(0,0)$

Two-Path Test: If we can find two paths to (a,b) on which a limit differs, the limit DNE.

1) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{-2x}{\sqrt{x^2+y^2}} = \lim_{(0,y)\to(0,0)} \frac{-2\cdot 0}{\sqrt{0+y^2}}$

along y-axis $(0,y)$

$$= \lim_{(0,y)\to(0,0)} 0 = 0$$



2) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{-2x}{\sqrt{x^2+y^2}} = \lim_{(x,0)\to(0,0)} \frac{-2x}{\sqrt{x^2+0}} = \lim_{(x,0)\to(0,0)} \frac{-2x}{|x|}$

along x-axis $(x,0)$

$$= \lim_{(x,0)\to(0,0)} \begin{cases} -2, & x>0 \\ 2, & x<0 \end{cases}$$

B/c we found two paths to $(0,0)$ where the limits disagreed,

$$\lim_{(x,y)\to(0,0)} \frac{-2x}{\sqrt{x^2+y^2}} \text{ DNE}$$

$$f(x,y) = \begin{cases} \dfrac{-2x}{\sqrt{x^2+y^2}} & \text{if } (x,y)\neq(0,0) \\ c & (x,y)=(0,0) \end{cases}$$

is never cts at $(0,0)$

ex: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$ along any line $y=mx$

1) $$\lim_{(x,mx)\to(0,0)} \frac{x^2(mx)}{x^4+(mx)^2} = \lim_{(x,mx)\to(0,0)} \frac{mx^3}{x^4+m^2x^2}$$

$$= \lim_{x\to 0} \frac{\cancel{x^2}(mx)}{\cancel{x^2}(m^2+x^2)}$$

$$= \frac{0}{m^2} = 0$$

2) Along $y=x^2$:

$$\lim_{(x,x^2)\to(0,0)} \frac{x^2(x^2)}{x^4+(x^2)^2} = \lim_{x\to 0} \frac{x^4}{x^4+x^4} = \frac{1}{2}$$

So $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$ DNE

MATH 2551 L - 9/22 - 14.3

Today: Partial Derivatives

- motivation
- geometrically
- algebraically

<u>Review</u>: $\lim_{(x,y)\to(1,1)} \frac{x^2-2xy+y^2}{x-y} = \frac{1-2+1}{1-1} = \frac{0}{0}$

$= \lim_{(x,y)\to(1,1)} \frac{(x-y)^2}{(x-y)}$

$= \lim_{(x,y)\to(1,1)} x-y = 1-1 = 0$

<u>Goal</u>: Describe how a function of mult. vars is changing at (a,b).

<u>ex</u>: Consider $f(x,y) = \frac{x^2 \sin(2y)}{9.8}$, the range in m of a projectile launched with speed x m/s at angle

of y rad. $\quad f(45, 0.6) = 142.6$ m

- If we fix angle of fire ($y = 0.6$ rad), what is the rate of change of the range as speed changes?

$$f(x, 0.6) = \frac{\sin(1.2)}{9.8} x^2$$ function of only x!

(trace with $y = 0.6$)

rate of change: $$\frac{d}{dx}\left(f(x, 0.6)\right) = \frac{\sin(1.2)}{4.9} x$$

At $(45, 0.6)$, this is ≈ 8.5 m/m/s

This is the partial derivative of f w.r.t. x, f_x

$$f_x(45, 0.6) = \frac{d}{dx}\left(f(x, 0.6)\right)\Big|_{x=45}$$

In general: $$f_x(a,b) = \frac{d}{dx}\left(f(x, b)\right)\Big|_{x=a}$$

$$= \lim_{h \to 0} \frac{f(a+h, b) - f(a,b)}{h}$$

The partial derivative of f w.r.t y, f_y

$$f_y(45, 0.6) = \frac{d}{dy}\left(f(45, y)\right)\Big|_{y=0.6}$$

$$= \frac{d}{dy}\left(\frac{45^2}{9.8} \sin(2y) \right) \Big|_{y=0.6}$$

$$= \frac{45^2}{4.9} \cos(2y) \Big|_{y=0.6}$$

$$\approx 190.9 \text{ m/rad}$$

<u>In general</u> $f_y(a,b) = \frac{d}{dy}\left(f(a,y) \right)\Big|_{y=b}$

<u>Notation</u>: $f_x = \frac{\partial f}{\partial x}$ $\quad f_y = \frac{\partial f}{\partial y}$

- $f_x(a,b)$ is the rate of change of f in x-direction at (a,b)
- $f_y(a,b)$ is the rate of change of f in y-direction at (a,b)

<u>ex</u>: $f(x,y) = 2y^2 - 4x^2$. Find $f_x(1,0)$, $f_y(1,0)$.

$$f_x(x,y) = \frac{\partial}{\partial x}(2y^2) - \frac{\partial}{\partial x}(4x^2)$$

treat y as a constant b/c y does not depend on x

$$= 0 - 8x$$

$$f_x(1,0) = -8$$

$$f_y(x,y) = \frac{\partial}{\partial y}(2y^2) - \frac{\partial}{\partial y}(4x^2)$$

treat x as a constant

$$= 4y - 0$$

$$f_y(1,0) = 0$$

ex Compute $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ for $f(x,y) = \frac{x}{x+y^2}$

$$\frac{\partial f}{\partial x} = \frac{\frac{\partial}{\partial x}(x)\cdot(x+y^2) - x\cdot\frac{\partial}{\partial x}(x+y^2)}{(x+y^2)^2}$$

$$= \frac{1(x+y^2) - x(1+0)}{(x+y^2)^2}$$

$$= \frac{y^2}{(x+y^2)^2}$$

$$\frac{x}{x+y^2} = \underline{x(x+y^2)^{-1}}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x(x+y^2)^{-1}\right) = x\frac{\partial}{\partial y}\left((x+y^2)^{-1}\right)$$

$$= x\left(-(x+y^2)^{-2}\underbrace{\frac{\partial}{\partial y}(x+y^2)}_{0+2y}\right)$$

$$= \frac{-2xy}{(x+y^2)^2}$$

Can compute multiple iterations

2nd order

$$(f_x)_x \quad (f_x)_y \quad (f_y)_x \quad (f_y)_y$$

1st 2nd

mixed: $(f_x)_y$, $(f_y)_x$

pure: $(f_x)_x$, $(f_y)_y$

$$(f_x)_x = f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$(f_x)_y = f_{xy} = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$(f_y)_x = f_{yx} = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$(f_y)_y = f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

- both inside → outside

ex: Find all 2^{nd} order partial derivatives for

$$g(x,y) = x^2y + \cos(y) + 2y\sin(x)$$

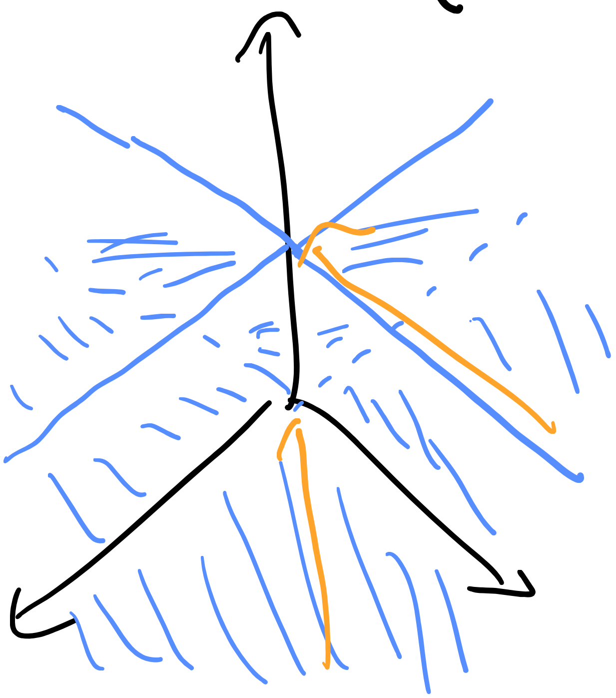
$$g_x = 2xy + 0 + 2y\cos(x)$$

$$g_y = x^2 \quad - \sin(y) + 2\sin(x)$$

$$g_{xx} = 2y - 2y\sin(x) \qquad g_{xy} = 2x + 2\cos(x)$$

$$g_{yx} = 2x + 0 + 2\cos(x) \qquad g_{yy} = 0 - \cos(y) + 0$$

- mixed partials are equal if f and all of these partials are continuous ex: $f(x,y) = \begin{cases} 0, & \text{if } xy \neq 0 \\ 1, & \text{if } xy = 0 \end{cases}$



- $f_x(0,0) = f_y(0,0) = 0$
- Is f is cts at $(0,0)$?

No, b/c

$\lim_{(x,y)\to(0,0)} f(x,y)$ DNE

MATH 2551 L – 9/27 – 14.6 (roughly)

Topics:
- tangent planes
- linearization
- differentiability & the total derivative

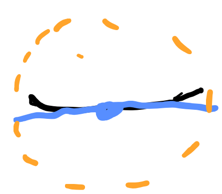
Recap: $f(x,y,z) = x^2 z + e^{yz} + z + x$

$$f_z = x^2 + e^{yz} \cdot \underbrace{\frac{\partial}{\partial z}(yz)}_{y} + 1 + 0$$

Last time: $f(x,y) = \begin{cases} 0, & \text{if } xy \neq 0 \\ 1, & \text{if } xy = 0 \end{cases}$

$0 = \frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0)$ but is not nice

Q: What is a differentiable function of multiple variables?



Tangent planes

ex: $f(x,y) = 2y^2 - 4x^2$

$f_x(x,y) = -8x$

$f_y(x,y) = 4y$

Goal: Find a plane tangent to the graph of f at $(1,1)$.

tell us slopes of tangent lines to traces of f

At $(1,1)$: $f(1,1) = -2$, $f_x(1,1) = -8$

$f_y(1,1) = 4$

Find eqns of these tangent lines:

1) line tangent to the trace $y=1$ at $(1,1)$

point: $(1,1,f(1,1)) = (1,1,-2)$

direction: $\langle 1,0,-8\rangle = \langle 1,0,f_x(1,1)\rangle$

$(1,1,-2) + t\langle 1,0,-8\rangle$

2) line tangent to the trace $x=1$ at $(1,1)$

point: $(1,1,-2)$

direction: $\langle 0,1,4\rangle = \langle 0,1,f_y(1,1)\rangle$

$(1,1,-2) + s\langle 0,1,4\rangle$

Find plane containing both lines:

$$\vec{n} = \begin{vmatrix} i & j & k \\ 1 & 0 & -8 \\ 0 & 1 & 4 \end{vmatrix} = \langle 8, -4, 1 \rangle \qquad P_0 = (1, 1, -2)$$

tangent plane is: $8(x-1) - 4(y-1) + 1\cdot(z+2) = 0$

solving for z: $z = -2 - 8(x-1) + 4(y-1)$

$f(1,1)$ $\quad f_x(1,1)$ $\quad (x-a)$ $\quad f_y(1,1)$ $\quad (y-b)$

For any $z=f(x,y)$, (a,b), the tangent plane to f at (a,b) is

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

We can use the equation of tangent plane to linearly approx. f near (a,b).

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

linearization of f near (a,b)

e.g. Approximate $f(x,y) = 2y^2 - 4x^2$ at $(1.1, 1.1)$ using linearization at $(1,1)$.

$f(1.1,1.1) \approx L(1.1,1.1)$ $\quad f(1.1,1.1) = -2.42$

$= -2 - \underbrace{8(1.1-1) + 4(1.1-1)}$

$= -2.4$

- <u>Differentials</u> (measure Δf)

$\Delta f = L(a,b) - f(a,b)$

$df = f_x(a,b)\,dx + f_y(a,b)\,dy$

ex: If we machine rectangles $x = 20$ cm by 10 cm $= y$ but can have errors of up to $\Delta x = .2$ cm, $\Delta y = .4$ cm. What is the max. error in area from a perfect 200 cm^2 rectangle?

$A(x,y) = xy$ at $(20,10)$

$A_x(20,10) = y\big|_{(20,10)} = 10$ $\quad A_y(20,10) = x\big|_{(20,10)} = 20$

$dA = 10\,dx + 20\,dy$ so $\Delta A \approx 10(.2) + 20(.4)$

$= 10\text{ cm}^2$

So the max. error in area is 10 cm^2

(areas between 190 & 210)

Def: A function of multiple variables is differentiable at $(a_1, a_2, \dots, a_n)$ if the linearization at the point is a good approx. of the function.

Thm: If f and its partial derivatives are continuous near $(a_1, \dots, a_n)$ then f is differentiable.

Total derivative

In 1-var: $f : \mathbb{R} \to \mathbb{R}$

linearization: $f(a) + \underbrace{f'(a)}_{\text{derivative of } f \text{ at } a} (x-a)$

derivative of f at a is the linear map that best approx. f near $(a, f(a))$

for multiple-variable:

$f : \mathbb{R}^n \to \mathbb{R}$

matrix vector

linearization: $f(\vec{a}) + \underbrace{Df(\vec{a})}_{\text{total derivative}} \cdot (\vec{x} - \vec{a})$

total derivative

linear map $\mathbb{R}^n \to \mathbb{R}$

ex f	$\mathbb{R}^n \to \mathbb{R}^m$	$\vec{a}$	Df (general)	$Df(\vec{a})$
$f(x) = x^2$	$\mathbb{R} \to \mathbb{R}$	2	$[f'(a)]$	$[4]$ $(2x\|_2)$
$\vec{r}(t) =$ $\langle \cos(t), \sin(t), 3t \rangle$	$\mathbb{R} \to \mathbb{R}^3$	π	$\begin{matrix}\to \\ \to \\ \to\end{matrix}\begin{bmatrix} x'(a) \\ y'(a) \\ z'(a) \end{bmatrix} = \vec{r}\,'(a)$ 1 row/component	$\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$
$f(x,y,z) = x^2yz$	$\mathbb{R}^3 \to \mathbb{R}$	$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ or $(1,2,3)$	$\begin{array}{ccc} 2xyz & x^2z & x^2y \end{array}$ $\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix}$ 1 col/variable	$\begin{bmatrix} 12 & 3 & 2 \end{bmatrix}$
$\vec{f}(s,t) = \langle \underset{f_1}{s+t}, \underset{f_2}{2t}, \underset{f_3}{3s} \rangle$	$\mathbb{R}^2 \to \mathbb{R}^3$	$\begin{array}{c} s \\ t \end{array}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$		

$$D\vec{f} = \begin{array}{c} \begin{array}{cc} s & t \end{array} \\ \begin{bmatrix} \frac{\partial f_1}{\partial s} \; 1 & \frac{\partial f_1}{\partial t} \; 1 \\ \frac{\partial f_2}{\partial s} \; 0 & \frac{\partial f_2}{\partial t} \; 2 \\ \frac{\partial f_3}{\partial s} \; 3 & \frac{\partial f_3}{\partial t} \; 0 \end{bmatrix} \end{array} \begin{array}{l} 1^{st} \text{ comp.} \\ 2^{nd} \text{ comp.} \\ 3^{rd} \text{ comp} \end{array} \qquad \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 3 & 0 \end{bmatrix}$$

MATH 2551 L - 9/29 - 14.4 & 14.5

Today: • Multivariable Chain Rule
• Directional Derivatives
• Gradient

Last time: $f: \mathbb{R}^n \to \mathbb{R}^m$ $(f(x_1, \dots, x_n) = \langle f_1(\vec{x}), f_2(\vec{x}), \dots, f_m(\vec{x}) \rangle$
then Df is the $m \times n$ matrix w/ 1 row per component of f and 1 col per variable whose entries are (partial) derivatives

From Calc I: If $h(x) = g(f(x))$, then
$$\to h'(x) = g'(f(x)) \cdot f'(x)$$

Multivariable: $f: \mathbb{R}^n \to \mathbb{R}^p$, $g: \mathbb{R}^p \to \mathbb{R}^m$
(n inputs, p outputs) (p inputs, m outputs)
and $h = g \circ f : \mathbb{R}^n \to \mathbb{R}^m$

Then $Dh(\vec{x}) = Dg(f(\vec{x}))\, Df(\vec{x})$

ex: Suppose we travel along the curve $\vec{f}(t) = \langle 2 - t^2, t^3 + 1 \rangle$ in $\mathbb{R}^2$. The altitude at (x, y) is $g(x, y) = 10 - \frac{1}{2}x^2 - \frac{1}{5}y^2$.

Then our altitude at time t is $h(t) = g(\vec{f}(t))$
$= g(x(t), y(t))$

What is $Dh(t) = [h'(t)]$?

- $Dg(\vec{f}(t)) = \left[\frac{\partial g}{\partial x} \quad \frac{\partial g}{\partial y}\right]\Big|_{\vec{f}(t)}$

$$= \left[-x \quad -\frac{2}{5}y\right]\Big|_{\vec{f}(t)}$$

$$= \left[-(2-t^2) \quad -\frac{2}{5}(t^3+1)\right]$$

- $D\vec{f}(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -2t \\ 3t^2 \end{bmatrix}$

$$\underset{(1\times 1)}{Dh(t)} = Dg(\vec{f}(t))\,D\vec{f}(t) = \underset{(1\times 2)}{\left[-(2-t^2) \quad -\frac{2}{5}(t^3+1)\right]}\underset{(2\times 1)}{\begin{bmatrix} -2t \\ 3t^2 \end{bmatrix}}$$

$$= \boxed{2t(2-t^2) - \frac{6}{5}t^2(t^3+1)}$$

$$\frac{dh}{dt} = \frac{\partial g}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial g}{\partial y}\cdot\frac{dy}{dt}$$

ex: Suppose $f(s,t) = \langle t-s^2, ts^2 \rangle$ gives position of a hiker depending on skills & time t.

At $(x,y) = (1,2)$, we measure $\frac{\partial g}{\partial x} = 10$

$\frac{\partial g}{\partial y} = -2$, $g(x,y)$ is altitude at (x,y)

If $h(s,t) = g(f(s,t))$ gives out altitude,

find $\frac{\partial h}{\partial s}(\underset{s,\ t}{1,2})$ and $\frac{\partial h}{\partial t}(\underset{s,\ t}{1,2})$.

$f(\underset{s,\ t}{1,2}) = \langle 2-1^2, 2\cdot 1^2 \rangle = \langle \underset{x,\ y}{1,2} \rangle$

$$\left[\frac{\partial h}{\partial s}(1,2) \quad \frac{\partial h}{\partial t}(1,2)\right] = Dh(1,2) = \underbrace{Dg(f(1,2))}\ \underbrace{Df(1,2)}$$

$\cdot\ Dg(f(1,2)) = \left[\frac{\partial g}{\partial x}(f(1,2)) \quad \frac{\partial g}{\partial y}(f(1,2))\right]$

$$= \left[\frac{\partial g}{\partial x}(1,2) \quad \frac{\partial g}{\partial y}(1,2)\right]$$

$$= \left[10 \quad -2\right]$$

$f(s,t) = \langle \underset{x}{t-s^2}, \underset{y}{ts^2} \rangle$

• $Df(1,2) = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{bmatrix}\Bigg|_{(1,2)}$

$= \begin{bmatrix} -2s & 1 \\ 2st & s^2 \end{bmatrix}\Bigg|_{(1,2)} = \begin{bmatrix} -2 & 1 \\ 4 & 1 \end{bmatrix}$

$Dh(1,2) = Dg(f(1,2))\,Df(1,2)$

$= \begin{bmatrix} 10 & -2 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 4 & 1 \end{bmatrix}$ $(1\times 2)\cdot(2\times 2)$
$\to (1\times 2)$

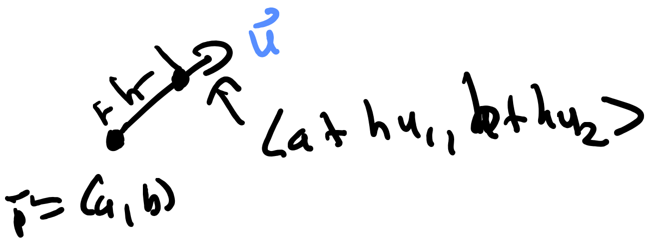
$= \begin{bmatrix} -20-8 & 10-2 \end{bmatrix} = \begin{bmatrix} -28 & 8 \end{bmatrix}$

$\frac{\partial h}{\partial s} = -28 \quad \frac{\partial h}{\partial t} = 8$

<u>Directional Derivative</u>: $\vec{u}$ a unit vector

$D_{\vec{u}} f(\vec{p}) = \lim_{h\to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$

" (a,b)



- rate of change of f at $\vec{p}$ in the direction of $\vec{u}$.

- If f is differentiable, then

  - $D_{\vec{u}} f(\vec{p}) = \underset{\text{matrix}}{Df(\vec{p})}\, \underset{\text{col vec}}{\vec{u}} = \underset{\text{vec}}{\nabla f(\vec{p})} \cdot \underset{\text{vec}}{\vec{u}}$

  - $\nabla f(\vec{p})$ is the <u>gradient of f at $\vec{p}$</u>

<u>ex</u>: If $f(x,y) = x^2 + y^2$, $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$

$$= \langle 2x, 2y \rangle$$

At $\vec{p} = \langle 1, 0\rangle$, compute $D_{\vec{u}} f(\vec{p})$ for

$\vec{u}_1 = \langle 1, 0\rangle$ $\quad \vec{u}_2 = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$, $\vec{u}_3 = \langle 0, 1\rangle$

$\vec{u}_4 = \langle -2, 0\rangle$

$\nabla f(\vec{p}) = \langle 2, 0\rangle$

$D_{\vec{u}_1} f(1,0) = \underbrace{\langle 2,0\rangle \cdot \langle 1,0\rangle}_{\nabla f(1,0)\, \cdot\, \vec{u}_1} = 2$

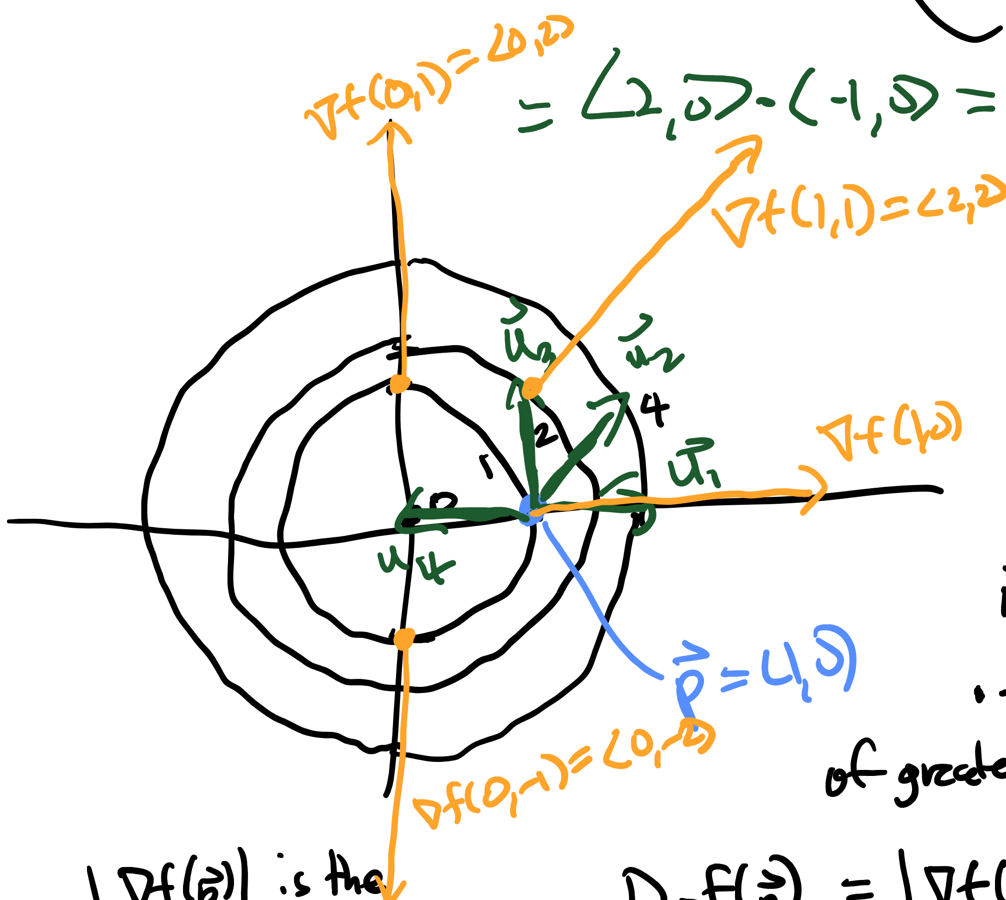
$D_{\vec{u}_2} f(1,0) = \langle 2,0\rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle = \frac{2}{\sqrt{2}} = \sqrt{2}$

$D_{\vec{u}_3} f(1,0) = \langle 2,0\rangle \cdot \langle 0,1\rangle = 0$

$\left. D_{\vec{u}_4} f(1,0) = \langle 2,0\rangle \cdot \langle -2,0\rangle = -4 \right] \times$

need unit vector

$= \langle 2,0\rangle \cdot \langle -1,0\rangle = -2$



$D_{\langle 1,0\rangle} f(\vec{p}) = f_x(\vec{p})$

- $\nabla f(\vec{p})$ is <u>always</u> the direction of greatest increase of f at $\vec{p}$.
- $-\nabla f(\vec{p})$ is the direction of greatest decrease of f at $\vec{p}$

- $|\nabla f(\vec{p})|$ is the maximum rate of change of f at p in any direction

$D_{\vec{u}} f(\vec{p}) = |\nabla f(\vec{p})| \cancel{|\vec{u}|}^{1} \cos\theta$

biggest if $\cos\theta = 1 \Rightarrow \theta = 0, \pi$

0 if $\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

- $\nabla f(\vec{p})$ is $\perp$ the level curve of f containing $\vec{p}$.

