MATH 2551 D - Dr. Hunter Lehmann

- Dr. Lehmann, Dr. H, Dr. Hunter, as you prefer

Daily Announcements & Reminders:

- Log into Canvas
- Meet your neighbors
- Practice Quiz 0 tomorrow in studio
- HW 12.2 & 12.3 due Th at 8pm

Class Values/Norms:

- Mistakes are a learning opportunity
- Mathematics is collaborative
- Make sure everyone is included
- Criticize ideas, not people
- Be respectful of everyone
- Ask for help
- Make sure your work is your own

Big Idea: Extend differential & integral calculus.

What are some key ideas from these two courses?

Differential Calculus

- Real-world application of derivatives
- optimization
- limits
- continuity
- related rates / Chain Rule

Integral Calculus

- area between curves
- infinite series
- FTC
- Riemann sums

name domain codomain

Before: we studied **single-variable functions** $f : \mathbb{R} \to \mathbb{R}$ like $f(x) = 2x^2 - 6$.

Now: we will study **multi-variable functions** $f : \mathbb{R}^n \to \mathbb{R}^m$: each of these functions is a rule that assigns one output vector with m entries to each input vector with n entries.

vector with n entries or n real #s

Common cases:

$n = 1, 2, 3$
$m = 1, 2, 3$

ex: linear transformation

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- $\vec{u}, \vec{v}$ in $\mathbb{R}^3 \to \vec{u} \times \vec{v}$
- $x^2 + y^2 + z^2 = 25$ (solutions to this are a sphere of radius 5)

Notation:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in \mathbb{R}^3 \text{ or } \langle 1, 2, 3 \rangle \text{ or } 1\vec{i} + 2\vec{j} + 3\vec{k}$$

- $f(x,y) = x^2 + y^2$
  $f(x,y,z) = x^2 + y^2 + z^2$
- (lat., long.) $\mapsto$ temp.
- electric field eqn