Ex: Find the tangent plane to $x^2+y^2+z^2=9$ at $(1,2,2)$.

$f(x,y,z) = \underline{x^2+y^2+z^2}$ so that ↑ is a level surface of f

↳ $f(x,y,z) = 9$

- $\nabla f(1,2,2) \perp$ level surface $f(x,y,z)=9$

(b/c $1^2+2^2+2^2=9$)

(1,2,2) is on this surface

$$\nabla f = \langle 2x, 2y, 2z \rangle \Big|_{(1,2,2)}$$

$$\vec{n} = \nabla f(1,2,2) = \langle 2,4,4 \rangle$$

So tangent plane is $2(x-1)+4(y-2)+4(z-2)=0$

- <u>normal line</u> at $\vec{p}$ is the line which is normal to the surface and goes through $\vec{p}$

e.g. $\ell(s) = \langle 1,2,2 \rangle + s\langle 2,4,4 \rangle$

for this sphere at (1,2,2)

If $\vec{r}(t)$ parameterizes a level curve of f

$$f(\vec{r}(t)) = c$$

$$\underbrace{Df(\vec{r}(t))}\, \underbrace{D\vec{r}(t)} = 0$$

$$\nabla f \cdot \vec{r}\,'(t) = 0$$

∇f is $\perp$ tangent to level curve

MATH 2551 L - 10/4 - 14.7 Optimization

Goals: - Identify critical points
- Classify critical points
- Find global extreme values on a closed & bounded domain

Recap poll

on Tuesday, $\langle f_x(a,b), f_y(a,b), -1\rangle$ is orthog. to $z=f(x,y)$ at $(a,b,f(a,b))$

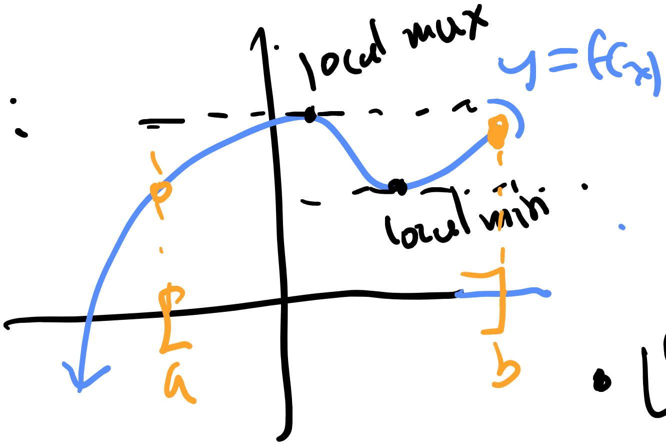
- $f_x(x,y) = 4x-3y$ at $(2,1)$:
  $f_y(x,y) = -3x$

  $f_x(2,1) = 5$
  $f_y(2,1) = -6$

$\langle 5,-6,-1\rangle$ is $\perp$ to $z=f(x,y)$ at $(2,1,2)$

$$z = 2x^2-3xy \Rightarrow 0 = \underbrace{2x^2-3xy-z}_{g(x,y,z)}$$

Recall:



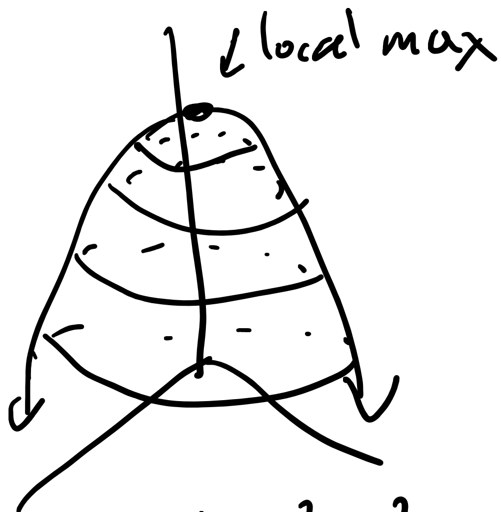
- critical pts: $f'(x)=0$ are places where tangent is horiz.
- Use 2^{nd} deriv to classify

- On $[a,b]$, compared f at crit. pts to f at endpoints

DEFINITIONS Let $f(x, y)$ be defined on a region R containing the point (a, b). Then

1. $f(a, b)$ is a **local maximum** value of f if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).
2. $f(a, b)$ is a **local minimum** value of f if $f(a, b) \leq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).

ex:



$z = 4 - x^2 - y^2$

$z = x^2 + y^2$

- $z = x^2 - y^2$ has a saddle point at $(0,0)$

in one direction a local max and in another a local min

- Tangent planes to these pts are horizontal
- At any local extrema/saddle point (a,b)

$$Df(a,b) = [0\ \ 0]\ ;\ \nabla f(a,b) = \vec{0}\ ;$$

$$f_x(a,b) = 0 = f_y(a,b)$$
(or one of these doesn't exist)

- Any pt (a,b) in the domain of f with $Df(a,b) = [0\ 0]$ or non-existent is a <u>critical point</u>

ex: ~~Find~~ crit. pts of $f(x,y) = x^2 - y^2$ & $\sqrt{x^2+y^2}$ $= f(x,y)$

$g(x,y) = x^3 + y^3 - 3xy$

f: 1) Find $Df(x,y)$. $Df(x,y) = [2x\ \ -2y]$

2) Set to $[0\ 0]$

3) Solve: $[2x\ \ -2y] = [0\ \ 0]$

$2x = 0$ & $-2y = 0$

$x = 0$ & $y = 0$

- f has only one crit. pt at $(0,0)$

g: 1) Find $Dg(x,y)$: $[3x^2 - 3y\ \ \ 3y^2 - 3x]$

2) Set to $[0\ 0]$

3) Solve $\begin{bmatrix} 3x^2-3y & 3y^2-3x \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$

$3x^2-3y=0 \qquad 3y^2-3x=0$

$x^2-y=0 \qquad y^2-x=0$

Solve for y

$y=x^2$ plugin

$(x^2)^2-x=0$

$x^4-x=0$

$x(x^3-1)=0$

$x=0$ or $x^3-1=0$

$\hookrightarrow y=0 \qquad x^3=1$

$x=1$

$\hookrightarrow y=1$

- g has 2 crit. pts, at $(0,0)$ & $(1,1)$

<u>Q</u>: How to classify? Use 2^{nd} derivative matrix

<u>Hessian</u>: $Hf(x,y) = \begin{bmatrix} f_{xx}(x,y) & f_{yx}(x,y) \\ f_{xy}(x,y) & f_{yy}(x,y) \end{bmatrix}$

2^{nd} derivative test [(a,b) crit pt of f]

- If $\det(Hf(a,b)) > 0$, $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local min
  - $\rightarrow$ f behaves same in all directions at (a,b)
- If $\det(Hf(a,b)) > 0$, $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local max
- If $\det(Hf(a,b)) < 0$, then f has saddle point at (a,b)
  - $\rightarrow$ f behaves differently in some direction
- If $\det(Hf(a,b)) = 0$, inconclusive

<u>ex</u>: Classify crit pts of $f(x,y) = x^2 - y^2$ & $g(x,y) = x^3 + y^3 - 3xy$.

f: $f_x = 2x$ $\quad f_y = -2y$ $\quad$ crit pt $(0,0)$

$$Hf = \begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

$\det(Hf(0,0)) = 2(-2) - 0 = -4$

Because $\det(Hf(0,0)) < 0$, f has a saddle point at $(0,0)$.

g: $g_x = 3x^2 - 3y \quad g_y = 3y^2 - 3x$, crit pts $(0,0), (1,1)$

$$H_g = \begin{bmatrix} g_{xx} & g_{yx} \\ g_{xy} & g_{yy} \end{bmatrix} = \begin{bmatrix} 6x & -3 \\ -3 & 6y \end{bmatrix}$$

At $(0,0)$:

$$\det(H_g(0,0)) = \det\left(\begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix}\right) = 0 \cdot 0 - (-3)(-3) = -9$$

Because $\det(H_g(0,0)) < 0$, g has a saddle point at $(0,0)$.

At $(1,1)$:

$$\det(H_g(1,1)) = \det\left(\begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}\right) = 36 - 9 = 27$$

Since $\det(H_g(1,1)) > 0$ and $g_{xx}(1,1) = 6 > 0$, g has a local min at $(1,1)$.

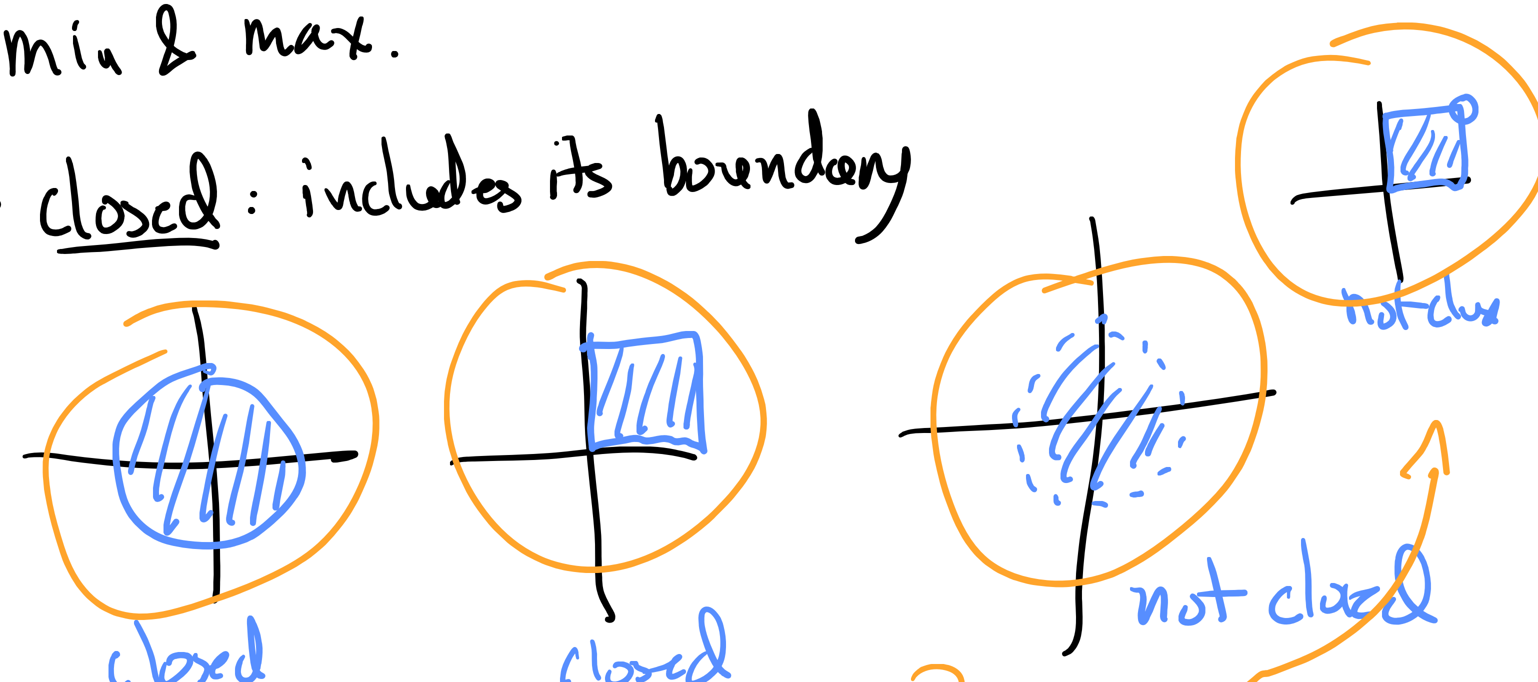
If $f_{xx}=0=f_{yy}=0$

$$Hf = \begin{bmatrix} 0 & f_{yx} \\ f_{xy} & 0 \end{bmatrix} \qquad \det(Hf) = -\underbrace{f_{xy} f_{yx}}$$

if $f_{xy}=f_{yx}$ (usually true) this <0

<u>Thm</u>: On a closed & bounded domain any continuous function $f(x,y)$ attains a global min & max.

- <u>closed</u>: includes its boundary

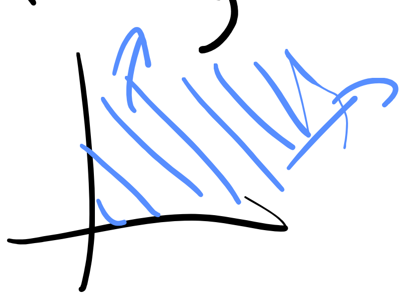


$\mathbb{R}^2$ is closed

not bounded

bounded

- <u>bounded</u>: can fit in a big enough circle

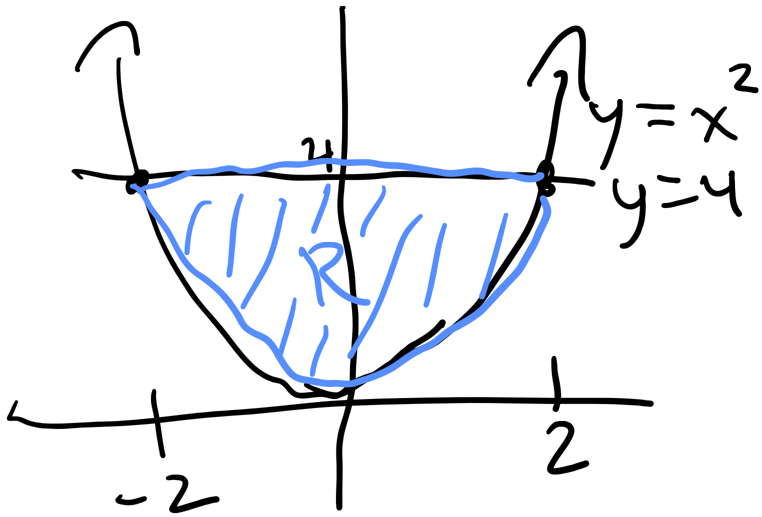


not bounded

To find the global min/max of $f(x,y)$ on a closed/bounded set R

1) Find all critical pts of f inside R
2) Find all critical pts/endpoints on boundary of R
3) Test value of f at all these points

ex) Find global min/max of $f(x,y) = 4x^2 - 4xy + 2y$ on the region bounded by $y = x^2$ & $y = 4$



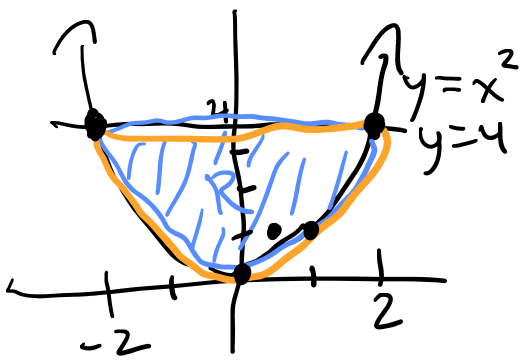
$4 = x^2 \Rightarrow x = \pm 2$

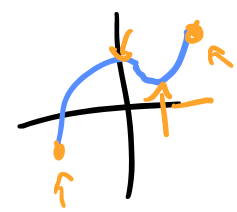
MATH 2551 L - 10/6 - 14.8 Lagrange Multipliers

• See Canvas announcement about Quiz 5

Today: – finish extreme values from Thurs.
– Lagrange multipliers

ex] Find global min/max of $f(x,y)=4x^2-4xy+2y$ on the region bounded by $y=x^2$ & $y=4$





$4=x^2 \Rightarrow x=\pm 2$

1) Find local min/max on interior (via crit. pts)

solve $Df=[0\ 0]$

$Df=[8x-4y \quad -4x+2]=[0\ 0]$

$8x-4y=0 \qquad -4x+2=0$

$4-4y=0 \qquad 4x=2$

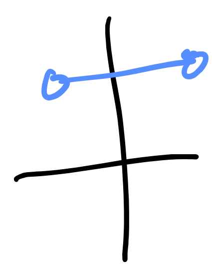
$y=1 \qquad x=\frac{1}{2}$

3] Evaluate f at points found

(x,y)	$f(x,y)$
$(\frac{1}{2},1)$	1
$(2,4)$	-8
$(0,0)$	0
$(1,1)$	2
$(-2,4)$	56

2) Find local min/max on boundary

a) On $y=4$; $-2\le x\le 2$



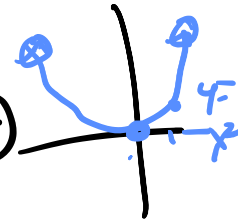
$f(x,y)=4x^2-4x(4)+2(4)$

$g(x)=4x^2-16x+8$

$g'(x)=8x-16=0 \qquad x=2, \; y=4$

c) Add boundary of the boundary (the intersection points of the edges)

b) On $y=x^2$; $-2 \le x \le 2$



$f(x,y) = 4x^2 - 4x(x^2) + 2(x^2)$

$h(x) = 6x^2 - 4x^3$

$h'(x) = 12x - 12x^2 = 0$

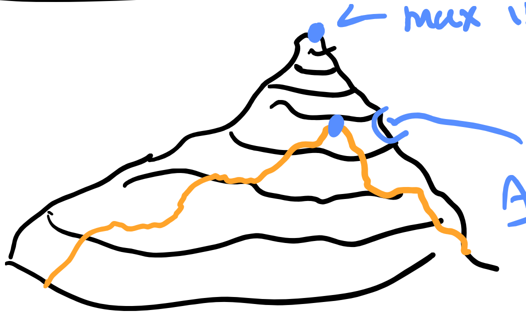
$12x(1-x) = 0$

$x=0 \qquad x=1$

$y=0 \qquad y=1$

The global max is 56 achieved at $(-2,4)$ and global min is -8 achieved at $(2,4)$.

Constrained Optimization

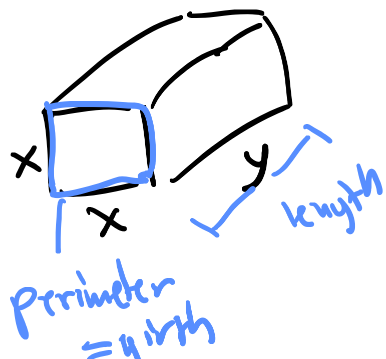


Q: What is highest pt on mtn

A: if we are constrained to stay on the trail?

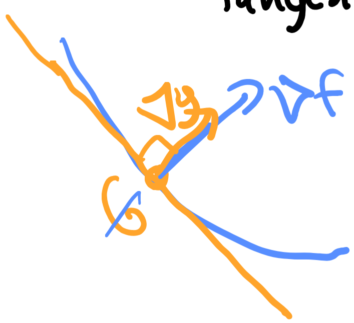
Goal: Maximize/minimize $f(x,y)$ subject to a constraint $g(x,y)=c$.

ex: Postal regulations requires the girth + length of a parcel to be at most 108 in. What is the largest volume of a rectangular parcel with a square end that can be mailed?



<u>Goal</u>: Maximize $f(x,y) = x^2 y$ subject to constraint $g(x,y) = 4x + y = 108$

↳ Find (x,y) where the constraint is tangent to a level curve since there must be a min/max.



↳ $\boxed{\text{Find } (x,y) \text{ where } \nabla f = \lambda \nabla g \text{ ; } g = c}$

Method of Lagrange Multipliers

$\nabla f = \langle 2xy, x^2 \rangle \quad \nabla g = \langle 4, 1 \rangle$

$\langle 2xy, x^2 \rangle = \lambda \langle 4, 1 \rangle \qquad 4x + y = 108$

$$\left.\begin{aligned} 2xy &= 4\lambda \\ x^2 &= \lambda \\ 4x + y &= 108 \end{aligned}\right\} \qquad \left.\begin{aligned} 2xy &= 4x^2 \\ 4x + y &= 108 \end{aligned}\right\} \rightarrow \begin{aligned} xy - 2x^2 &= 0 \\ x(y - 2x) &= 0 \end{aligned}$$

$x = 0$

$4(0) + y = 108$

$y = 108$

$y - 2x = 0$

$y = 2x$

$4x + 2x = 108$

$x = 18$

$y = 36$

$f(0, 108) = 0 \text{ in}^3$

$f(18, 36) = 11664 \text{ in}^3$

Max volume is 11,664 for a 18" x 18" x 36" box ex: Find points on surface $z^2 = xy + 4$ closest to origin.

Goal: minimize $d = \sqrt{x^2+y^2+z^2}$ subject to constraint

dist from $(0,0,0)$ to (x,y,z)

$g(x,y,z) = z^2 - xy = 4$

Actually work with $f = d^2 = x^2+y^2+z^2$

Solve $\nabla f = \lambda \nabla g$; $g = 4$

$$\langle 2x, 2y, 2z \rangle = \lambda \langle -y, -x, 2z \rangle$$

$2x = -\lambda y$

$2y = -\lambda x$

$2z = \lambda \cdot 2z$

$z^2 - xy = 4$

$z = \lambda z$

$z - \lambda z = 0$

$z(1-\lambda) = 0$

Case 1: $\lambda = 1$

$2x = -y$

$2y = -x$

$z^2 - xy = 4$

$-4x = -x$

$x = 0$

so $y = 0$

$z^2 = 4$ so $z = \pm 2$

$\boxed{(0,0,\pm 2)}$

Case 2: $z = 0$

$2x = -\lambda y$; $\lambda = \frac{-2x}{y}$

$2y = -\lambda x$; $\lambda = \frac{-2y}{x}$

$-xy = 4 \Rightarrow x, y \neq 0$

$\frac{-2x}{y} = \frac{-2y}{x}$

$x^2 = y^2$

$x = \pm y$

if ~~$x = y$: $-(y^2) = 4$~~

if $x = -y$: $y^2 = 4$ $y = \pm 2$

$\boxed{(-2,2,0), (2,-2,0)}$

$$d(0,0,\pm 2) = \sqrt{0^2+0^2+(\pm 2)^2} = 2$$

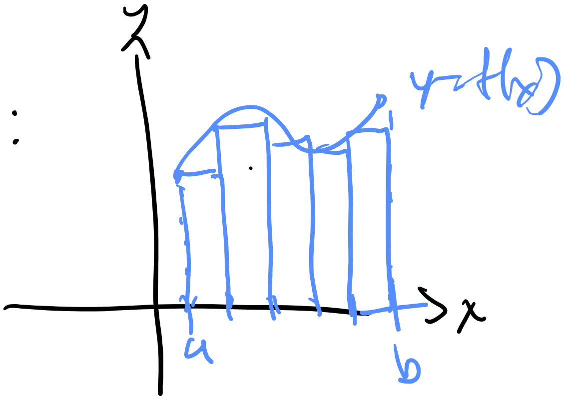
$$d(\mp 2,\pm 2,0) = \sqrt{2^2+2^2+0^2} = \sqrt{8}$$

The points on $z^2 = xy+4$ closest to origin are $(0,0,\pm 2)$.

MATH 2551 L - 10/13 - 15.1

- No class Tuesday - Fall Break
- <u>Today</u>: - Double Integrals
  - Iterated Integrals

<u>Idea</u>:

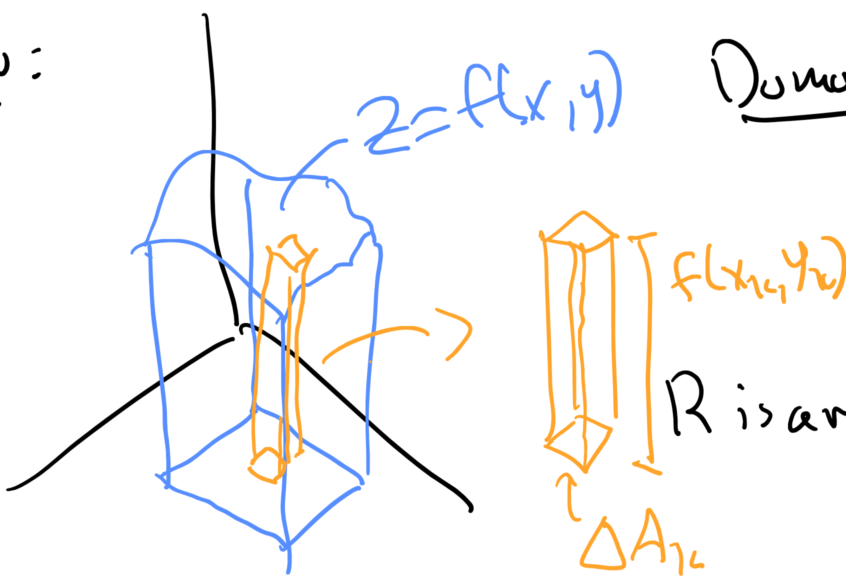


limit of Riemann sums $\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x$

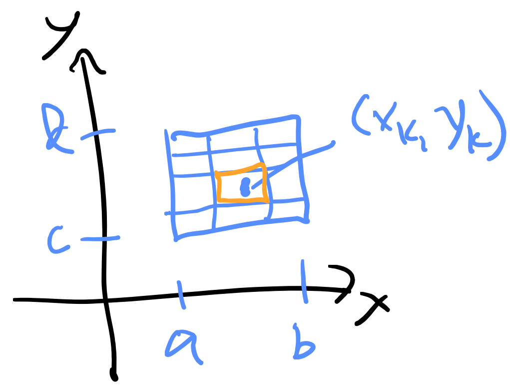
to get $\int_a^b f(x)\,dx$

"signed area under $y=f(x)$ from $x=a$ to $x=b$"

<u>New</u>:



<u>Domain</u>:



R is a rectangle : $a \le x \le b$
$c \le y \le d$

OR $[a,b]\times[c,d]$

$$V \approx \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k$$

<u>Def</u>: The <u>double integral</u> of $f(x,y)$ over rectangle R

is $\iint_R f(x,y)\,dA = \lim_{|A_k| \to 0} \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k$,

if the limit exists.

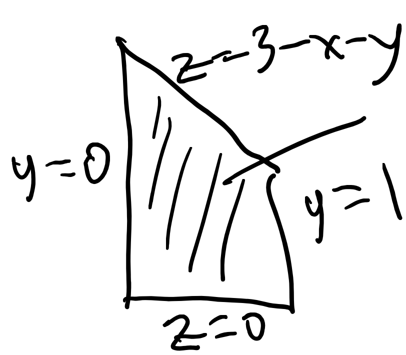
- If f is continuous on R, the limit exists.
- $\iint_R f(x,y)\,dA$ is the signed volume of the solid between $z = f(x,y)$ & $z = 0$ over rectangle R

<u>Q</u>: How to compute?

<u>ex</u>: Compute $\iint_R 3 - x - y\,dA$ for $R: 0 \le x \le 1,\ 0 \le y \le 1$

- $\iint_R 3 - x - y\,dA = V = \int_0^1 A(x)\,dx$

For a fixed x:



$$A(x) = \int_0^1 3 - x - y\,dy$$

$$= 3y - xy - \frac{1}{2}y^2 \Big|_{y=0}^{y=1}$$

$$= 3 - x - \frac{1}{2} - (0 - 0 - 0)$$

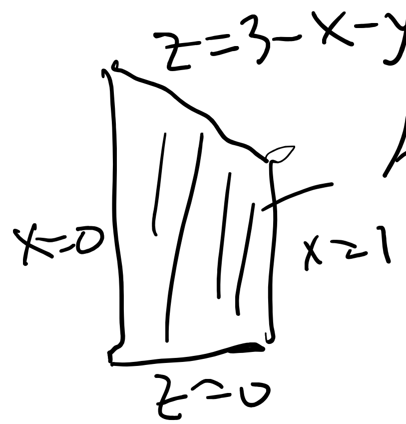
- treat x as a constant

$$= \frac{5}{2} - x$$

$$\iint_R 3-x-y\,dA = \int_{x=0}^{x=1} \frac{5}{2} - x\,dx = \frac{5}{2}x - \frac{1}{2}x^2\Big|_0^1$$

$$= \frac{5}{2} - \frac{1}{2} = \boxed{2}$$

For a fixed y: $\displaystyle \iint_R 3-x-y\,dA = \int_{y=0}^{y=1} A(y)\,dy$



$$A(y) = \int_{x=0}^{x=1} 3-x-y \;\; \underline{dx}$$ • y is constant

$$= 3x - \frac{1}{2}x^2 - yx \Big|_{x=0}^{x=1}$$

$$= 3 - \frac{1}{2} - y = \frac{5}{2} - y$$

$$\iint_R 3-x-y\,dA = \int_{y=0}^{y=1} \frac{5}{2} - y\,dy = \boxed{2}$$

<u>Fubini's Thm</u>: If $f(x,y)$ is continuous on R

then $\displaystyle \iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$ ⇐ 1^{st} comp. above

$$= \underbrace{\int_c^d \int_a^b f(x,y)\,dx\,dy}_{\text{iterated integral}}$$ ⇐ 2^{nd}

$\underline{\text{ex}}$: Compute $\iint_R 25-2x^2-y\,dA$, $R: [-1,1]\times[0,2]$

$$= \int_{-1}^{1}\int_0^2 25-2x^2-y\,dy\,dx$$

$$= \int_0^2\int_{-1}^1 25-2x^2-y\,\underline{dx}\,dy$$

$$= \int_0^2 25x-\frac{2}{3}x^3-yx \Big|_{x=-1}^{x=1} dy$$

$$= \int_0^2 \left(25-\frac{2}{3}-y\right)-\left(-25+\frac{2}{3}+y\right) dy$$

$$= \int_0^2 50-\frac{4}{3}-2y\,dy$$

$$= 50y-\frac{4}{3}y-y^2\Big|_0^2 = 100-\frac{8}{3}-4$$

$$= \boxed{96-8/3}$$

$\underline{\text{ex}}$: Compute $\iint_R xe^{e^{e^y}}\,dA$ on $[-1,1]\times[0,4]$

$$= \int_{-1}^{1}\int_0^4 xe^{e^{e^y}}\,dy\,dx \quad \Big] \text{ hard!!!}$$

$$= \int_0^4\int_{-1}^1 xe^{e^{e^y}}\,dx\,dy$$

$$\hookrightarrow \int_0^4 \frac{1}{2}x^2 e^{e^{e^y}} \Big|_{x=-1}^{x=1} dy$$

$$= \int_0^4 \frac{1}{2}e^{e^{e^y}} - \frac{1}{2}e^{e^{e^y}} dy$$

$$= \int_0^4 0 \, dy$$

$$= 0$$

---

ex: $\int_0^1 \int_0^1 3x^2y^3 \, dx \, dy$

$$= \int_0^1 x^3y^3 \Big|_{x=0}^{x=1} dy$$

$$= \int_0^1 y^3 \, dy$$

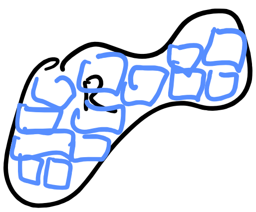
$$= \frac{1}{4}y^4 \Big|_0^1 = \boxed{\frac{1}{4}}$$

MATH 2551 L - 10/20, 15.2/15.3

Today:
- Integrals over non-rectangular regions
- Double integrals as area

Last time: $\iint_R f(x,y)\,dA$ computes volume under $z=f(x,y)$ over a rectangle R

Q: What if R is not a rectangle?

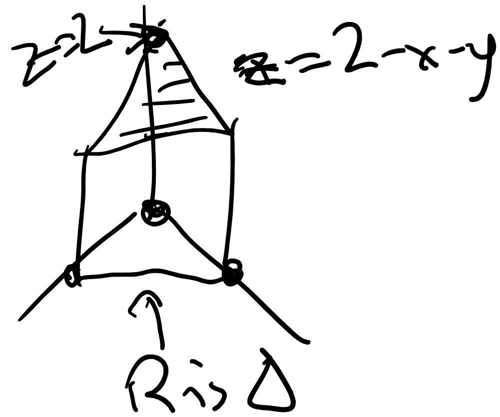


- divide into rectangles
- write Riemann sum
- take a limit

- if $f(x,y)$ is cts and R is bounded by smooth curves, the limit exists and we have

$\iint_R f(x,y)\,dA$ is the volume under $z=f(x,y)$ on R

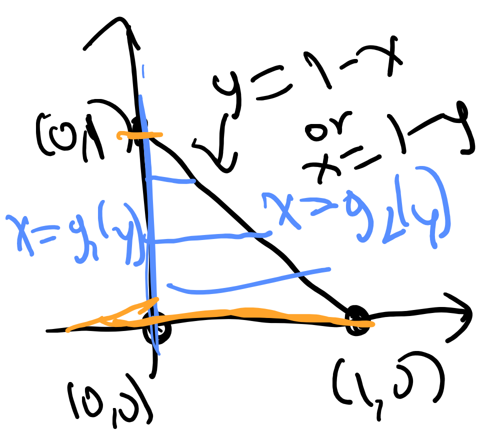
- To compute: rewrite as iterated integral ex: Compute volume of the solid whose base is the triangle with vertices (0,0), (0,1), (1,0) in the xy-plane and whose top is $z=2-x-y$.



$$V=\iint_R 2-x-y\,dA$$

1) Slice horizontally: $V=\int_c^d \underbrace{\int_{x=g_1(y)}^{x=g_2(y)} f(x,y)\,dx}_{A(y)}\,dy$

a) Sketch region



b) Determine bounds

$x=g_1(y)=0$

$x=g_2(y)=1-y$

left & right bounds

$c=0$

$d=1$

smallest & largest values of y in region

c) Integrate: $V=\int_0^1\int_0^{1-y} 2-x-y\,dx\,dy$

- outer bounds always constants
- inner bounds always constants or expressions w/ remaining variable

$$= \int_0^1 2x - \frac{x^2}{2} - xy \Big|_0^{1-y} dy$$

$$= \int_0^1 2(1-y) - \frac{(1-y)^2}{2} - (1-y)y - (0-0-0)\,dy$$

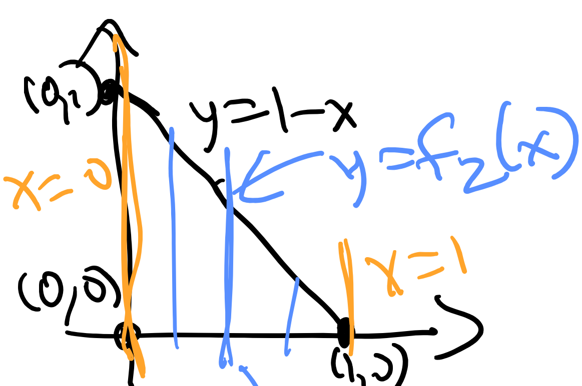
$$= \int_0^1 \frac{3}{2} - 2y + \frac{1}{2}y^2\,dy$$

$$= \frac{3}{2}y - y^2 + \frac{1}{6}y^3 \Big|_0^1$$

$$= \frac{3}{2} - 1 + \frac{1}{6} - 0 + 0 - 0 = \boxed{\frac{2}{3}}$$

2) Vertical slice : $V = \int_a^b \underbrace{\int_{f_1(x)}^{f_2(x)} f(x,y)\,dy}_{A(x)}\,dx$

a) Sketch region



b) Determine bounds:

$$\left.\begin{aligned} f_1(x) &= 0 \\ f_2(x) &= 1-x \end{aligned}\right\} \text{bottom/top of slices}$$

$y = f_1(x) = 0$

$a = 0$, $b = 1$ } smallest & largest values of x

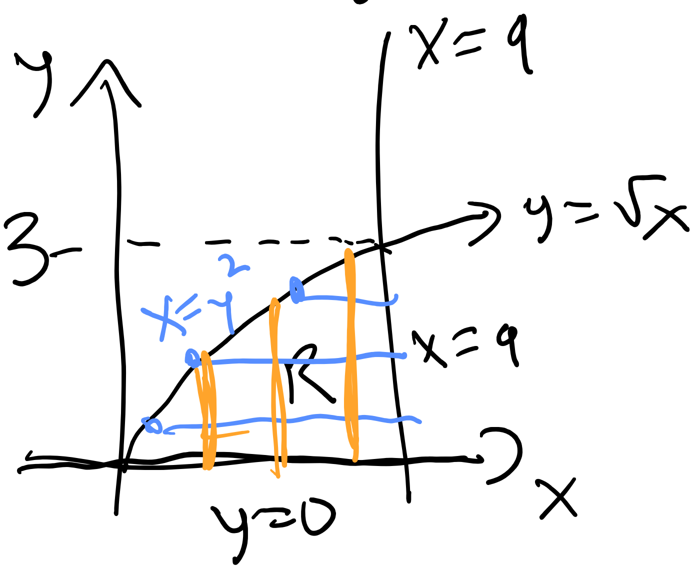
(1) Integrate: $\int_0^1 \int_0^{1-x} 2-x-y \, dy \, dx = \frac{2}{3}$

• Fubini's Thm still true: if f cts,

$$\iint_R f(x,y)\, dx\, dy = \iint_R f(x,y)\, dy\, dx$$

ex: Write the two iterated integrals for $\iint_R 1 \, dA$ for R bounded by $y=\sqrt{x}$, $y=0$, $x=9$.

a) Sketch region



Horiz. slices:

$$\int_0^3 \int_{y^2}^9 1 \, dx \, dy$$

Vert. slices

$$\int_0^9 \int_0^{\sqrt{x}} 1 \, dy \, dx$$

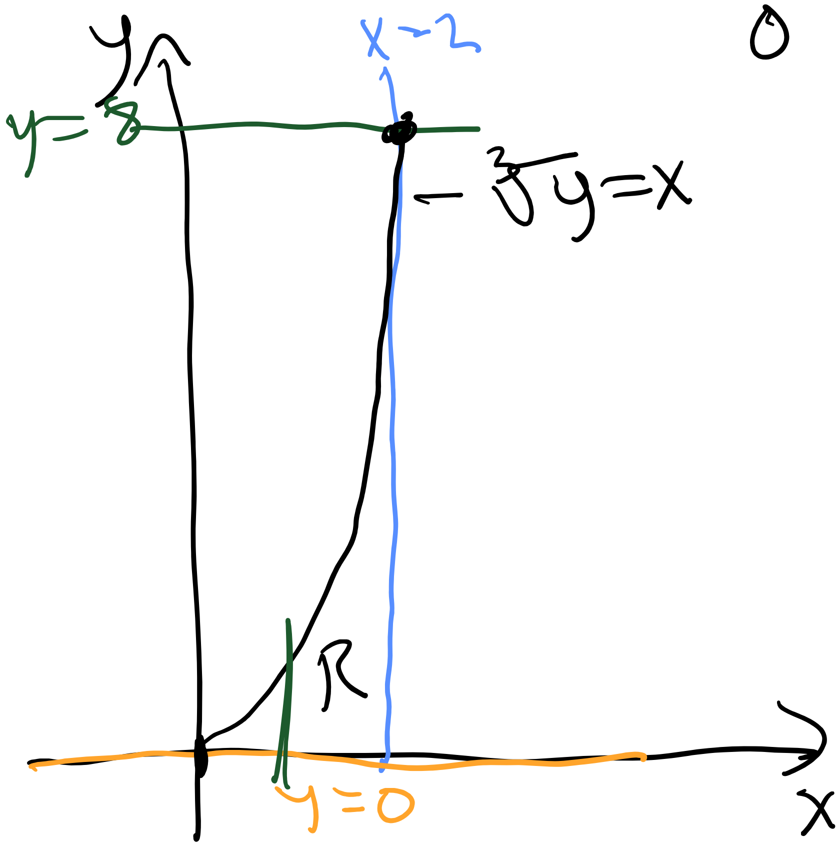
ex: Reverse order of integration to evaluate

$$\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\, dx\, dy$$

• this order is impossible by hand

a) Sketch region: $\sqrt[3]{y} \leq x \leq 2$

$0 \leq y \leq 8$



$x=2$

$y=0$

$y=8$

$\sqrt[3]{y}=x \iff y=x^3$

b) New bounds: vertical slices

$$\int_0^2 \int_0^{x^3} e^{x^4}\, dy\, dx = \int_0^2 e^{x^4} y \Big|_0^{x^3} dx$$

$$= \int_0^2 x^3 e^{x^4} - 0\, dx$$

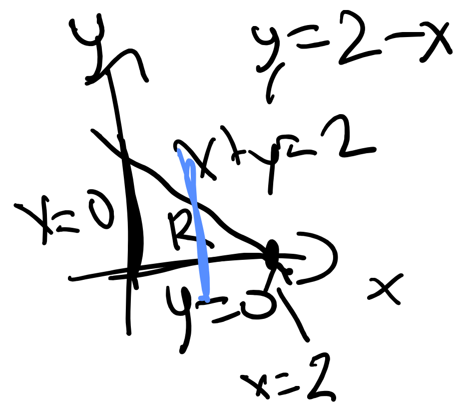
$$u=x^4 \quad du=4x^3\,dx$$

$$= \frac{1}{4} e^{u}\Big|_{x=0}^{x=2}$$

$$= \frac{1}{4} e^{x^4}\Big|_{x=0}^{x=2}$$

$$= \frac{1}{4}\left(e^{16}-1\right)$$

ex: Find the volume of the wedge cut from the 1st octant by the cylinder $z=12-3y^2$ and the plane $x+y=2$.



$$V=\iint_R 12-3y^2\,dA$$

$$=\int_0^2\int_0^{2-x} 12-3y^2\,dy\,dx$$

$$=\int_0^2 12y-y^3\Big|_0^{2-x}\,dx$$

$u=2-x$

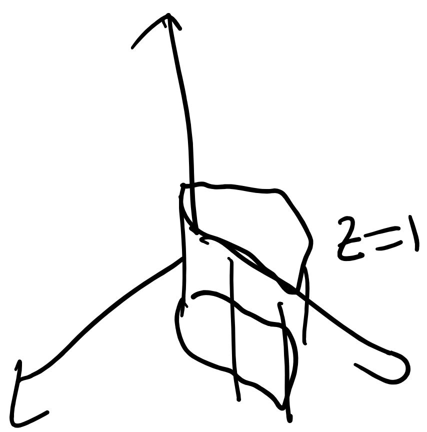
$$=\int_0^2 12(2-x)-(2-x)^3-(0-0)\,dx$$

$$= -6(2-x)^2 + \frac{1}{4}(2-x)^4 \Big|_0^2$$

$$= 20$$

- $\iint_R 1 \, dA$ = volume under $z=1$ on region R

How is $\iint_R 1 \, dA$ related to area of R?



$$V = 1 \cdot \text{area of } R$$
$$= \iint_R 1 \, dA$$

ex: Compute area of the region R bounded by $y=\sqrt{x}$, $y=0$, $x=9$.

$$\text{Area} = \iint_R 1 \, dA = \iint_R dA = \int_0^3 \int_{y^2}^9 1 \, dx \, dy$$
$$= \int_0^3 9 - y^2 \, dy$$

- Average value of $f(x,y)$ on R

is $f_{avg} = \frac{1}{\text{area of } R} \iint_R f(x,y)\,dA$

$= 9y - \frac{y^3}{3} \Big|_0^3$

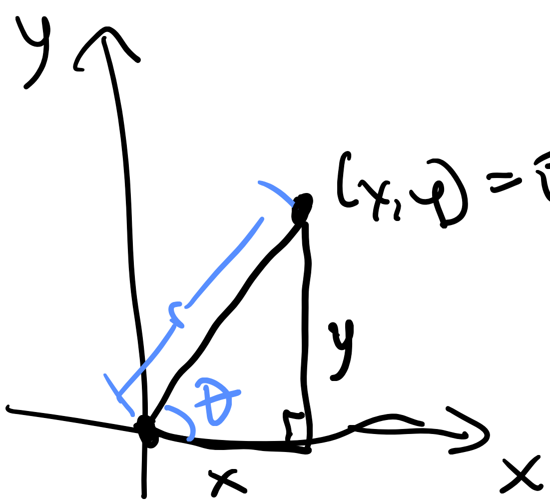
$= 27 - 9 = \boxed{18}$

MATH 2551 L - 10/25 - Sections 15.4, 15.5

- Midterm 2 grades released, see Canvas announcement

Today:
- Polar Coordinates
- Double Integrals in Polar Coordinates
- Start Triple Integrals (more on Th)

Polar coordinates



- Cartesian coords: give distances in $\vec{i}$ and $\vec{j}$ directions from $(0,0)$
- Polar coordinates:

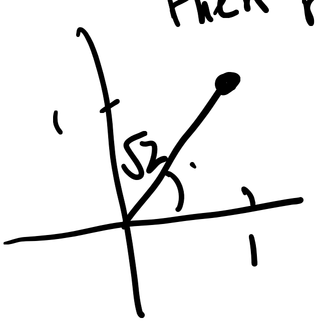
r = distance from $(0,0)$ to point

θ = angle between ray $\overrightarrow{OP}$ and pos. x-axis

e.g.: If $x=1, y=1$

then $r = \sqrt{1^2+1^2} = \sqrt{2}$

$\theta = \pi/4$



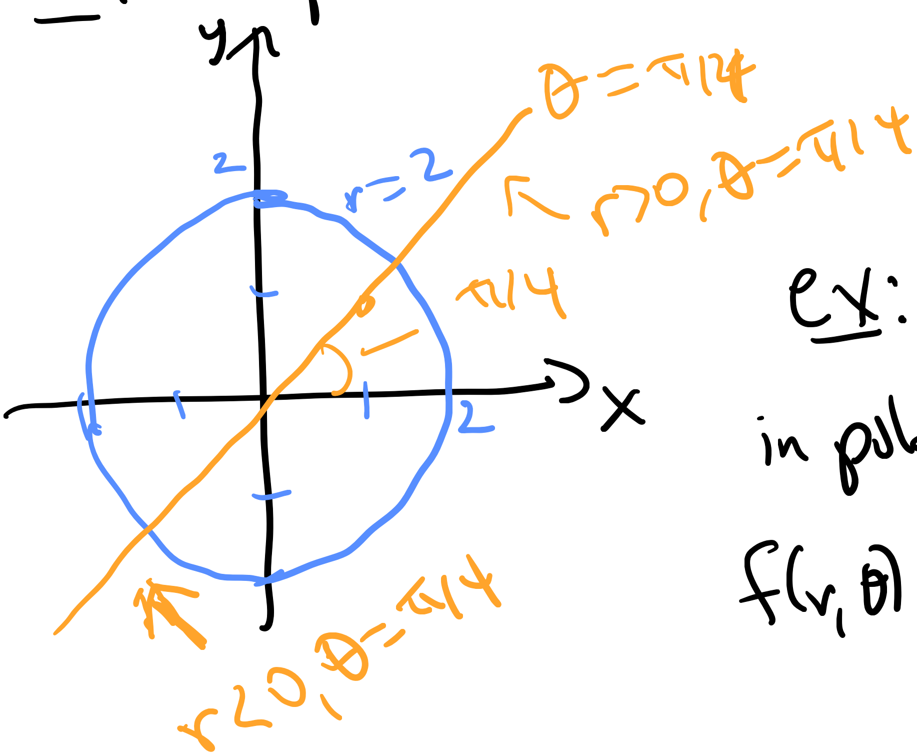
- (r, θ)

Polar → Cartesian: $x = r\cos\theta \quad y = r\sin\theta$

Cartesian → Polar: $r^2 = x^2 + y^2 \quad \tan\theta = \frac{y}{x}$

- for us $r > 0$ in integrals

ex: Graph $r = 2$ & $\theta = \pi/4$ in xy-plane.



ex: Write $f(x,y) = \sqrt{x^2+y^2}$ in polar coordinates.

$$f(r,\theta) = \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2}$$

$$= \sqrt{r^2} = r$$

Write $r = 2\sin\theta$ in Cartesian coords

$$r^2 = 2r\sin\theta$$

$$x^2 + y^2 = 2y$$

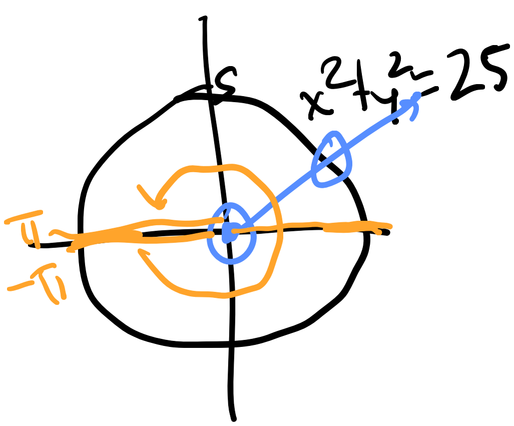
$$x^2 + y^2 - 2y + 1 = 0 + 1$$

$$x^2 + (y-1)^2 = 1$$

Double Integrals in Polar Coordinates

• Given a region R in xy-plane described in polar coordinates and a function $f(r,\theta)$, what is $\iint_R f(r,\theta)\, dA$?

ex: Compute the area of the disk of radius 5 centered at $(0,0)$.



$$A = \iint_R 1\, dA$$

$$= \int_{-5}^{5} \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} dy\, dx$$

• annoying

Work in polar coords:

$0 \leq r \leq 5$

$0 \leq \theta \leq 2\pi$

$$\int_{\theta_1}^{\theta_2} \int_{r=f_1(\theta)}^{r=f_2(\theta)} dA$$

$$A = \int_0^{2\pi} \int_0^5 dr\, d\theta = \int_0^{2\pi} r\Big|_0^5 d\theta$$

this is not dA

$$= 5\theta\Big|_0^{2\pi} = 10\pi$$

not $\pi(5)^2$

$$\Delta A = \frac{1}{2}r_2^2\Delta\theta - \frac{1}{2}r_1^2\Delta\theta$$

$$= \frac{r_2+r_1}{2}\Delta r\Delta\theta$$

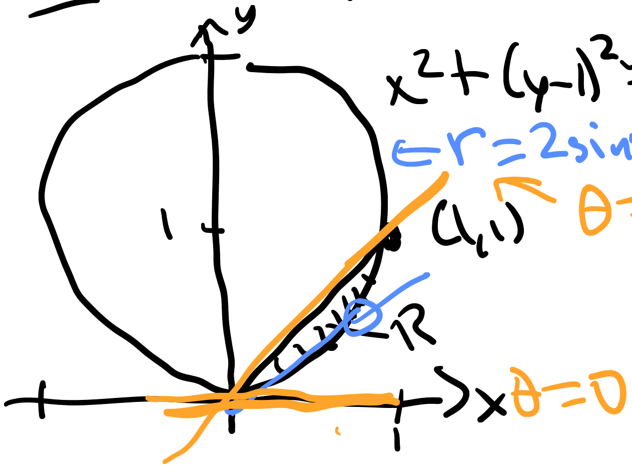
as $\Delta r, \Delta\theta \to 0$ $\boxed{dA = r\, dr\, d\theta}$

$$A = \int_0^{2\pi}\int_0^5 r\,dr\,d\theta = \int_0^{2\pi} \boxed{\frac{1}{2}r^2}\Big|_0^5 d\theta$$

$$= \frac{25}{2}\theta\Big|_0^{2\pi}$$

$$= \boxed{25\pi}$$

ex: Compute the area of the shaded region.



$$A = \iint_R dA$$

In polar coords:

$0 \le r \le 2\sin\theta$

$0 \le \theta \le \pi/4$

- $\sin^2\theta = \frac{1}{2}(1-\cos(2\theta))$
- $\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$

$$A = \int_0^{\pi/4}\int_0^{2\sin\theta} r\,dr\,d\theta$$

$$= \int_0^{\pi/4} \frac{1}{2}r^2\Big|_0^{2\sin\theta} d\theta$$

$$= \int_0^{\pi/4} 2\sin^2\theta\, d\theta$$

$$= \int_0^{\pi/4} 1-\cos(2\theta)\,d\theta$$

$$= \theta - \frac{1}{2}\sin(2\theta)\Big|_0^{\pi/4}$$

$$= \boxed{\pi/4 - \frac{1}{2}}$$

ex: Compute $\iint_D e^{-(x^2+y^2)}\,dA$ on the unit disk

$D = \{(x,y) \mid x^2+y^2 \le 1\}$.

- In Cartesian coords, impossible!

In polar coords:

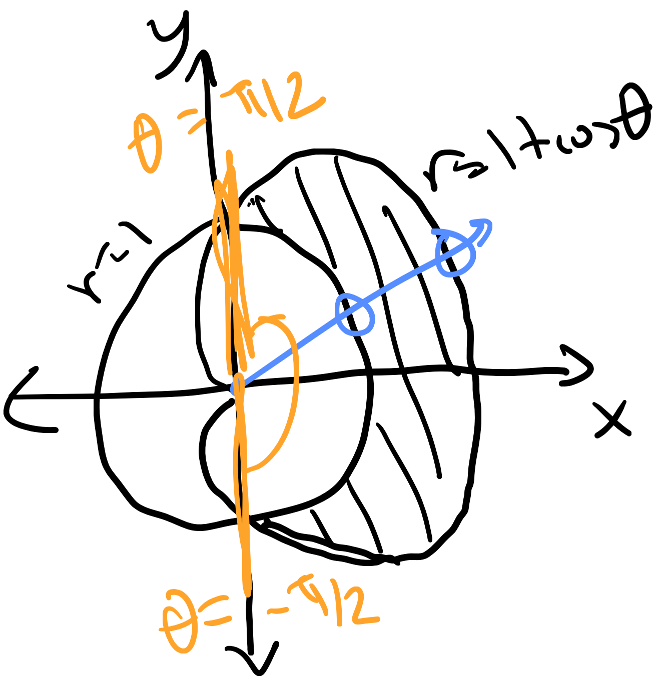
$$e^{-(x^2+y^2)} \mapsto e^{-r^2}$$

$$\int_0^{2\pi}\int_0^1 e^{-r^2}\, r\,dr\,d\theta$$

- let $u=-r^2$

$$= \pi(e-1)$$

ex: Write an integral for volume under $z=x$ on the region between $r=1+\cos\theta$ (a cardioid) and the circle $r=1$, where $x\geq 0$.



$$V=\iint_R x\,dA$$

$$=\int_{-\pi/2}^{\pi/2}\int_1^{1+\cos\theta}(r\cos\theta)\,r\,dr\,d\theta$$

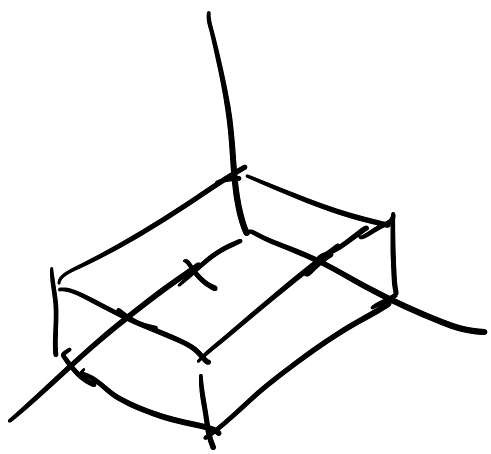
Triple Integrals

If D is a volume in $\mathbb{R}^3$:

$$\text{Volume of } D=\iiint_D 1\,dV$$ => compute w/ triple iterated integral

e.g $\int_0^1 \int_0^2 \int_0^3 dx\,dy\,dz$ computes

the volume of the box $0 \le x \le 3$, $0 \le y \le 2$,
$0 \le z \le 1$



$V = 6$

MATH 2551 L - 10/27 - 15.5/15.6

Today: • Triple Integrals & Volume
• Applications to mass & moments

$$\iiint_D dV = \text{volume of } D \text{ in } \mathbb{R}^3$$

$$= \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} dz\, dy\, dx$$

must be constants

must use only remaining variables

• 6 orders of integration
• focus on 1st step

ex: 1) Mechanics: Compute $\int_0^1 \int_0^{2-x} \int_0^{2-x-y} dz\, dy\, dx$

$$= \int_0^1 \int_0^{2-x} (2-x)-y \, dy\, dx$$

$$= \int_0^1 (2-x)y - \tfrac{1}{2}y^2 \Big|_0^{2-x} dx$$

$$= \int_0^1 (2-x)^2 - \tfrac{1}{2}(2-x)^2 \, dx$$

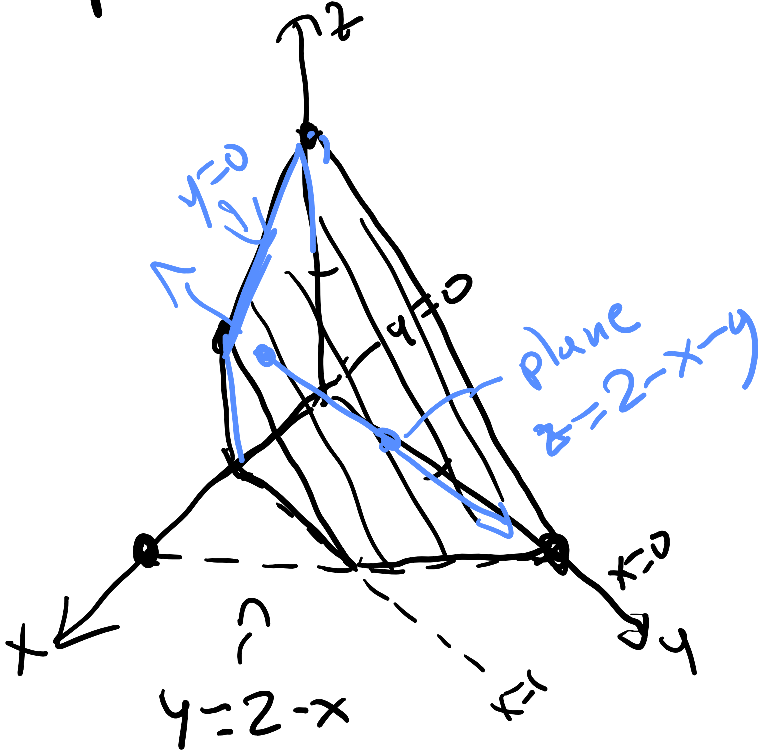
$$= \int_0^1 \tfrac{1}{2}(2-x)^2 dx = -\tfrac{1}{6}(2-x)^3 \Big|_0^1 = \boxed{\tfrac{7}{6}}$$

2) Interpretation: What shape is this the volume of?

$0 \le z \le 2-x-y$

$0 \le y \le 2-x$

$0 \le x \le 1$



3) Rearrange

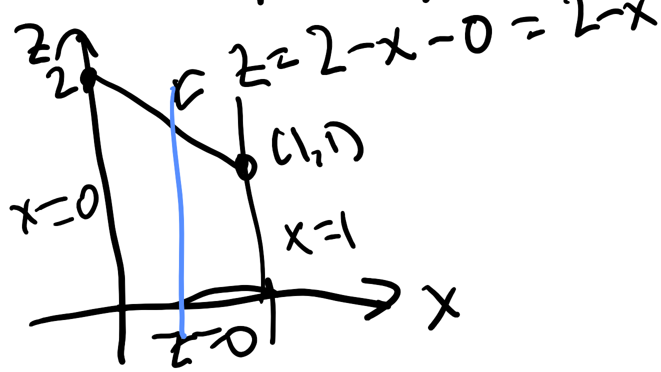
Write an equivalent $dy\,dz\,dx$ integral.

a) Draw a line parallel to the 1st axis of integration and see where it enters/leaves D: those are bounds for 1st integral

enters: $y=0$ leaves: $y=2-x-z$ (solving for y)

b) Project onto corresponding coord.-plane

Onto xz-plane ($y=0$)



$$\boxed{\int_0^1 \int_0^{2-x} \int_0^{2-x-z} dy\,dz\,dx}$$

ex: Find an integral for the volume of the solid in the first octant bounded by $z = 3 - x^2 - y^2$ and $z = 2y$.

a) Integrating wrt to z first

b) Integrating wrt to x first

c) Integrating wrt to y first

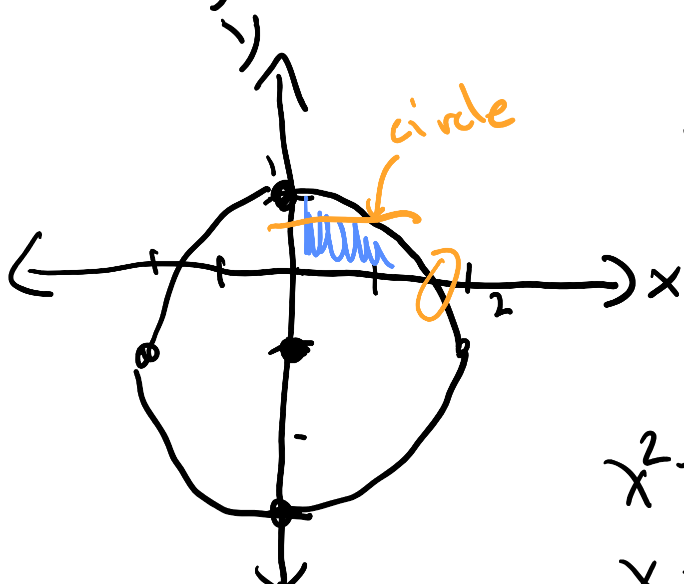
a) z: enters at $z = 2y$

leaves at $z = 3 - x^2 - y^2$

Project onto xy-plane: need to find curve of intersection

$$2y = 3 - x^2 - y^2 \Rightarrow x^2 + y^2 + 2y = 3$$

$$x^2 + (y+1)^2 = 4$$ (complete the square)



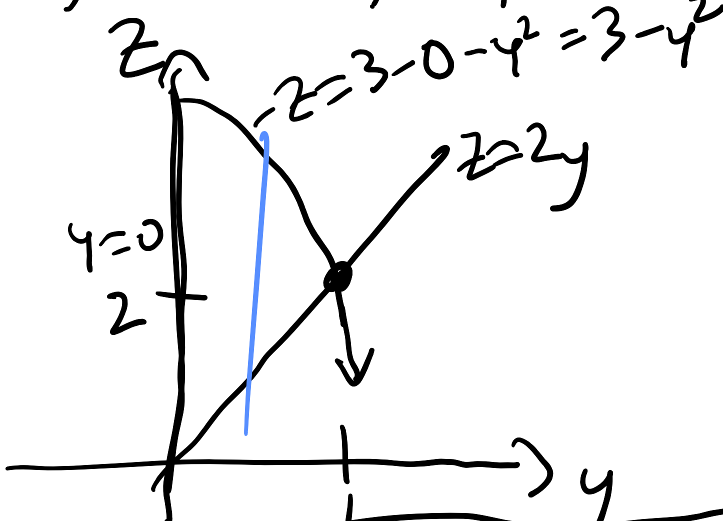
$$\boxed{\int_0^1 \int_0^{\sqrt{4-(y+1)^2}} \int_{2y}^{3-x^2-y^2} dz\, dx\, dy}$$

$$x^2 = 4 - (y+1)^2$$

$$x = \sqrt{4-(y+1)^2}$$

b) line in x direction
enters at $x=0$ leaves at $x=\sqrt{3-y^2-z}$

Project onto yz-plane ($x=0$)



$2y=3-y^2$

$y^2+2y-3=0$

$(y+3)(y-1)=0$

$y=\cancel{-3}, 1$

$$\boxed{\int_0^1\int_{2y}^{3-y^2}\int_0^{\sqrt{3-y^2-z}} dx\, dz\, dy}$$

__Applications__: $\rho(x,y,z)=$ mass density (mass/unit vol) in a region D

__Compute mass__: $M=\iiint_D \rho(x,y,z)\,dV$

__Moments__ is an integral or sum of integrals of the form

$$\iiint_D x^a y^b z^c \rho(x,y,z)\,dV$$

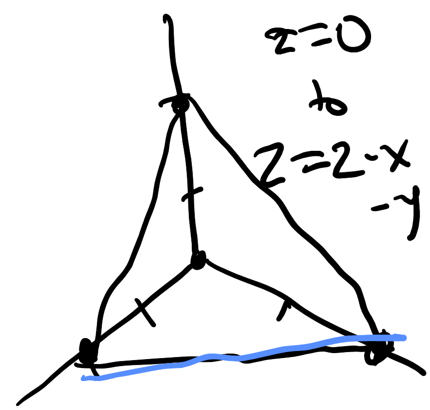
- 1^{st} moments (a,b,c are $0,1$) to compute center of mass
- 2^{nd} moments (a,b,c are $0,1,2$) compute difficulty of rotation around an axis

ex: A solid region in 1^{st} octant is bounded by the plane $x+y+z=2$. The density is $\rho(x,y,z)=2x$. Compute

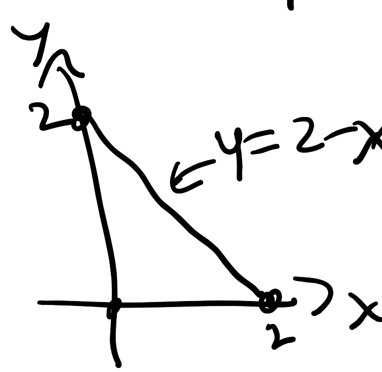
a) mass of the solid

b) center of mass

c) moment of inertia about y-axis



Project to xy-plane



a) $M=\iiint_D \rho\, dV$

$$=\int_0^2\int_0^{2-x}\int_0^{2-x-y} 2x\,dz\,dy\,dx$$

$= 4/3$

b) $M_{yz} = \iiint_D x\rho \, dV = \int_0^2 \int_0^{2-x} \int_0^{2-x-y} 2x^2 \, dz\,dy\,dx$

$= 16/15$

$M_{xz} = \iiint_D y\rho \, dV = \int_0^2 \int_0^{2-x} \int_0^{2-x-y} 2xy \, dz\,dy\,dx$

$= 8/15$

$M_{xy} = \iiint z\rho \, dV = 8/15$

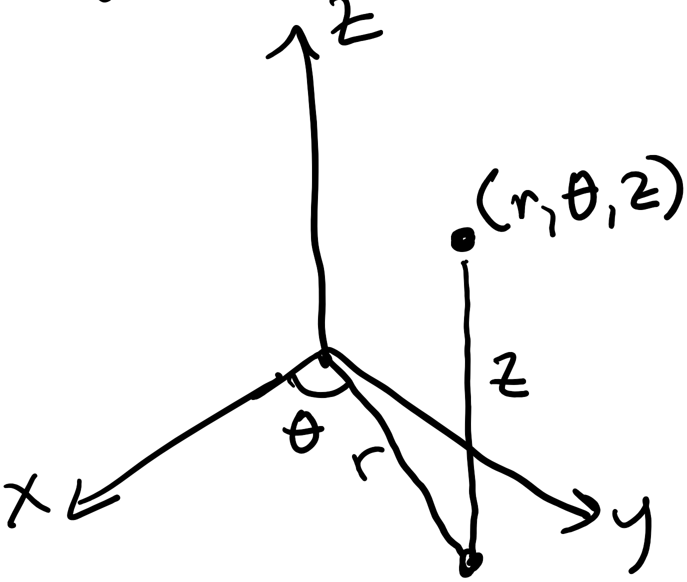
center of mass: $(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right)$

$= \left(\frac{4}{5}, \frac{2}{5}, \frac{2}{5}\right)$

MATH 2551 L - 11/1 - 15.7

Today:
- Cylindrical Coordinates
- Spherical Coordinates
- Triple Integrals using them

Cylindrical Coordinates (polar w/ z-axis)

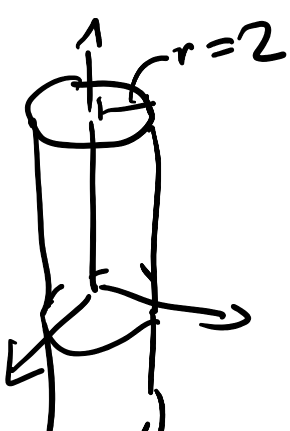


$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

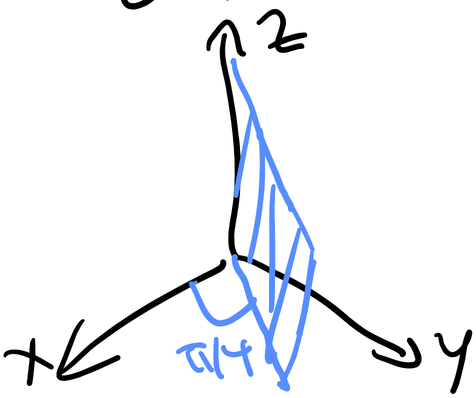
$x^2 + y^2 = r^2$
$\tan\theta = \frac{y}{x}$
$z = z$

$r \geq 0$, $0 \leq \theta \leq 2\pi$, z anything

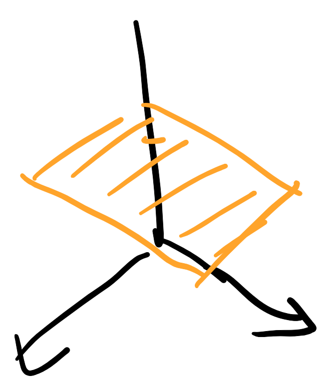
ex: What do $r=2$, $\theta = \pi/4$, $z=1$ look like?



$\theta = \pi/4$:



$z=1$

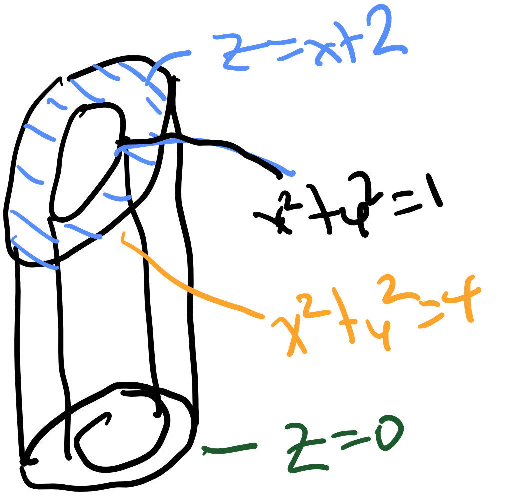


Triple Integrals in Cylindrical Coordinates

$$\iiint_D h(r,\theta,z)\,dV = \int_{\theta_1}^{\theta_2}\int_{g_1(\theta)}^{g_2(\theta)}\int_{f_1(r,\theta)}^{f_2(r,\theta)} h(r,\theta,z)\,dV$$

$$\boxed{dV = r\,dz\,dr\,d\theta}$$

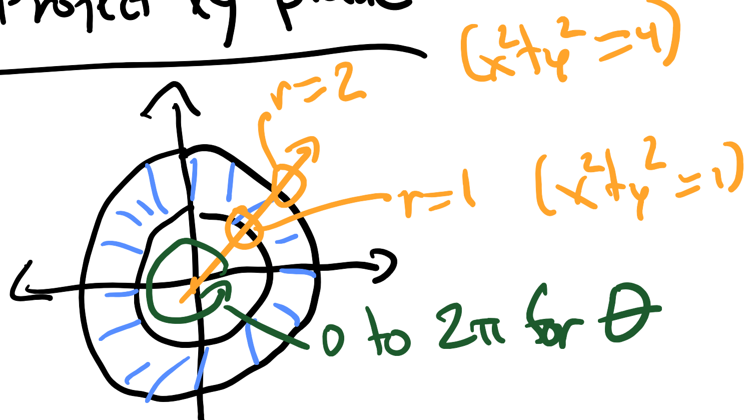
ex: Setup a triple integral in cylindrical coords for volume of the region D lying below $z=x+2$, above xy-plane and between $x^2+y^2=1$ and $x^2+y^2=4$.



__z-bounds__: $\underset{\text{enter}}{0} \le z \le \underset{\text{leave}}{x+2}$ convert to cylindrical

$$0 \le z \le r\cos\theta + 2$$

Project xy plane



$$V = \int_0^{2\pi}\int_1^2\int_0^{r\cos\theta+2} \underbrace{r\,dz\,dr\,d\theta}_{1\cdot dV}$$

ex: Convert $\boxed{\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz\, dz\, dx\, dy$

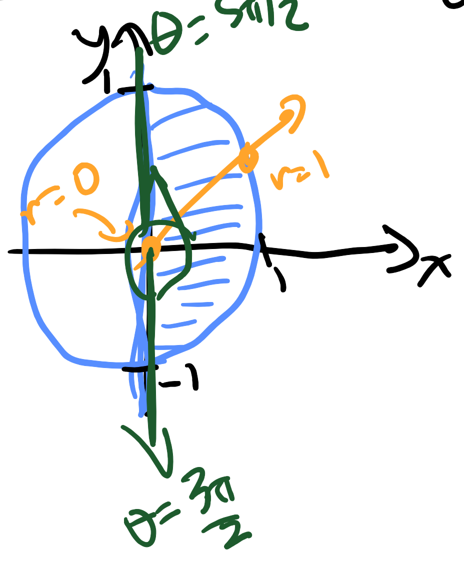
into cylindrical coords

- xy region is circular
- x^2+y^2, $\sqrt{x^2+y^2}$ appear

z-bounds: $x^2+y^2 \le z \le \sqrt{x^2+y^2}$

$$r^2 \le z \le \sqrt{r^2} = r$$

r, θ-bounds



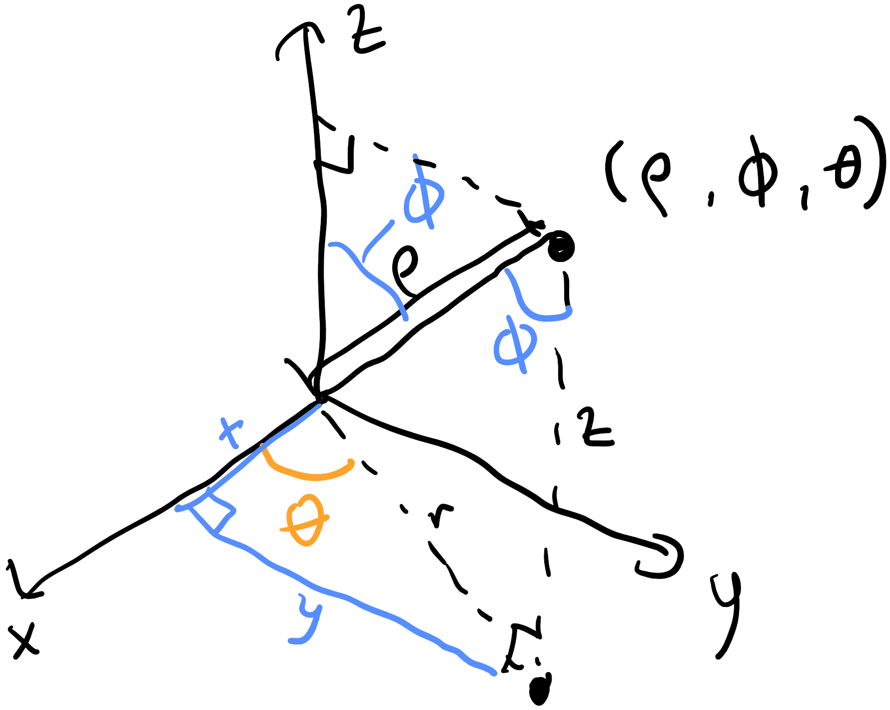
$0 \le x \le \sqrt{1-y^2}$ $\qquad$ $0 \le r \le 1$

$-1 \le y \le 1$ $\qquad$ $\frac{3\pi}{2} \le \theta \le \frac{5\pi}{2}$

$$\int_{3\pi/2}^{5\pi/2}\int_{0}^{1}\int_{r^2}^{r} (r\cos\theta)(r\sin\theta)z\, r\, dz\, dr\, d\theta$$

Spherical Coordinates

$\phi = \varphi$



- ρ = dist to origin
- ϕ = angle from pos. z-axis to the ray $\overrightarrow{OP}$
- θ = same as cylindrical

- $\rho \geq 0$, $0 \leq \phi \leq \pi$, $0 \leq \theta \leq 2\pi$

Spherical → Cylindrical

$r = \rho \sin\phi$

$\theta = \theta$

$z = \rho\cos\phi$

Spherical → Cartesian

$x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi$

Cylindrical → Spherical

$\rho^2 = r^2 + z^2$

Cartesian → Spherical

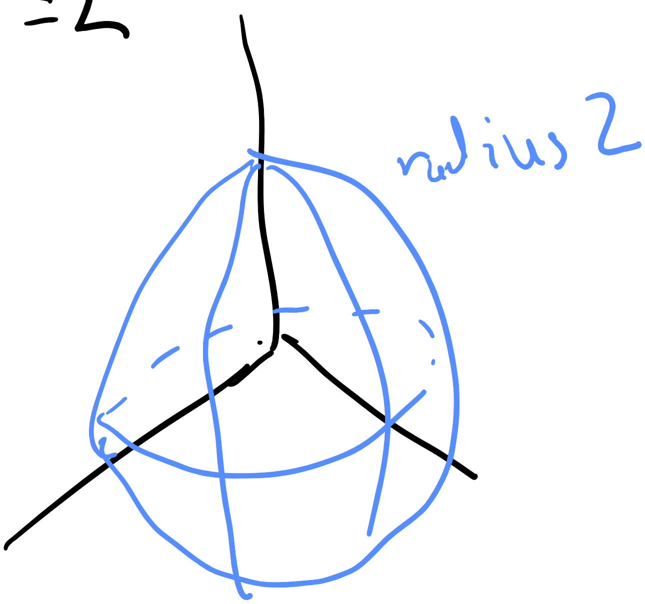
$\rho^2 = x^2 + y^2 + z^2$

$\theta = \theta$ $\qquad$ $\tan\theta = y/x$

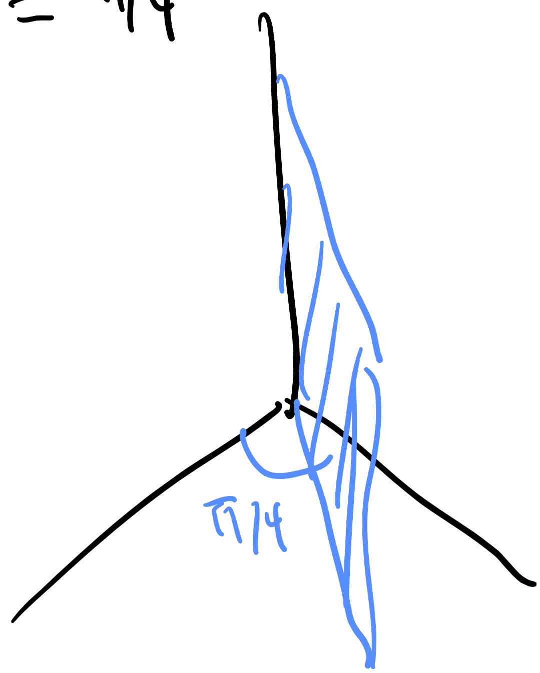
$\tan\phi = \frac{r}{z}$ $\qquad$ $\tan\phi = \frac{\sqrt{x^2+y^2}}{z}$

<u>ex</u>: What do $\rho = 2$, $\theta = \pi/4$, $\phi = \pi/4$ look like?

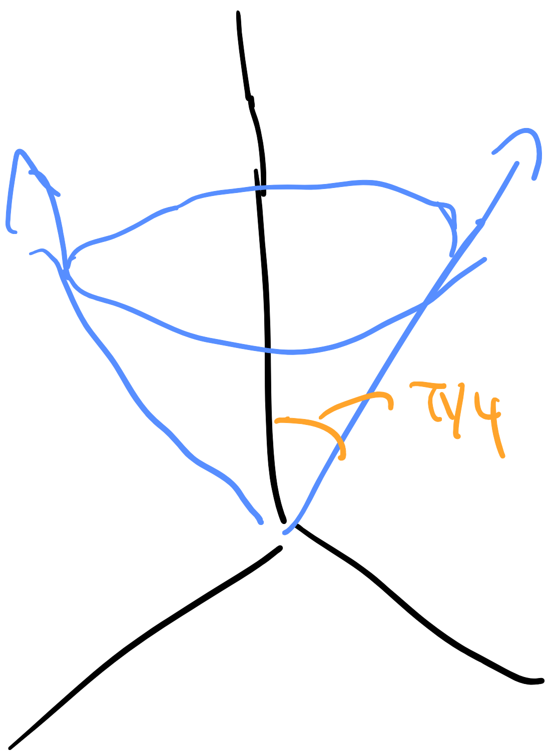
$\rho = 2$



$\theta = \pi/4$



$\phi = \pi/4$

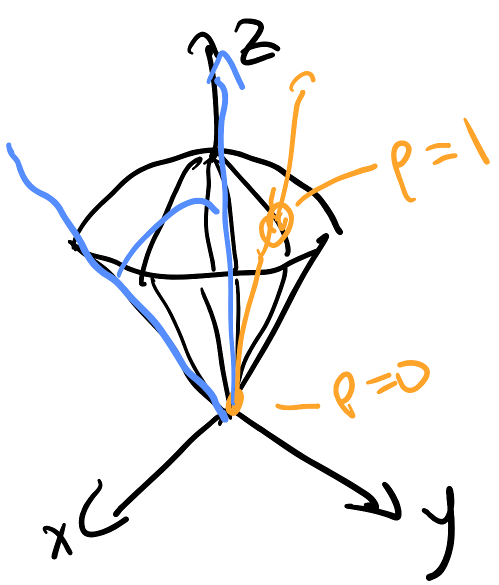


Triple Integrals in Spherical Coords

$$\iiint_D h(\rho,\phi,\theta)\,dV = \int_{\theta_1}^{\theta_2}\int_{f_1(\theta)}^{f_2(\theta)}\int_{g_1(\phi,\theta)}^{g_2(\phi,\theta)} h(\rho,\phi,\theta)\,dV$$

$$\boxed{dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}$$

ex: Find the volume of "ice cream cone" D bounded by $x^2+y^2+z^2=1$ and $z=\sqrt{3}\sqrt{x^2+y^2}$.



ρ-bounds: ray enters at $\rho=0$
leaves at sphere
$\rho=1$

ϕ-bounds: $\phi=0$ lower bound

ϕ upper bd is cone:

$z=\sqrt{3}\sqrt{x^2+y^2}$

OR

$$\tan\phi = \frac{\sqrt{x^2+y^2}}{z}$$

$$= \frac{\cancel{\sqrt{x^2+y^2}}\ 1}{\sqrt{3}\cancel{\sqrt{x^2+y^2}}}$$

$$\cancel{\rho}\cos\phi = \sqrt{3}\sqrt{r^2}$$

$$= \sqrt{3}\sqrt{\rho^2\sin^2\phi}$$

$$= \sqrt{3}\,\cancel{\rho}\sin\phi$$

$$\cos\phi = \sqrt{3}\sin\phi$$

$$\frac{1}{\sqrt{3}} = \tan\phi \quad \phi = \frac{\pi}{6}$$

$\underline{\theta\text{-bounds}}$: 0 to 2π

$$\boxed{V = \int_0^{2\pi}\int_0^{\pi/6}\int_0^1 \rho^2\sin\phi\, d\rho\, d\phi\, d\theta}$$

$$\frac{\frac{1}{2}}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}$$

MATH 2551 L - 11/13 - 15.8

Next week: • 15.5-15.6 & 15.7-15.8 HW due T
• Midterm 3 on W in studio, see Canvas

Today: • Recap spherical coords
• Change of variables
- generalize u-sub, trig sub, polar/cylindrical/spherical coords!

Spherical Coords (ρ, ϕ, θ)

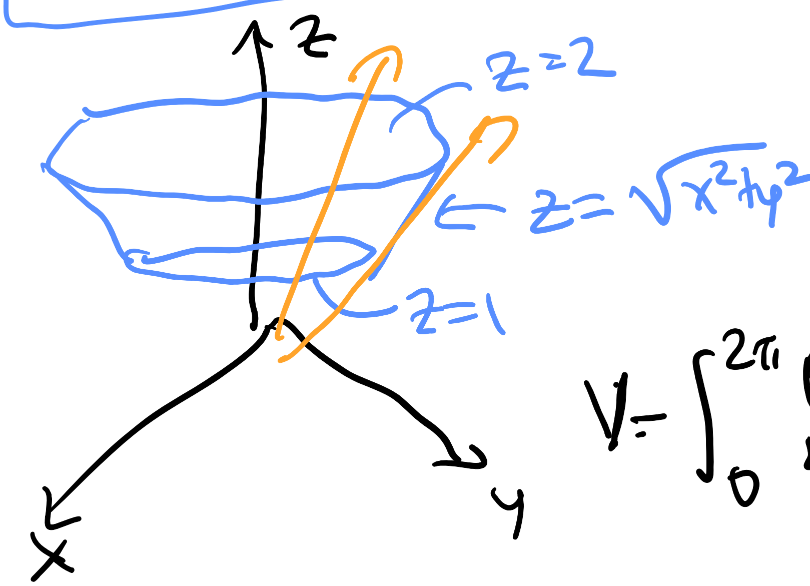
$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

ex: Write an integral for the volume of the part of the cone $\boxed{z = \sqrt{x^2+y^2}}$ between $z=1$, $z=2$.



• cones: $\phi = c$
so spherical coords good!

$$V = \int_0^{2\pi} \int_0^{\pi/4} \int_{\sec\phi}^{2\sec\phi} 1 \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

ρ-bounds: enter $z=1$ leave $z=2$

$\rho\cos\phi = 1$ $\qquad$ $\rho\cos\phi = 2$

$\rho = \sec\phi$ $\qquad$ $\rho = 2\sec\phi$

ϕ-bounds $\phi = 0$ to $\phi = c$ (cone) $z = \sqrt{x^2+y^2}$

$$\cancel{\rho}\cos\phi = \sqrt{\rho^2\sin^2\phi}$$
$$= \cancel{\rho}\sin\phi$$

θ-bounds: $\theta = 0$ to $\theta = 2\pi$

$\tan\phi = 1$ $\qquad$ $\phi = \pi/4$

$\phi = \arctan(1)$

Change of Variables

In single var calc: $\displaystyle\int_1^{\sqrt{3}} \frac{1}{1+x^2}\,dx = \int_{\pi/4}^{\pi/3} d\theta$

Trig sub: $x = \tan\theta$ $\qquad$ $dx = \sec^2\theta\, d\theta$

$x = f(\theta)$ $\qquad$ $dx = f'(\theta)\, d\theta$

$x = 1$ $\qquad$ $\tan\theta = 1$ $\qquad$ $\theta = \arctan(1) = \pi/4$

$x = \sqrt{3}$ $\qquad$ $\tan\theta = \sqrt{3}$ $\qquad$ $\theta = \arctan\sqrt{3} = \frac{\pi}{3}$

$$\frac{1}{1+x^2} = \frac{1}{1+\tan^2\theta}$$

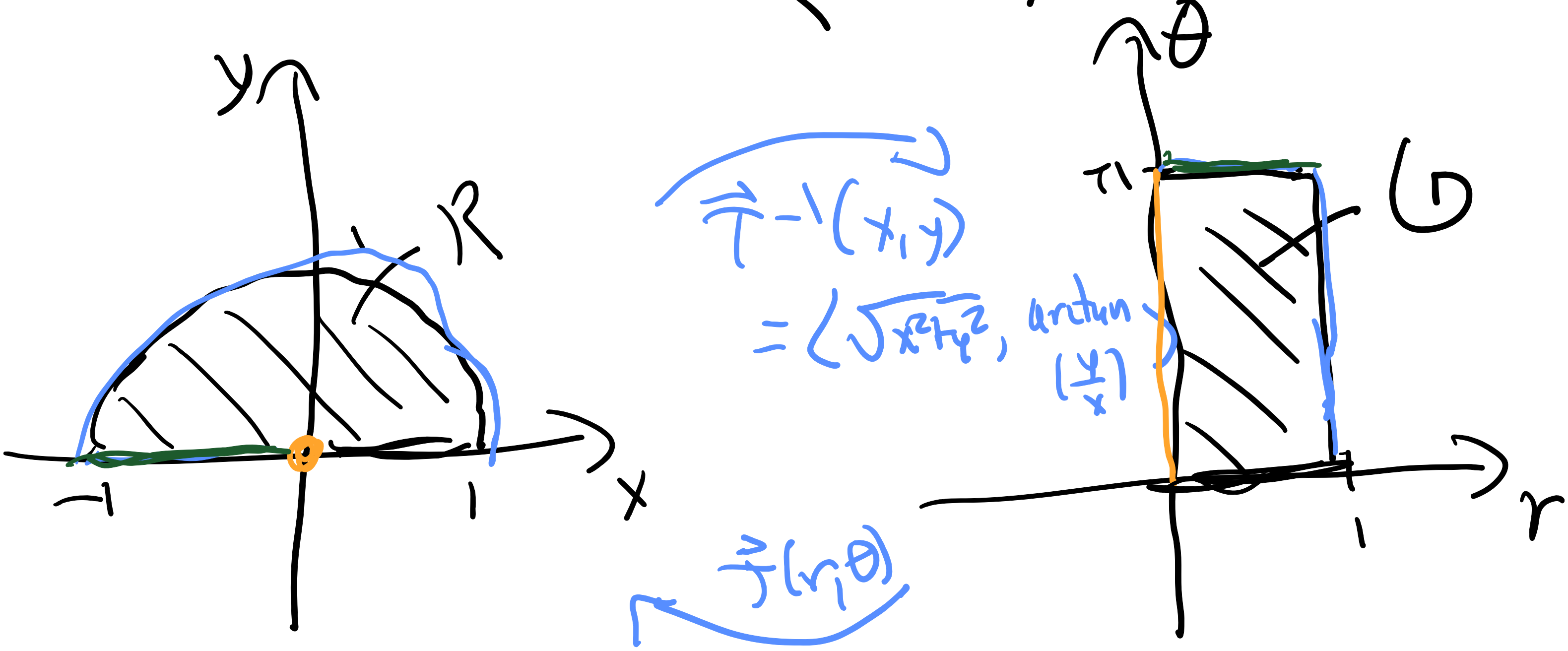
$$\int_{f^{-1}(1)}^{f^{-1}(\sqrt{3})} \frac{1}{1+(f(\theta))^2} f'(\theta)\, d\theta$$

Polar coords again:

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} \sqrt{x^2+y^2}\, dy\, dx$$ convert to polar

using $x = r\cos\theta$, $y = r\sin\theta$.

$$\vec{T}(r,\theta) = \langle x(r,\theta), y(r,\theta)\rangle$$
$$= \langle r\cos\theta, r\sin\theta\rangle$$



$$D\vec{T}(r,\theta) = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$\begin{aligned} |D\vec{T}(r,\theta)| &= r\cos^2\theta + r\sin^2\theta \\ &= r(\cos^2\theta + \sin^2\theta) \\ &= r \end{aligned}$$

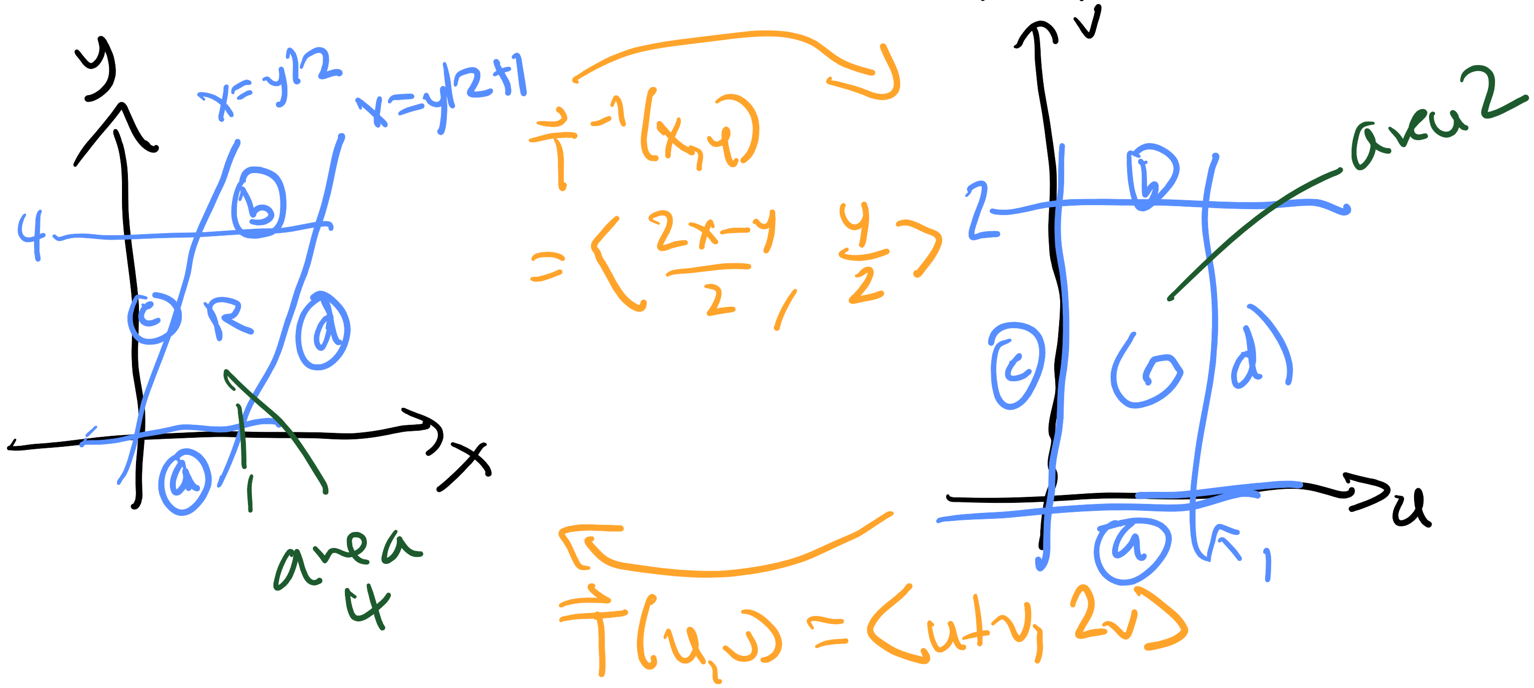
Polar integral: $$\underbrace{\int_0^\pi \int_0^1}_{D = \vec{T}^{-1}(R)} \underbrace{r}_{f(x(r,\theta),\,y(r,\theta))} \big(\overbrace{r}^{|D\vec{T}(r,\theta)|}\, dr\, d\theta\big)$$

Jacobian determinant: $$\begin{aligned} &|D\vec{T}(r,\theta)| \\ &= J(r,\theta) \\ &= |J(r,\theta)| \\ &= \frac{\partial(x,y)}{\partial(r,\theta)} \end{aligned}$$

<u>Theorem</u>: $\vec{T}(u,v)$ is a 1-to-1, differentiable transformation taking G in uv-plane to a region R in xy-plane. Then (under mild conditions)

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(\vec{T}(u,v))\,|D\vec{T}(u,v)|\,du\,dv$$

<u>ex</u>: Evaluate $\int_0^4 \int_{y/2}^{y/2+1} \frac{2x-y}{2}\,dx\,dy$ via the transformation $u = \frac{2x-y}{2}$, $v = \frac{y}{2}$.



1) Invert to find $\vec{T}$:

$$u = \frac{2x-y}{2} \longrightarrow u = x - \frac{2v}{2} \rightarrow \underline{x = u+v}$$

$$v = \frac{y}{2} \longrightarrow \underline{y = 2v}$$

2) Find G

a) $y=0 \rightarrow 2v=0 \rightarrow v=0$

b) $y=4 \rightarrow 2v=4 \rightarrow v=2$

c) $x=\frac{y}{2} \rightarrow u+v=\frac{2v}{2} \rightarrow u=0$

d) $x=\frac{y}{2}+1 \rightarrow u+v=\frac{2v}{2}+1 \rightarrow u=1$

3) Find Jacobian

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$|D\vec{T}(u,v)| = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2 - 0 = 2$$

- Jacobian is the scaling factor for area under $\vec{T}$

4) Convert integrand

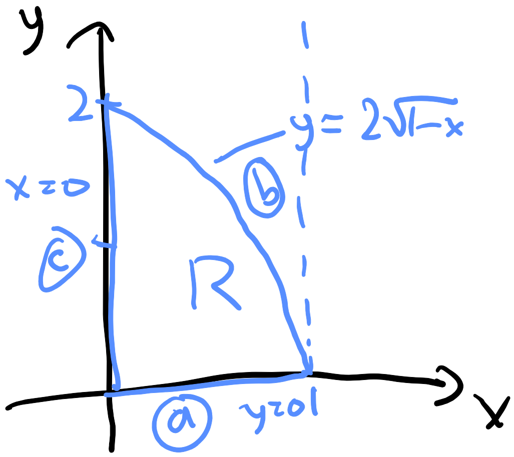
$$\frac{2x-y}{2} = u \quad \checkmark$$

$$\frac{2(u+v) - 2v}{2} = \frac{2u}{2} = u$$

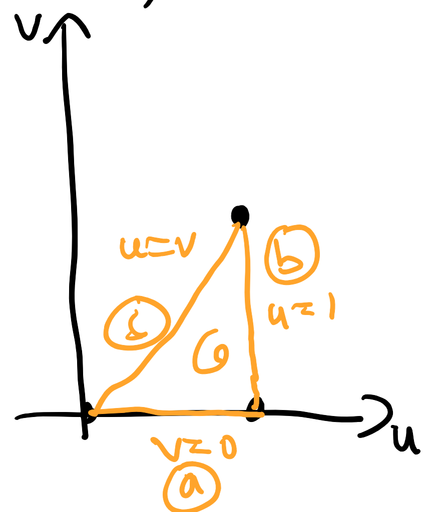
$$\iint_R \frac{2x-y}{2}\,dx\,dy = \underbrace{\int_0^2 \int_0^1}_{G} \underbrace{u}_{f(\vec{T}(u,v))} \cdot \underbrace{2}_{|D\vec{T}(u,v)|} \cdot du\,dv$$

$$\begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix} = 3 - (-4) = 7$$

ex: Evaluate $\int_0^1 \int_0^{2\sqrt{1-x}} \sqrt{x^2+y^2}\, dy\, dx$ using the transformation $\vec{T}(u,v) = \langle u^2-v^2, 2uv \rangle$ by showing that if G is the triangle with vertices $(0,0)$, $(1,0)$, & $(1,1)$ in the uv-plane, then $\vec{T}(G) = R$



$0 \le y \le 2\sqrt{1-x}$
$0 \le x \le 1$



1) We need to show $T(G) = R$, so let's convert the boundary equations for G

a) $v=0 \rightarrow \begin{array}{l} x = u^2 - 0 \\ y = 0 \end{array}$ so we get $\boxed{y=0}$

b) $u=1 \rightarrow \begin{array}{l} x = 1-v^2 \\ y = 2v \end{array} \rightarrow \begin{array}{l} v = \sqrt{1-x} \\ y = 2v \end{array} \rightarrow \boxed{y = 2\sqrt{1-x}}$

c) $u=v \rightarrow \begin{array}{l} x = u^2-u^2 = 0 \\ y = 2u^2 \end{array}$ so we get $\boxed{x=0}$

• Note: in a) & c) the eqns $x=u^2$ and $y=2u^2$ are telling us how to map the points with $v=0$ and $u=v$ resp. onto the points with $y=0$ and $x=0$.

2) Find Jacobian: $|D\vec{T}(u,v)| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix}$

$= 4u^2 + 4v^2$

3) Convert integrand: $f(\vec{T}(u,v))$

$= \sqrt{(u^2-v^2)^2 + (2uv)^2}$

$= \sqrt{u^4 - 2u^2v^2 + v^4 + 4u^2v^2}$

$$= \sqrt{u^4 + 2u^2v^2 + v^4}$$
$$= \sqrt{(u^2+v^2)^2}$$
$$= u^2 + v^2$$

4) Apply change of vars:

$$\int_0^1 \int_0^{2\sqrt{1-x}} \sqrt{x^2+y^2}\, dy\, dx = \int_0^1 \int_0^u (u^2+v^2)(4u^2+4v^2)\, dv\, du$$
$$= 4\int_0^1 \int_0^u u^4 + 2u^2v^2 + v^4\, dv\, du$$
$$= 4\int_0^1 u^4 v + \tfrac{2}{3}u^2v^3 + \tfrac{1}{5}v^5 \Big|_0^u du$$
$$= 4\int_0^1 u^5 + \tfrac{2}{3}u^5 + \tfrac{1}{5}u^5\, du$$
$$= 4\int_0^1 \tfrac{28}{15}u^5\, du$$
$$= \frac{\cancel{4}^{2}\cdot 28}{15\cdot \cancel{6}_{3}} u^6 \Big|_0^1 = \boxed{\frac{56}{45}}$$

<u>Note</u>: This integral is diffiult in polar & Cartesian coords, but not bad at all in these uv-coords.

e.g. in polar we get r-bounds of 0 & $r\sin\theta = 2\sqrt{1-r\cos\theta}$

$$\rightarrow r^2\sin^2\theta = 4 - 4r\cos\theta$$
$$\rightarrow r^2\sin^2\theta + 4r\cos\theta - 4 = 0$$
$$\rightarrow r = \frac{-4\cos\theta + \sqrt{16\cos^2\theta + 16\sin^2\theta}}{2\sin^2\theta}$$
$$= -2\cot\theta\csc\theta + 2\csc^2\theta$$

$$\boxed{\int_0^1 \int_0^{-2\cot\theta\csc\theta + 2\csc^2\theta} r^2\, dr\, d\theta}$$

MATH 2551 L - 11/10 - 16.1/16.2

- Exam 3 was too long - I will take this into account
- Tuesday 11/22 we will not have in-person class
  - lecture will be recorded & posted
  - virtual office hour 10 - noon (will post on Canvas)

Unit 4 Big Ideas

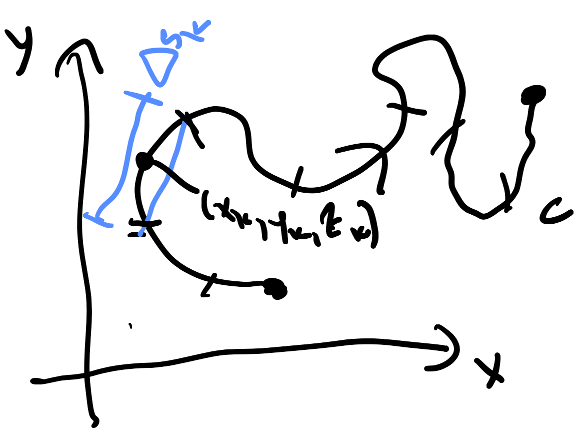
- Extend 1D/2D integrals to 1D/2D objects in 2D/3D space
- Extend fundamental theorem of calculus

Line Integrals

- compute mass of wire along a curve
- work done by a force as an object moves along a curve

Setup: $\int_C f(x,y,z)\, ds$

- C is a curve in space



- $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, $a \le t \le b$
  - tips of these vectors trace out C
  - usually want smooth
  - need orientation

$f(x,y,z) = \overset{\text{linear}}{\text{density}}$ (g/cm) $\to$ want to know mass along C

$\sum_{k=1}^{n} f(x_k, y_k, z_k) \cdot \Delta s_k$ is approx mass $\qquad \downarrow$ $\int_C f(x,y,z)\,ds$

Def: $\int_C f(x,y,z)\,ds = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k, z_k)\,\Delta s_k.$

To compute:

1) Parameterize C

2) $ds = |\vec{r}\,'(t)|\,dt$

3) $\int_C f(x,y,z)\,ds$

$= \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$

4) Integrate

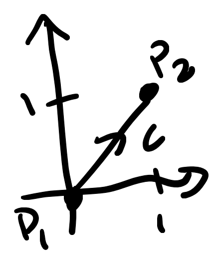
Q: What is $\int_C 1\,ds$?

A: arc length of C

$= \int_a^b |\vec{r}\,'(t)|\,dt$

ex: Suppose density of a wire along the straight line from $(0,0)$ to $(1,1)$ is $\rho(x,y) = 2x + y^2$ g/cm. Find the mass of this wire.

$\text{mass} = \int_C \rho(x,y)\,ds$



$\vec{r}(t) = \overrightarrow{P_1P_2}\,t + \overrightarrow{OP_1}$

$0 \le t \le 1$

$\vec{r}(t) = \langle 1,1\rangle t + \langle 0,0\rangle$

$= \langle t, t\rangle \quad 0 \le t \le 1$

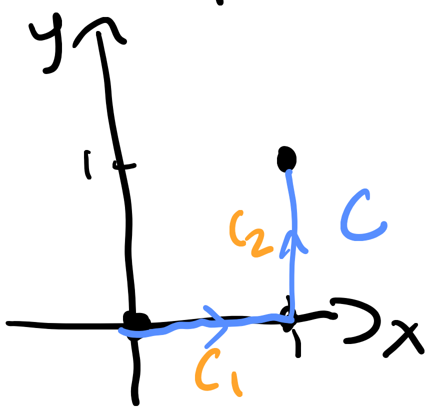
$|\vec{r}\,'(t)| = |\langle 1,1\rangle| = \sqrt{2}$

$\text{mass} = \int_0^1 (2t + t^2)\sqrt{2}\,dt$

$= \left(t^2 + \frac{t^3}{3}\right)\sqrt{2}\,\Big|_0^1$

$= \boxed{4\sqrt{2}/3 \quad g}$

ex: Compute $\int_C 2x+y^2\,ds$ along the curve C below



$$\int_C 2x+y^2\,ds = \int_{C_1} 2x+y^2\,ds + \int_{C_2} 2x+y^2\,ds$$

C_1: $\vec{r}_1(t) = \langle 1,0\rangle t + \langle 0,0\rangle,\ 0\le t\le 1$
$= \langle t,0\rangle$
$|\vec{r}_1{}'(t)| = |\langle 1,0\rangle| = 1$

C_2: $\vec{r}_2(t) = \langle 0,1\rangle t + \langle 1,0\rangle,\ 0\le t\le 1$
$= \langle 1,t\rangle$
$|\vec{r}_2{}'(t)| = |\langle 0,1\rangle| = 1$

$$\int_C 2x+y^2\,ds = \int_0^1 (2t+0^2)\cdot 1\,dt + \int_0^1 (2(1)+t^2)\cdot 1\,dt$$
$$= t^2\Big|_0^1 + \left(2t+\frac{t^3}{3}\right)\Big|_0^1 = \boxed{\frac{10}{3}}$$

- line integrals usually depend on path, not just endpoints

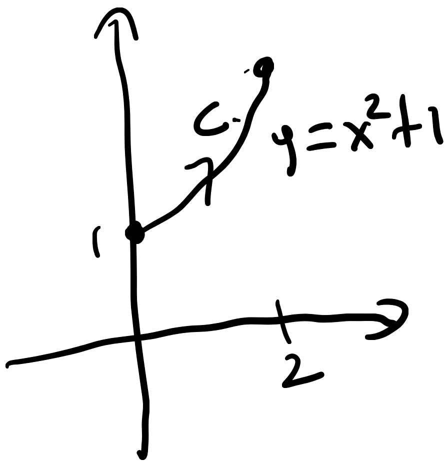
Parametrizations

oriented in +x direction

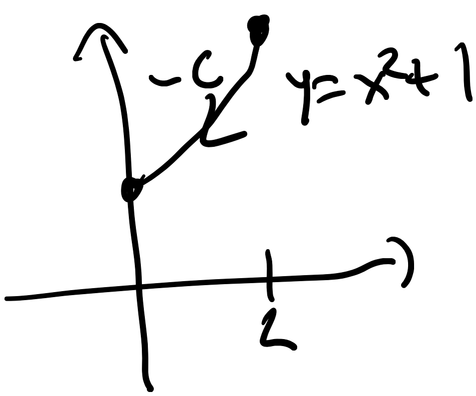
- C is a portion of a graph $y=f(x)$, $a \le x \le b$

$$\vec{r}(t)=\langle t, f(t)\rangle, \quad a \le t \le b$$

ex:



$$\vec{r}(t)=\langle t, t^2+1\rangle \quad 0\le t\le 2$$



$$\vec{r}_2(t)= \vec{r}(-t) \quad -2\le t\le 0$$

$$= \langle -t, (-t)^2+1\rangle$$

or

$$= \vec{r}(2-t) \quad 0\le t\le 2$$

$$= \langle 2-t, (2-t)^2+1\rangle$$

- $\int_{-C} f(x,y,z)\,ds = -\int_C f(x,y,z)\,ds$

- C is part of a circle/ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$\vec{r}(t)= \langle a\cos(t), b\sin(t)\rangle, \quad \alpha\le t\le \beta$$

— ccw

Vector Fields

- a function that associates a vector to every point in its domain
  - grav/electric fields
  - slope fields
  - tangent vectors along a curve
  - normal vectors to a surface
  - $\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \rangle$

$$\vec{F}(x,y,z) = P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + R(x,y,z)\,\vec{k}$$

- graphically: draw a vector $\vec{F}(a,b,c)$ with base at (a,b,c)

MATH 2551 - L - 11/15 - 16.2

- Quiz 9 will cover 16.1: parameterizations/scalar line integrals

Last time

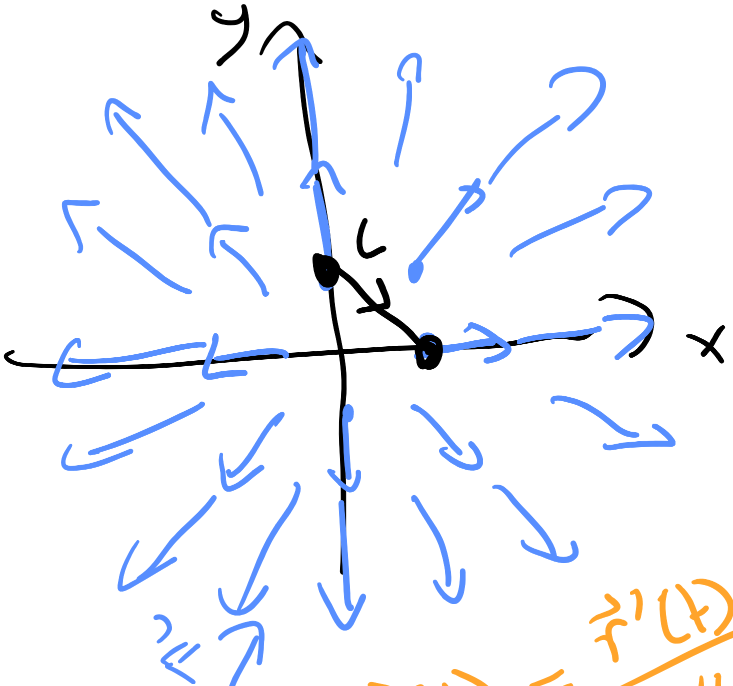
- scalar line integral: $\int_C f(x,y,z)\,ds$ adds up value of f along C
- vector fields: $\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$

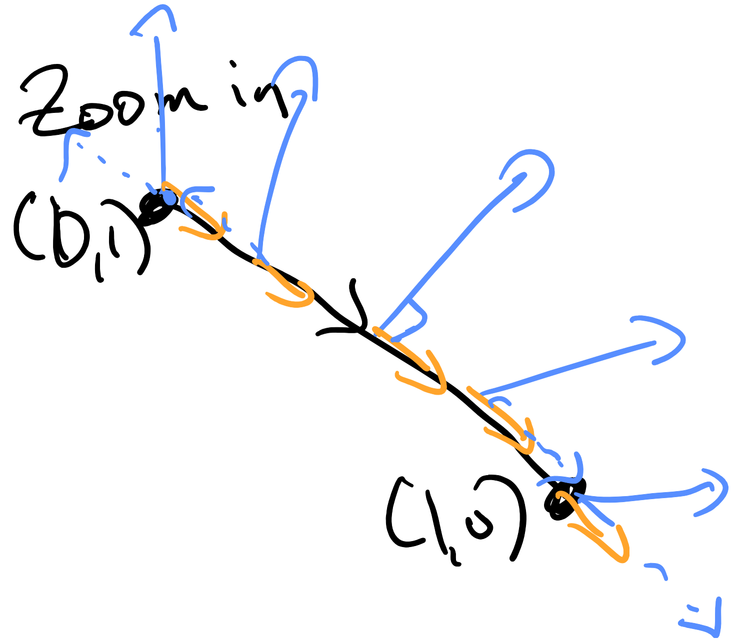
Today:
- line integrals of vector fields
- work/flow/flux

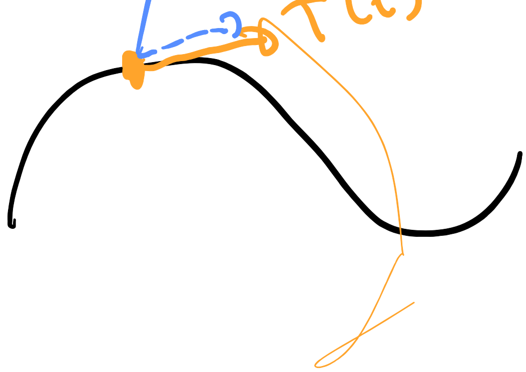
Idea: work done by a force $\vec{F}$ as an object moves along a curve C

work = Force $\cdot$ displacement

ex: Suppose we have a force field $\vec{F}(x,y) = \langle x, y\rangle$ N. Find work done by $\vec{F}$ on a moving object from $(0,1)$ to $(1,0)$ in a straight line (x, y in meters).







$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

Idea:

$$\boxed{\text{work} = \int_C \underbrace{\vec{F}\cdot\vec{T}}_{\text{force along } C}\ \overbrace{ds}^{\text{small displacement}}}$$

scalar

$$\int_C \vec{F}\cdot\vec{T}\,ds = \underbrace{\int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt}_{\text{for computation}}$$

ex: cont

$$\vec{r}(t) = \langle 1,-1\rangle t + \langle 0,1\rangle$$
$$0 \le t \le 1$$

$$\vec{r}\,'(t) = \langle 1,-1\rangle$$

$$\vec{F}(x,y) = \langle \underbrace{x}_{P(x,y)}, \underbrace{y}_{Q(x,y)} \rangle$$

$$\text{work} = \int_0^1 \underbrace{\langle t, 1-t\rangle}_{\vec{F}(\vec{r}(t))} \cdot \langle \underbrace{1}_{x'(t)}, \underbrace{-1}_{y'(t)} \rangle \, dt$$

$$= \int_0^1 t + (t-1) \, dt$$

$$= t^2 - t \Big|_0^1 = 0 \ \text{Nm}$$

$P(x,y,z)\,x'(t) = P\,dx$

$Q\,dy = Q(x,y,z)\,y'(t)$

<u>Notation</u>:

$$\int_C (\vec{F}\cdot\vec{T})\,ds = \int_C \vec{F}\cdot d\vec{r}$$

$$= \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

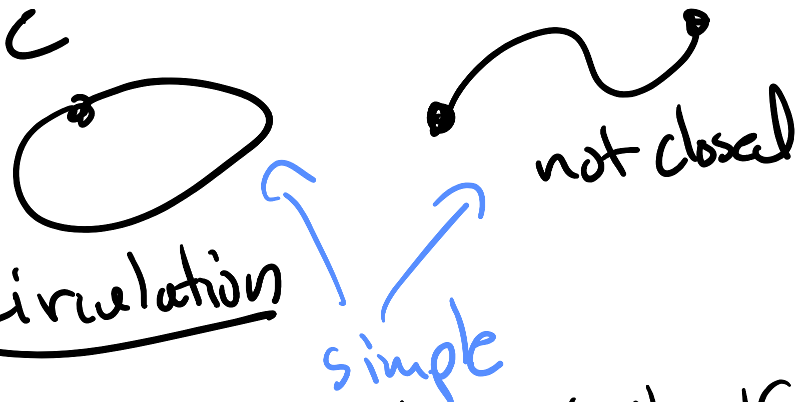
$$= \int_C P\,dx + Q\,dy + R\,dz$$

- If $\vec{F}(x,y,z)$ represents velocity vectors for a fluid in motion, the <u>flow</u> of a vector field along a curve C

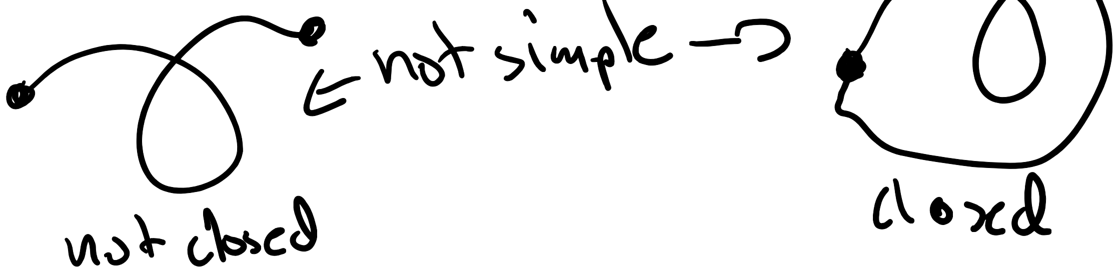
is the total flow rate of fluid along C

flow along $C = \int_C \vec{F} \cdot \vec{T}\, ds$

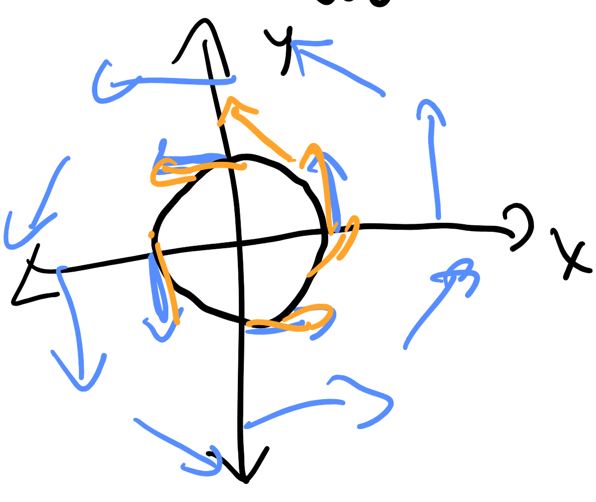
- If C is <u>closed</u> this is called <u>circulation</u>



- C is <u>simple</u> if it doesn't intersect itself



<u>ex</u>: Find the circulation of the velocity field $\vec{F}(x,y) = \langle -y, x \rangle$ cm/s around the unit circle, parameterized CCW.



1) Identify equation

$$\text{flow} = \int_C \vec{F} \cdot \vec{T}\, ds$$

2) Parameterize C

$\vec{r}(t) = \langle \cos(t), \sin(t) \rangle \quad 0 \le t \le 2\pi$

$\vec{r}\,'(t) = \langle -\sin(t), \cos(t) \rangle$

3) Plug In:

$$\int_0^{2\pi} F(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

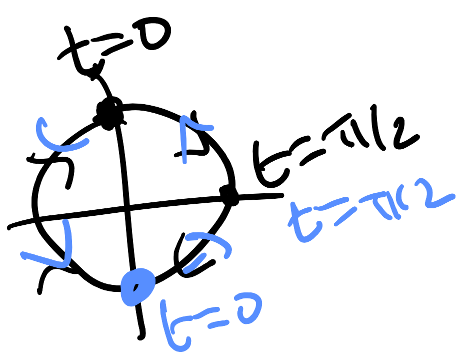
$$= \int_0^{2\pi} \langle -\sin(t), \cos(t) \rangle \cdot \langle -\sin(t), \cos(t) \rangle\, dt$$

$$= \int_0^{2\pi} \sin^2(t) + \cos^2(t)\, dt$$

$$= t \Big|_0^{2\pi} = \boxed{2\pi} \text{ cm}^2/\text{s}$$

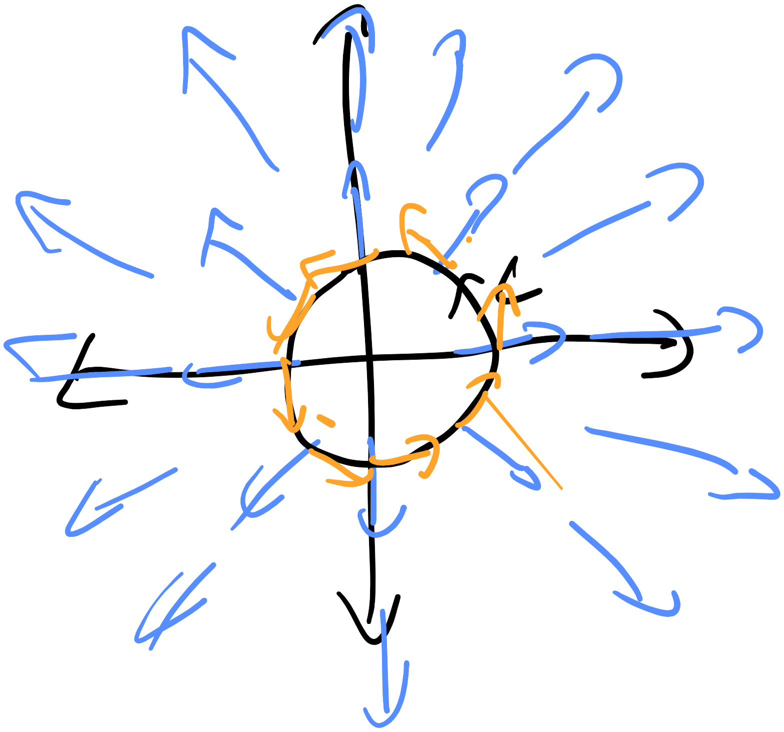
To go CW: $\vec{r}(t) = \langle \sin(t), \cos(t) \rangle$

$0 \le t \le 2\pi$



$\langle \sin(t), \cos(t) \rangle$

ex: What is the circulation of $\vec{F}(x,y) = \langle x, y \rangle$ around the unit circle?

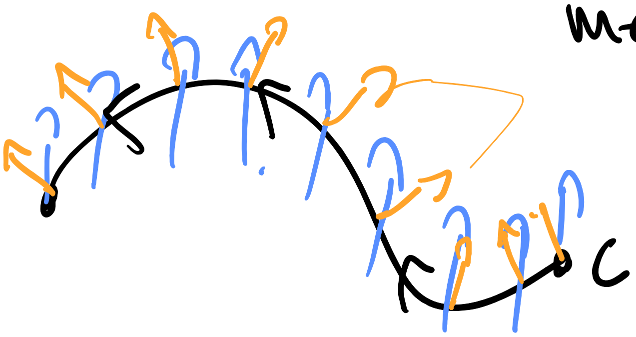


$\vec F \perp \vec T$ everywhere,

so $\vec F \cdot \vec T = 0$

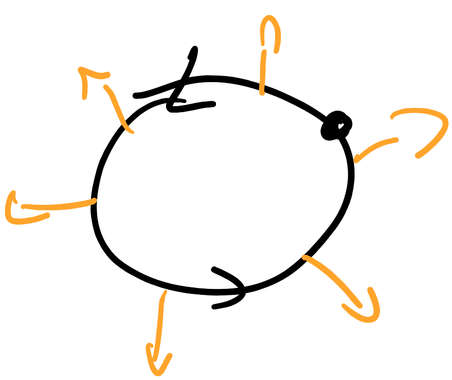
so $\int_C \vec F \cdot \vec T \, ds = 0$

- **Flux** across a plane curve of a vector field measures flow across the curve.



$$\int_C \vec F \cdot \vec n \, ds$$

outward (right-hand) unit normal



- if CCW: $\vec n = \vec T \times \vec k$

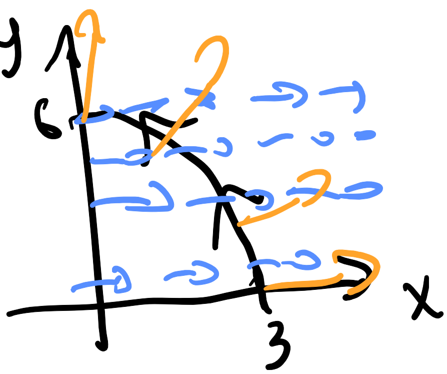
  CW: $\vec n = \vec k \times \vec T$

$$\vec n = \overset{\text{ccw}}{\underset{\text{cw}}{\pm}} \frac{\langle y'(t), x'(t) \rangle}{|\vec r\,'(t)|}$$

So $\int_C \vec{F}\cdot\vec{n}\,ds = \int_a^b F(\vec{r}(t))\cdot\langle -y'(t), x'(t)\rangle\,dt$

$$= \int_C Q\,dx - P\,dy$$

<u>ex</u>: Compute flux of $\vec{v} = \langle 3+2y-\frac{y^2}{3}, 0\rangle$ cm/s across the quarter ellipse



$\vec{r}(t) = \langle 3\cos(t), 6\sin(t)\rangle$

$0 \le t \le \pi/2$

$\vec{r}\,'(t) = \langle -3\sin(t), 6\cos(t)\rangle$

$$\text{flux} = \int_C \vec{v}\cdot\vec{n}\,ds = \int_0^{\pi/2} 0(-3\sin(t)) - \left(3+2(6\sin(t)) - \frac{(6\sin(t))^2}{3}\right)\cdot 6\cos(t)\,dt$$

$$= 30\ \text{cm}^2/\text{s}$$

MATH 2551 L - 11/17 - 16.3/16.4

<u>Reminder</u>: No lecture on Tues 11/22 - recording will be posted.
- virtual office hours 10am-noon on 11/22, link will be in a Canvas announcement.

<u>Last time</u>: Circulation/work/flux line integrals of vector fields

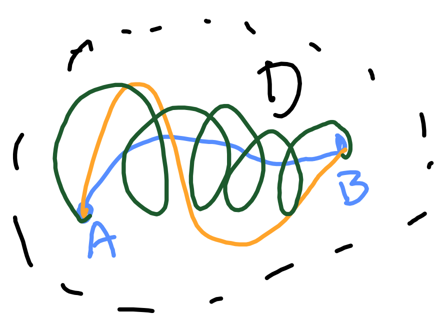
<u>Today</u>:
- Conservative vector fields
- Fundamental Theorem of Line Integrals
- Potential functions
- How is a vector field changing?

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C P\,dx + Q\,dy + R\,dz$$

$$\int_C \vec{F}\cdot\vec{n}\,ds = \int_C P\,dy - Q\,dx$$

<u>Def</u>: A vector field $\vec{F}$ is <u>conservative</u> on an open set D if the value of $\int_C \vec{F}\cdot d\vec{r}$ is the same for <u>any</u> path C from A to B in D.

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r} = \int_{C_3} \vec{F}\cdot d\vec{r}$$



ex: $\vec{F} = \langle x, y \rangle$ is conservative

$f(x,y) = \frac{x^2+y^2}{2}$ so that $\nabla f = \langle x, y \rangle$

Fundamental Theorem of Line Integrals

If C is a path from A to B, then

$$\int_C \nabla f \cdot d\vec{r} = f(B) - f(A)$$

FTC: $\int_a^b f'(x)\,dx = f(b) - f(a)$

- $\Rightarrow$ All gradient vector fields are conservative

Why? $\int_C \nabla f \cdot d\vec{r} = \int_a^b \underbrace{\nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)}_{\text{chain rule}}\,dt = \int_a^b \frac{d}{dt}\left(f(\vec{r}(t))\right) dt$

$\Updownarrow$ FTC

$= f(\vec{r}(b)) - f(\vec{r}(a))$

$= f(B) - f(A)$

$\vec{r}(t)$ parameterizes C

$a \leq t \leq b$

- $\vec{F}$ is conservative $\iff$ $\vec{F} = \nabla f$ for some f
  - f called potential function for $\vec{F}$

- If $\vec{F}$ is conservative: what is $\oint_C \vec{F} \cdot \vec{T}\,ds$?

$= 0$

b/c $= f(B) - f(A)$
and $B = A$

ex: Last time, $\oint_C \vec{F}\, d\vec{r} = 2\pi$, for
$\vec{F} = \langle -y, x \rangle$, $C =$ unit circle.
so $\vec{F}$ is not conservative and there is no
$f(x,y)$ such that $\vec{F} = \nabla f$.

If $\nabla f = \vec{F}$: $f_x = -y \Rightarrow f(x,y) = \int -y\, dx$
$= -xy + g(y)$

$f_y = x \Rightarrow f(x,y) = \int x\, dy$
$= xy + h(x)$

$f_{xy} = -1$

$f_{yx} = 1$

Not possible

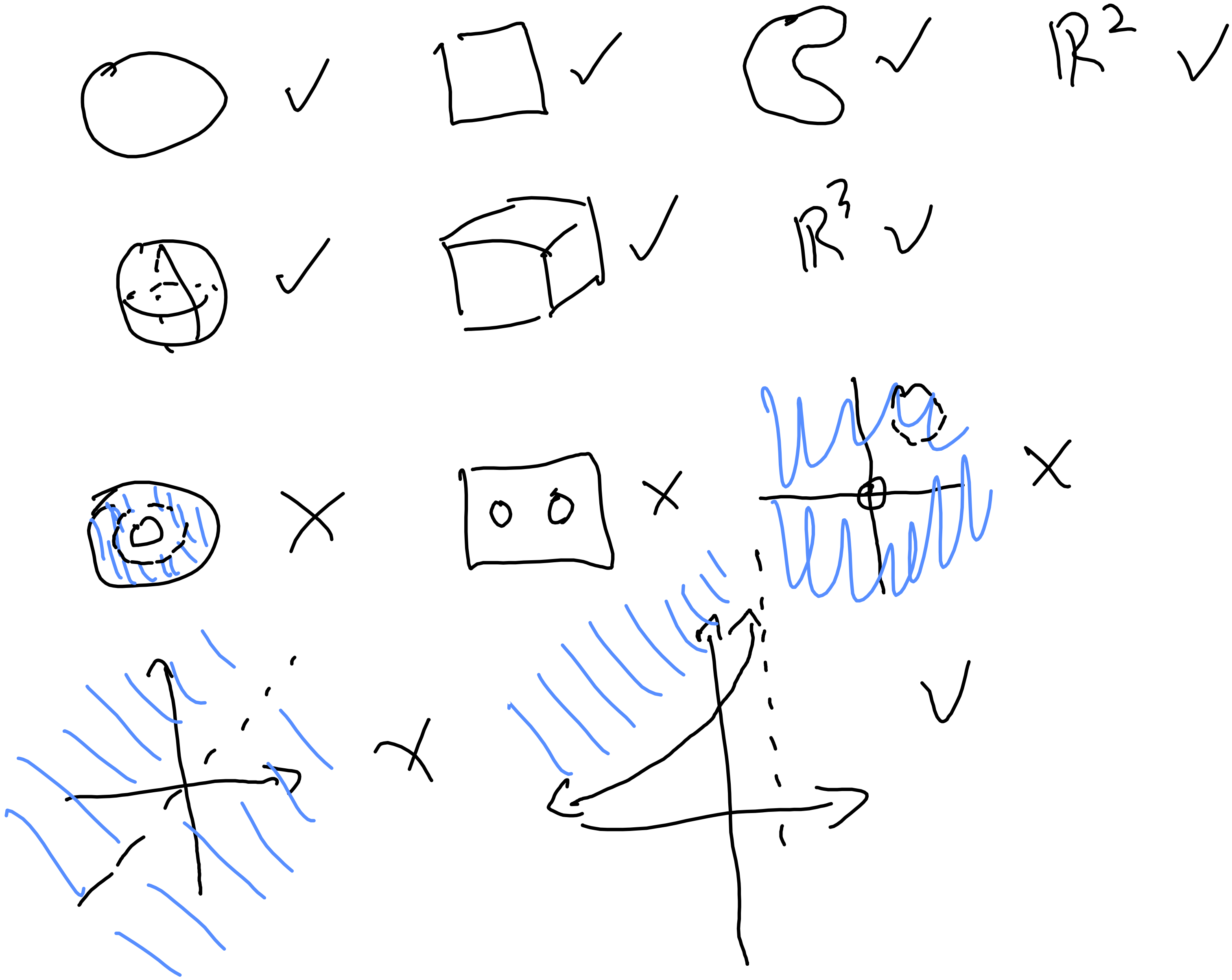
$f_x \quad f_y \quad f_z$

Mixed Partials Test: $\vec{F} = \langle P, Q, R \rangle$

$\vec{F} = \nabla f \iff P_z = R_x$ and $Q_z = R_y$ and $P_y = Q_x$

$f_{xz} \qquad f_{zx}$

as long as our domain is simply-connected
no holes



ex: Find a potential for $\vec{F} = \langle \underbrace{6x^2 - 2xy^2 + \frac{y}{2\sqrt{x}}}_{f_x},\ \underbrace{-2x^2y + 4 + \sqrt{x}}_{= f_y} \rangle$

on $\{(x,y) \mid x > 0\}$

$P_y = -4xy + \frac{1}{2\sqrt{x}}$ $\qquad$ $Q_x = -4xy + 0 + \frac{1}{2\sqrt{x}}$

1) Pick a variable and take antiderivative

In $\mathbb{R}^3$ this step gives $g(x,z)$

$$f(x,y) = \int f_y(x,y)\,dy = \int -2x^2y + 4 + \sqrt{x}\,dy$$

$$= \underline{-x^2y^2 + 4y + \sqrt{x}\,y + g(x)}$$

2) Take other partial derivative & compare

$$6x^2 - \cancel{2xy^2} + \cancel{\frac{y}{2\sqrt{x}}} = f_x = \cancel{-2xy^2} + 0 + \cancel{\frac{y}{2\sqrt{x}}} + g'(x)$$

$$6x^2 = g'(x)$$

3) Take antiderivative:

$$g(x) = \int 6x^2\,dx = 2x^3 + C$$

So potential function = $\boxed{-x^2y^2 + 4y + \sqrt{x}\,y + 2x^3 + C}$

<u>How do we measure change of a vector field?</u>

1) <u>Divergence</u> (flux density)

- measures expansion (+) / compression (−)
- inst. rate of change of strength of field in

direction of flow

$$-\operatorname{div}\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \qquad \text{(\underline{scalar})}$$

$$= \underbrace{\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle}_{\nabla} \cdot \vec{F} = \nabla \cdot \vec{F}$$

2) <u>Curl</u> (circulation density)

– measures how a vector field twists

$$\operatorname{curl} F = \nabla \times \vec{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

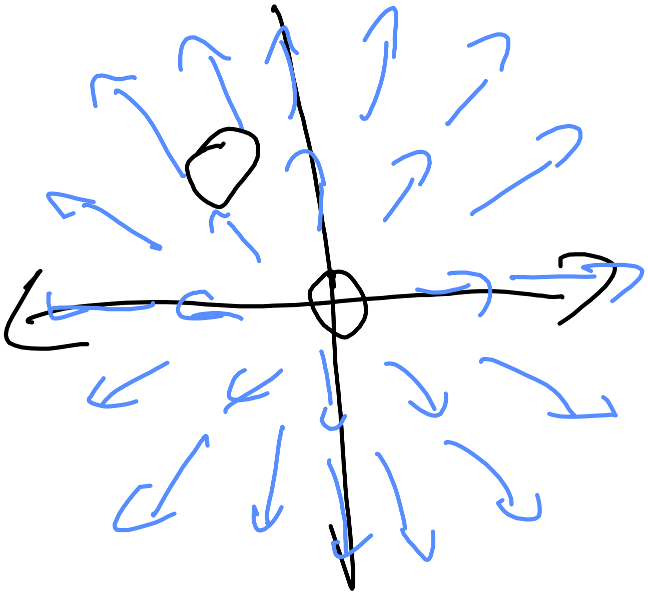
- RHR direction of axis of spin

$|\operatorname{curl} F| =$ rate of spin

$$\text{If } F(x,y) = \langle P, Q \rangle \to \operatorname{curl} F = \nabla \times \langle P, Q, 0 \rangle$$
$$= \langle 0, 0, Q_x - P_y \rangle$$

$$\text{Scalar curl} = \operatorname{curl}\vec{F} \cdot \vec{k} = Q_x - P_y$$

<u>ex</u>: $\vec{F} = \langle x, y \rangle$



$$\operatorname{div} F > 0$$

$$\operatorname{div} F = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y)$$

$$= 2$$

$$\operatorname{curl} F \cdot \vec{k} = \frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)$$

$$= 0 - 0 = 0$$

MATH 2551 K12 – 11/22 – More Curl & Divergence

1) Divergence (flux density)

∴ measures how much the field is expanding/compressing

$$\Rightarrow \operatorname{div}\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \quad \text{(scalar)}$$

$$= \underbrace{\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle}_{\nabla} \cdot \vec{F} = \nabla \cdot \vec{F}$$

2) Curl (circulation density / spin)

$$\operatorname{curl}\vec{F} = \nabla \times \vec{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

if $\vec{F}(x,y) = \langle P, Q \rangle$

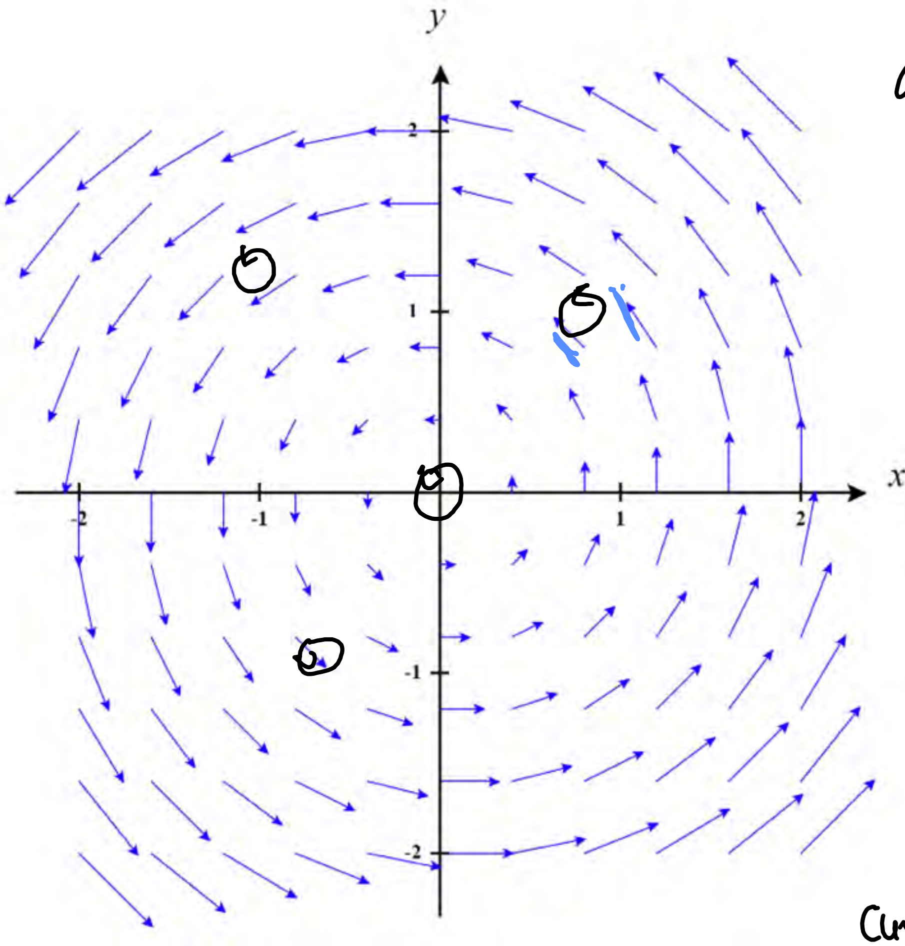
$$\rightarrow \operatorname{curl}\vec{F} = \nabla \times \langle P, Q, 0 \rangle$$

$$= \langle 0, 0, Q_x - P_y \rangle$$

scalar curl: $\operatorname{curl}\vec{F} \cdot \vec{k} = Q_x - P_y$

- $\operatorname{curl}\vec{F}$ points RH rule direction of axis of spin.
- $|\operatorname{curl}\vec{F}|$ tells us spin rate

Ex: $\vec{F}(x,y) = \langle -y, x \rangle$. Use the picture to decide if $\operatorname{div}\vec{F}$ and $\operatorname{curl}\vec{F}\cdot\vec{k}$ are $+, -, 0$, then confirm with the formula.



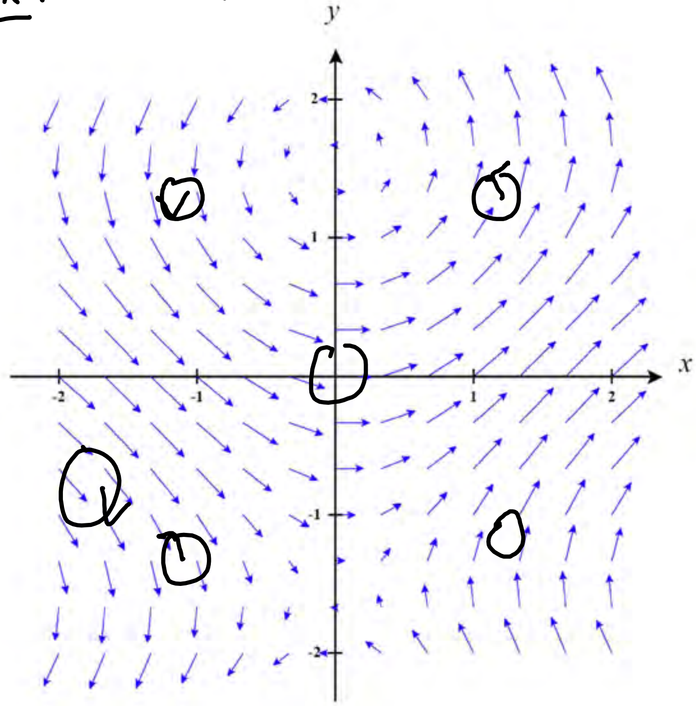
Expect: $\operatorname{div}\vec{F} = 0$

$\operatorname{curl}\vec{F}\cdot\vec{k} > 0$

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x)$$
$$= 0 + 0$$
$$= 0$$

$$\operatorname{curl}\vec{F}\cdot k = \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)$$
$$= 1 - (-1)$$
$$= 2$$

Ex: $\vec{F}(x,y) = \langle \cos(y), \sin(x) \rangle$. Again, use the picture to decide if div F, curl F are +, −, 0 and use formulas to confirm.



Expect:

div $\vec{F} = 0$

curl $\vec{F} \cdot \vec{k} = ?$

$$\text{div}\,\vec{F} = \frac{\partial}{\partial x}(\cos(y)) + \frac{\partial}{\partial y}(\sin(x)) = 0 + 0 = 0$$

$$\text{curl}\,\vec{F} \cdot \vec{k} = \frac{\partial}{\partial x}(\sin(x)) - \frac{\partial}{\partial y}(\cos(y))$$

$$= \cos(x) + \sin(y)$$

MATH 2551 - K/L - 11/22 - Green's Theorem

Last time: We saw two measures of how vector fields change

$$\text{div}\,\vec{F} = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \quad \text{(flux density)}$$

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle \quad \text{(spin)}$$

Today: Relate rates of change of vector field inside a region to vector field on boundary.

- If $\vec{F}(x,y) = \langle P, Q \rangle$, $\text{curl}\,\vec{F} \cdot \vec{k} = Q_x - P_y$ measures circulation density

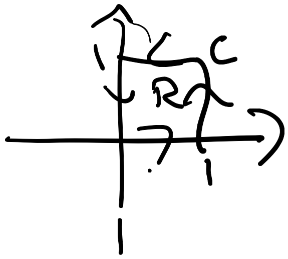
Green's Thm: Suppose C is a piecewise smooth, simple, closed curve enclosing on its left a region R in the plane. If $\vec{F} = \langle P, Q \rangle$ has continuous partial derivatives around R. Then

a) $\displaystyle\oint_C \vec{F} \cdot \vec{T}\, ds = \int_C P\,dx + Q\,dy = \iint_R \text{curl}\,\vec{F} \cdot \vec{k}\, dA = \iint_R Q_x - P_y\, dA$

b) $\displaystyle\oint_C \vec{F} \cdot \vec{n}\, ds = \int_C P\,dy - Q\,dx = \iint_R \text{div}\,\vec{F}\, dA = \iint_R P_x + Q_y\, dA$

"integrating circulation/flux density over the inside of a region gives the net circulation/flux on boundary"

ex: Evaluate line integral $\displaystyle\oint_C xy\,dy - y^2\,dx$ on the square bounded by $x=0, x=1, y=0, y=1$.



- Can use either form:
  - If $\vec{F}=\langle -y^2, xy\rangle$, this is $\oint_C \vec{F}\cdot\vec{T}\,ds$
  - If $\vec{F}=\langle xy, y^2\rangle$, this is $\oint_C \vec{F}\cdot\vec{n}\,ds$

$$\oint_C xy\,dy - y^2\,dx = \iint_R \frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(-y^2)\,dA$$

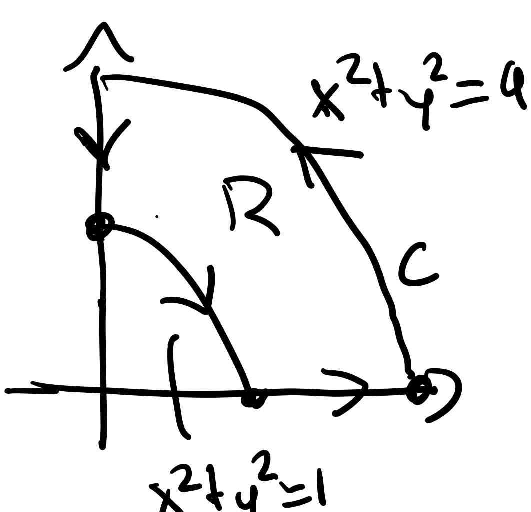
OR

$$\iint_R \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(y^2)\,dA$$

$$= \int_0^1 \int_0^1 y+2y\,dy\,dx$$

$$= \int_0^1 \frac{3}{2}y^2\Big|_0^1 dx = \frac{3}{2}x\Big|_0^1 = \boxed{\frac{3}{2}}$$

ex: Compute flux out of the region R for $\vec{F}=\langle \frac{1}{3}x^3, \frac{1}{3}y^3\rangle$.



$$\text{flux} = \oint_C \vec{F}\cdot\vec{n}\,ds$$

$$= \oint_C P\,dy - Q\,dx$$

$$= \iint_R \operatorname{div}\vec{F}\,dA$$

$$= \iint_R x^2+y^2\,dA$$

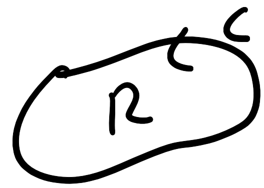
$$= \int_0^{\pi/2}\int_1^3 r^2 \cdot r \, dr \, d\theta$$

$$= \int_0^{\pi/2} \frac{1}{4} r^4 \Big|_1^3 \, d\theta$$

$$= 20\theta \Big|_0^{\pi/2}$$

$$= \boxed{10\pi}$$

Green's Thm and Area



If $\vec{F} = \frac{1}{2}\langle x, y\rangle$ then

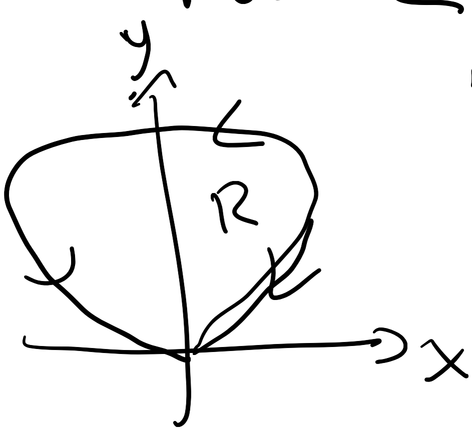
$$\oint_C \vec{F}\cdot\vec{n}\, ds = \boxed{\oint_C \frac{1}{2}(x\,dy - y\,dx)}$$

$$= \frac{1}{2}\iint_R 1+1 \, dA = \iint_R 1 \, dA$$

$= \text{area of } R$

ex: Let R be bounded by

$\vec{r}(t) = \langle \sin(2t), \sin(t)\rangle, \quad 0 \le t \le \pi$



Find area of R.

$$\text{area of } R = \frac{1}{2}\int_0^{\pi} \sin(2t)\cos(t) - \sin(t)\, 2\cos(2t)\, dt$$

$$= \frac{1}{2}\int_0^{\pi} 2\sin(t)\cos^2(t) - \sin(t)(4\cos^2(t)-2)\,dt$$

$$= \frac{1}{2}\int_{-1}^{1} 2u^2 - (4u^2-2)\,du$$

$$= \int_{-1}^{1} 1-u^2\,du$$

$$= \boxed{\frac{4}{3}}$$

MATH 2551 - K1L - 11/22 - Part 2 - 16.5

<u>Goal</u>: Describe surfaces in $\mathbb{R}^3$ parametrically in order to compute integrals on surfaces.

Curves

- <u>Explicit</u>: $y = f(x)$

  e.g. $y = \sin(x)$

  - familiar

- <u>Implicit/Level Curve</u>: $F(x,y) = 0$

  e.g. $F(x,y) = \sin(x) - y = 0$

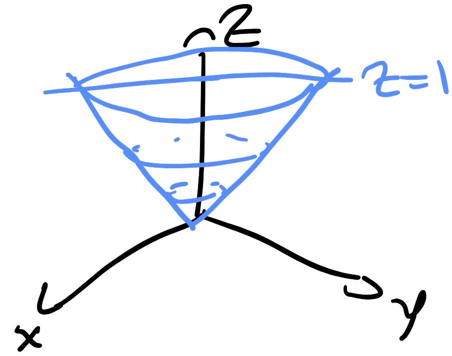
  - sometimes solving for explicit form is impossible

- <u>Parametric Vector Form</u>: $\vec{r}(t) = \langle f(t), g(t) \rangle$
  $a \le t \le b$

  e.g. $\vec{r}(t) = \langle t, \sin(t) \rangle$

  - gives orientation
  - use for arc length, curvature, line integrals

Surfaces



$z = r$

- <u>Explicit form</u>: $z = f(x,y)$

  e.g. $z = f(x,y) = \sqrt{x^2+y^2}$

- **Implicit / Level Surface form:** $F(x,y,z)=0$

  e.g. $F(x,y,z)=\dfrac{\sqrt{x^2+y^2}}{z}-1=0$

- **Parametric Vector Form** $\vec{r}(u,v)=\langle f(u,v),\ g(u,v),\ h(u,v)\rangle$

  e.g. $\vec{r}(u,v)=\langle u,\ v,\ f(u,v)\rangle=\langle u, v, \sqrt{u^2+v^2}\rangle$

  $u^2+v^2\le 1$

  e.g. $\vec{r}(r,\theta)=\langle r\cos\theta,\ r\sin\theta,\ r\rangle,\quad 0\le r\le 1,\quad 0\le\theta\le 2\pi$

ex: Give parametric representation for the surfaces

a) $x=5y^2+2z^2-10$

b) $x=5y^2+2z^2-10$ that is in front of yz-plane

c) $x^2+y^2+z^2=9$

d) $x^2+y^2=25$

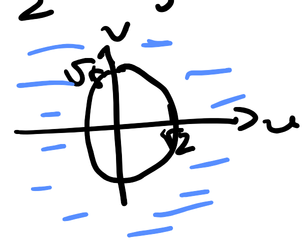
a) $\vec{r}(u,v)=\langle 5u^2+2v^2-10,\ u,\ v\rangle \qquad u,v\in\mathbb{R}$

b) $\vec{r}(u,v)=\langle 5u^2+2v^2-10,\ u,\ v\rangle,$

$5u^2+2v^2-10>0$

$5u^2+2v^2>10$

$\dfrac{u^2}{2}+\dfrac{v^2}{5}>1$



c) In spherical coords: $\rho=3$

$\vec{r}(\phi,\theta)=\langle 3\sin\phi\cos\theta,\ 3\sin\phi\sin\theta,\ 3\cos\phi\rangle$

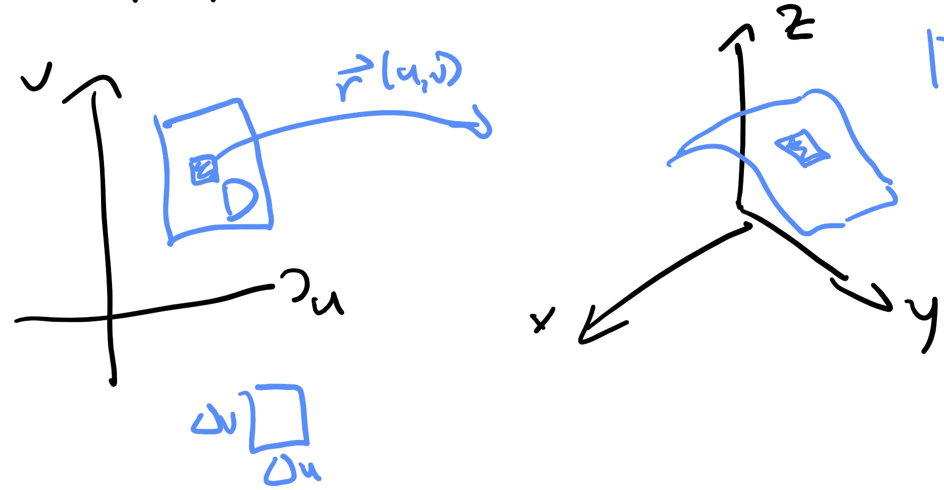
$0\le\phi\le\pi,\quad 0\le\theta\le 2\pi$

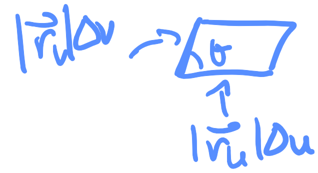
d) In cylindrical coords: $r=5$

$$\vec{r}(\theta,z) = \langle 5\cos\theta,\ 5\sin\theta,\ z\rangle \qquad 0\le\theta\le 2\pi,\ \ z\in\mathbb{R}$$

Why?

- Computing surface area
- $\vec{r}(u,v)$ to be smooth ($\vec{r}_u, \vec{r}_v$ not parallel inside domain)





Area: $|\vec{r}_u||\vec{r}_v|\sin\theta\,\Delta u\Delta v$

$|\vec{r}_u\times\vec{r}_v|\,\Delta u\Delta v$

- $\vec{n} = \vec{r}_u\times\vec{r}_v$ is normal to surface
- $\text{Area} = \iint_D |\vec{r}_u\times\vec{r}_v|\,dA$

ex: Compute the area of the portion of the cylinder $x^2+y^2=25$ between $z=0$ and $z=1$.

$$\vec{r}(\theta,z) = \langle 5\cos\theta, 5\sin\theta, z\rangle,\ \underbrace{0\le\theta\le 2\pi,\ 0\le z\le 1}_{D}$$

Area: $\iint_D |\vec{r}_\theta\times\vec{r}_z|\,dA$

$\vec{r}_\theta = \langle -5\sin\theta, 5\cos\theta, 0\rangle$

$\vec{r}_z = \langle 0, 0, 1\rangle$

$= \iint_D |\langle 5\cos\theta, 5\sin\theta, 0\rangle|\,dA$

$$= \int_0^{2\pi} \int_0^1 5 \, dz \, d\theta$$

$$= \int_0^{2\pi} 5z \Big|_0^1 \, d\theta = 5\theta \Big|_0^{2\pi} = \boxed{10\pi}$$

MATH 2551 L - 11/29 - 16.6

- Quiz 9 - circles are <u>not</u> straight lines!

  $\vec{r}(t) = \overrightarrow{P_1P_2}\, t + \overrightarrow{OP_1}$ is <u>only</u> a parameterization of a line.

  $\vec{r}(t) = \langle a\cos(t), a\sin(t)\rangle$

  a = radius

- Final Exam info on Canvas

- Fill out CIOS, currently at 33%

<u>Today</u>: - more surface parameterizations

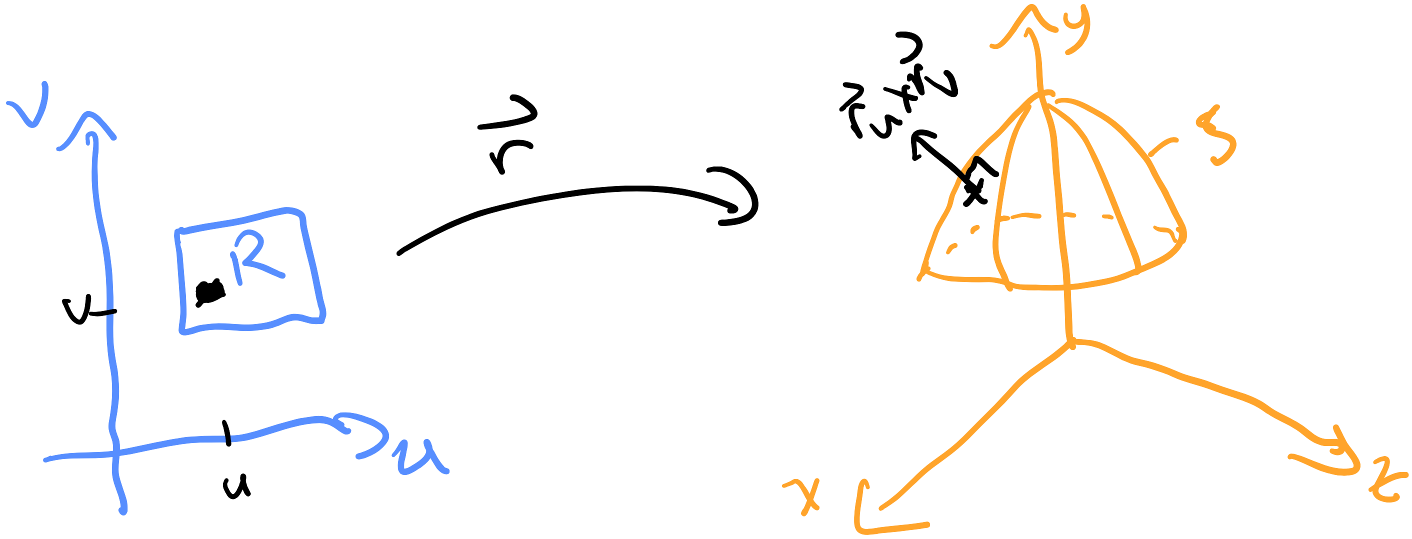
- surface integrals of scalar functions
- flux through a surface

<u>Last time</u>: Discussed parameterizing surfaces in $\mathbb{R}^3$

$\vec{r} : \mathbb{R}^2 \to \mathbb{R}^3$

$$\vec{r}(u,v) = \langle \underbrace{x(u,v), y(u,v), z(u,v)}_{\text{point on surface } S \text{ in } \mathbb{R}^3} \rangle$$

$\uparrow$ point in uv-plane



- Nice parameterizations
  - $\vec{r}_u, \vec{r}_v$ exist, cts, everywhere in R
  - $\vec{r}_u \times \vec{r}_v$ should be consistently/continuously oriented

- if S is $z = f(x, y)$ with domain R in xy-plane
  $(x = f(y, z))$
  then $\vec{r}(x, y) = \langle x, y, f(x, y) \rangle$, domain R
  is a parameterization of S

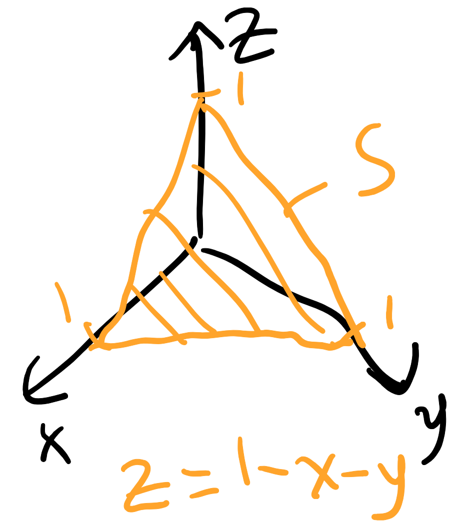
- if S is part of a sphere/cone/cylinder/paraboloid, use cylindrical/spherical coords.

- $\iint_R |\vec{r}_u \times \vec{r}_v| \, dA$ = surface area of S

<u>Surface Integral</u> of $f(x,y,z)$ over S

$$\iint_S f(x,y,z)\, \underset{dS}{d\sigma} = \iint_R f(\vec{r}(u,v)) \underbrace{|\vec{r}_u \times \vec{r}_v| \, dA}_{\text{surface area differential}}$$

ex: Suppose density of a plate S in the shape of portion of plane $x+y+z=1$ in the 1st octant is $\rho(x,y,z) = 6xy$. Find mass of S.

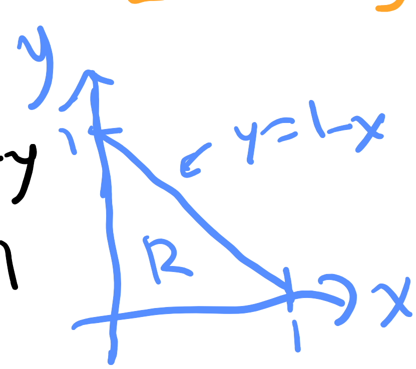


$$\text{mass} = \iint_S 6xy \, d\sigma$$

1) Parameterize S

$$\vec{r}(x,y) = \langle x, y, 1-x-y \rangle$$

Domain R: $0 \le y \le 1-x$, $0 \le x \le 1$ | $0 \le x \le 1-y$, $0 \le y \le 1$



$\vec{r}_x = \langle \frac{\partial}{\partial x}(x), \frac{\partial}{\partial x}(y), \frac{\partial}{\partial x}(1-x-y) \rangle = \langle 1, 0, -1 \rangle$

$\vec{r}_y = \langle 0, 1, -1 \rangle$

$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \langle 1, 1, 1 \rangle$

$|\vec{r}_x \times \vec{r}_y| = \sqrt{3}$

2) Plug in: $f(\vec{r}(x,y)) = 6xy$

3) Evaluate: $\text{mass} = \iint_S \rho(x,y,z)\, d\sigma =$

$= \iint_R \rho(\vec{r}(x,y))\, |\vec{r}_x \times \vec{r}_y|\, dA$

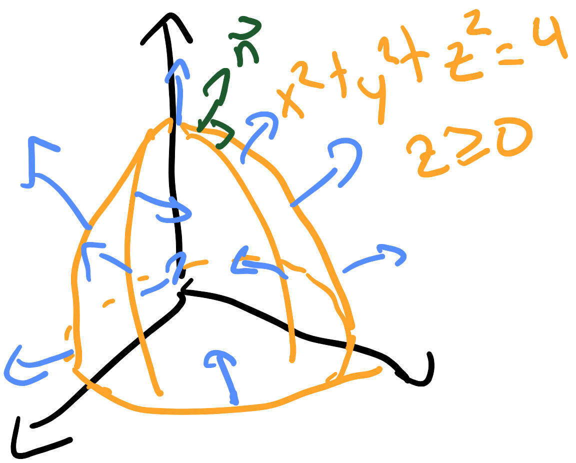
$= \int_0^1 \int_0^{1-x} 6xy\, \sqrt{3}\, dy\, dx$

$= \boxed{\sqrt{3}/4}$

Surface Integrals of Vector Fields $\vec{F}$

Goal: Find the flux of $\vec{F}$ through S

- $\vec{r}(u,v)$ paramaterizes S with domain R
- $\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$



$\vec{F}$

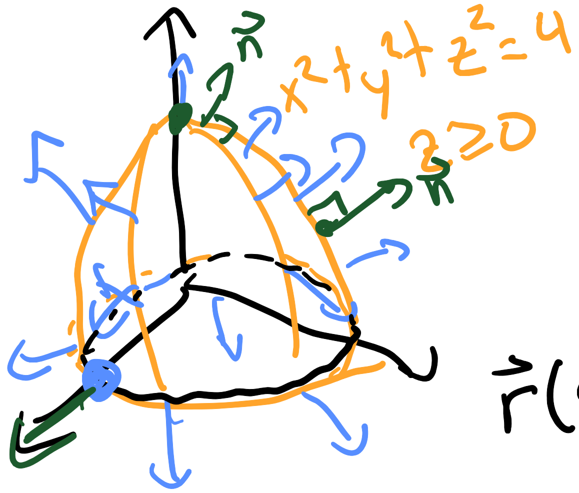
flux > 0 means field is moving through S in the $\vec{n}$ direction

$$\text{flux} = \iint_S (\vec{F} \cdot \vec{n}) \, d\sigma = \iint_R \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) \, dA$$

- $\vec{n}$ = unit normal to S

ex: Find the flux of $\vec{F} = \langle x,y,z \rangle$ through the upper hemisphere of $x^2+y^2+z^2=4$, oriented away from origin.

$\rho = 2$

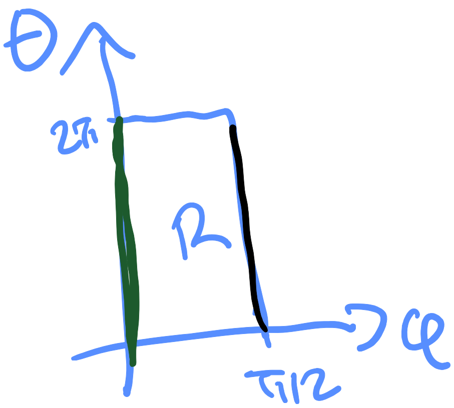


flux > 0

1) Parameterize S

$$\vec{r}(\varphi,\theta) = \langle 2\sin\varphi\cos\theta,\ 2\sin\varphi\sin\theta,\ 2\cos\varphi\rangle$$

$$0 \leq \varphi \leq \pi/2 \quad 0 \leq \theta \leq 2\pi$$



$$\vec{r}_\varphi = \langle 2\cos\varphi\cos\theta,\ 2\cos\varphi\sin\theta,\ -2\sin\varphi\rangle$$

$$\vec{r}_\theta = \langle -2\sin\varphi\sin\theta,\ 2\sin\varphi\cos\theta,\ 0\rangle$$

$$\vec{r}_\varphi \times \vec{r}_\theta = \langle 4\sin^2\varphi\cos\theta,\ 4\sin^2\varphi\sin\theta,\ 4\cos\varphi\sin\varphi\rangle$$

- check orientation $(\varphi,\theta) = (\pi/2, 0)$

$\vec{n}(\pi/2, 0) = \langle 4, 0, 0\rangle$

2) Plug In:

$$\vec{F} = \langle x, y, z\rangle$$

$$\vec{F}(\vec{r}(\varphi,\theta)) = \langle 2\sin\varphi\cos\theta,\ 2\sin\varphi\sin\theta,\ 2\cos\varphi\rangle$$

$$\vec{F}(\vec{r}(\varphi,\theta)) \cdot \vec{r}_\varphi \times \vec{r}_\theta = 8\sin\varphi$$

3) Evaluate: flux $= \iint_S (\vec{F} \cdot \vec{n})\, d\sigma$

$$= \iint_R \vec{F}(\vec{r}(\varphi, \theta)) \cdot (\vec{r}_\varphi \times \vec{r}_\theta)\, dA$$

$$= \int_0^{2\pi} \int_0^{\pi/2} 8 \sin \varphi \, d\varphi \, d\theta$$

$$= 16\pi$$

MATH 2551 L - 12/1 - 16.7/16.8

Reminders

- See Canvas announcement for final exam info
  - more practice problems posted in Sample Exams (this is the Studypalooza set)
- Fill out CIOS if you have not yet done so
  - Thanks to everyone who has!
  - Section L has 49% completion
- Last homeworks are due Tuesday

Today: Generalize Green's Theorem to $\mathbb{R}^3$ in two ways
- Stokes' theorem
- Divergence theorem

$\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$

Recall: $\vec{F} = \langle P, Q, R \rangle$ a vector field

divergence: $\text{div}\,\vec{F} = \nabla \cdot \vec{F} = P_x + Q_y + R_z$
- measures flux density / rate at which field strength is changing
- scalar

Curl: $\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$

- measures circulation density / how field rotates at a point
- vector, oriented in RHR direction of axis rotation

<u>Stokes' Thm:</u> Let S be a smooth oriented surface and C be its compatibly oriented boundary. Let $\vec{F}$ be a vector field with cts partial derivatives.

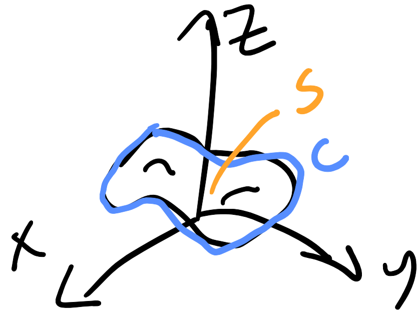
$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, d\sigma = \oint_C (\vec{F} \cdot \vec{T})\, ds$$

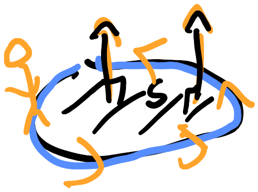
"flux of curl $\vec{F}$ across S = circulation of $\vec{F}$ along boundary"

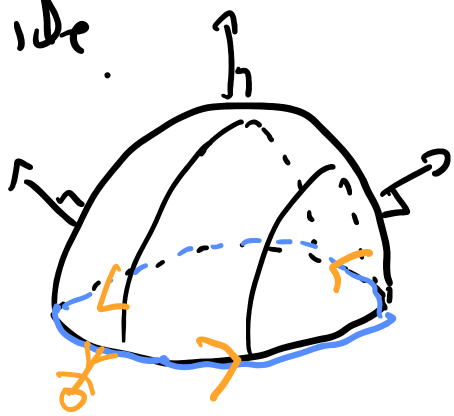
- If S = a region R in xy-plane, this is literally Green's Thm:

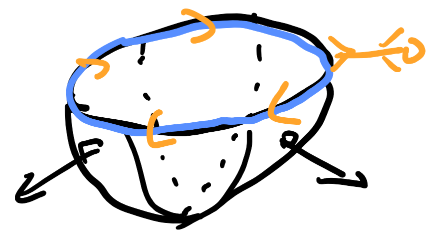
get $\iint_R (\nabla \times \vec{F}) \cdot \vec{k}\, dA$

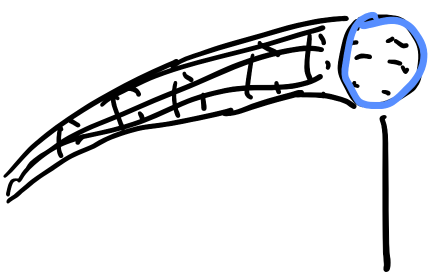
• S and C are oriented compatibly if walking along C with your head in the direction of normals to S has S on your left side.



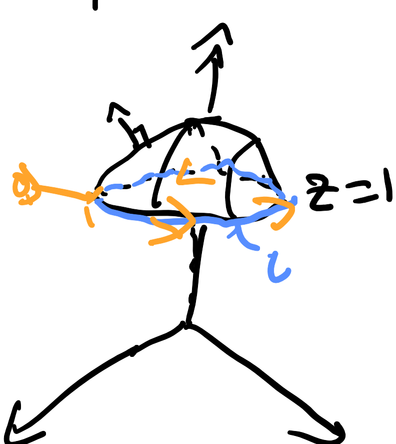








ex: $\vec{F} = \langle -y,\ x + \underline{(z-1)}\, x^{x\sin(x)},\ x^2+y^2 \rangle$. Find $\iint_S (\text{curl}\, F)\cdot\vec{n}\, d\sigma$ over S the part of the sphere $x^2+y^2+z^2=2$ above $z=1$, oriented outward.



<u>Option 1</u>: Do it directly.

$$\text{curl}\,\vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x+(z-1)x^{x\sin(x)} & x^2+y^2 \end{vmatrix}$$

$$= \left\langle —, —, \underbrace{\frac{\partial}{\partial x}\left(x + (z-1)\,x^{x\sin(x)}\right)}_{\text{NO}} \right\rangle$$

Option 2: Use Stokes' Thm

- S & C oriented compatibly ✓
- $\vec{F}$ has cts partials ✓

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, d\sigma = \int_C \vec{F} \cdot \vec{T}\, ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

- C is $x^2 + y^2 + 1^2 = 2$; $z = 1$

$$x^2 + y^2 = 1 \ ; \ z = 1$$

$$\vec{r}(t) = \langle \cos(t), \sin(t), 1 \rangle, \quad 0 \le t \le 2\pi$$

$$\vec{r}\,'(t) = \langle -\sin(t), \cos(t), 0 \rangle$$

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = \langle -\sin(t), \cos(t) + 0, \overset{1}{\cancel{\cos^2(t) + \sin^2(t)}} \rangle$$

$$\cdot \langle -\sin(t), \cos(t), 0 \rangle$$

$$= \sin^2(t) + \cos^2(t) + 0 = 1$$

$$= \int_0^{2\pi} 1\, dt = \boxed{2\pi}$$

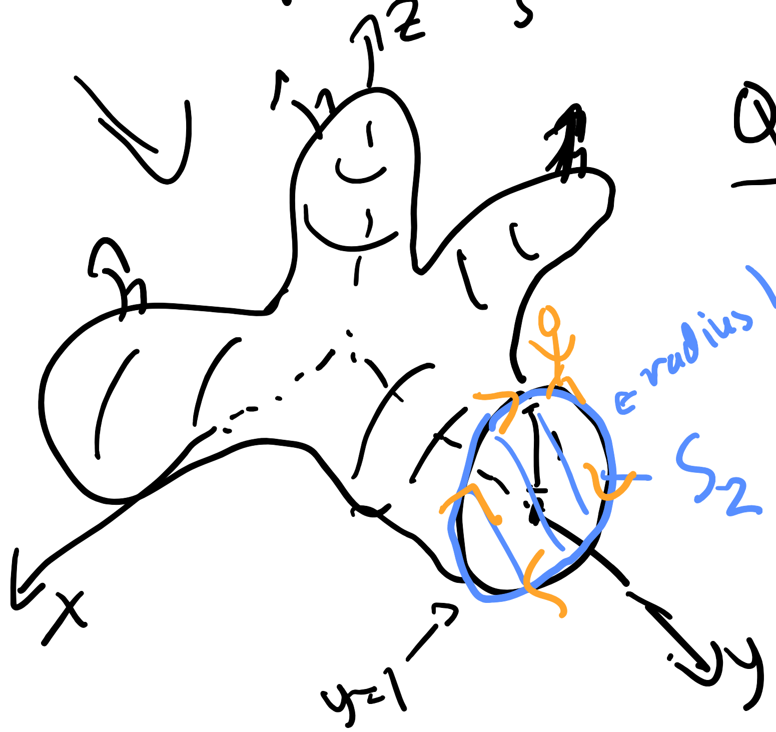
- If S_1 and S_2 have the same boundary C

then

$$\iint_{S_1} (\text{curl}\,\vec{F})\cdot\vec{n}\,d\sigma = \iint_{S_2} (\text{curl}\,\vec{F})\cdot\vec{n}\,d\sigma$$

ex: $\text{curl}\,\vec{F} = \langle y^y \sin(z^2),\ (y-1)e^{x+x+x}+2,\ ze^{x+xy} \rangle$

and compute $\iint_S (\text{curl}\,\vec{F})\cdot\vec{n}\,d\sigma$ over the surface

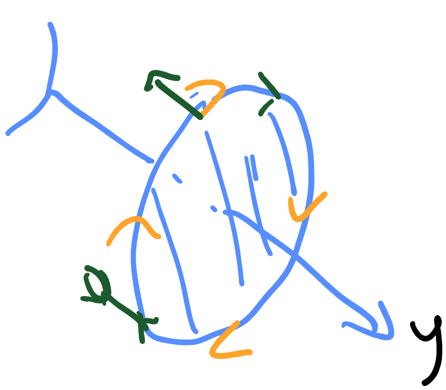


Option 1: Do it. No.

Option 2: Stokes Thm.

- Don't have $\vec{F}$
- How to get $\vec{F}$ from curl $\vec{F}$?

Option 3: Replace S with S_2



S_2 is a disk of radius 1 centered on y-axis in y=1 plane:

$\vec{n} = \langle 0, -1, 0 \rangle$ $y=1$

$\text{curl}\,\vec{F}\cdot\vec{n} = 0 - 2 + 0$

$$\iint_S \operatorname{curl}\vec{F}\cdot\vec{n}\,d\sigma = \iint_{S_2} \operatorname{curl}\vec{F}\cdot\vec{n}\,d\sigma = \iint_{S_2} -2\,d\sigma$$

$$= -2\cdot(\text{area of } S_2)$$

$$= \boxed{-2\pi}$$

<u>Divergence Thm:</u>

Let S be a <u>closed</u> surface, D the region inside S, oriented outward, and $\vec{F}$ = vector field with cts partials.

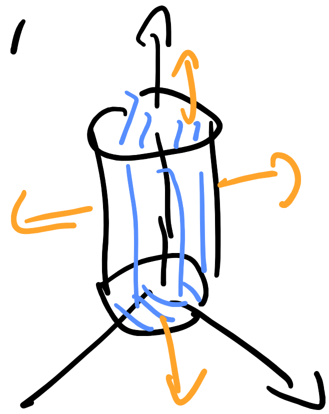
$$\iiint_D \nabla\cdot\vec{F}\,dV = \iint_S \vec{F}\cdot\vec{n}\,d\sigma$$

"the sum of flux into/out of each point inside S = net flux of $\vec{F}$ out of S

ex: $\vec{F} = \langle y^{123} e^{\sin(yz)},\ y - x^{z^x},\ z^2 - z \rangle$

S: cylinder $r=1$ from $z=0$, $z=3$ together with top/bottom disks, oriented outward

Find $\iint_S \vec{F}\cdot\vec{n}\, d\sigma$.



Apply Div Thm?

- closed? ✓
- F has cts partials ✓
- oriented outward ✓

$$\iint_S \vec{F}\cdot\vec{n}\, d\sigma = \iiint_D \nabla\cdot\vec{F}\, dV$$

$$= \int_0^{2\pi}\int_0^1\int_0^3 (0 + \cancel{1} + 2z - \cancel{1})\, r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^1 r\, dr \cdot \int_0^3 2z\, dz$$

$$= 2\pi \cdot \frac{1}{2} \cdot 9 = \boxed{9\pi}$$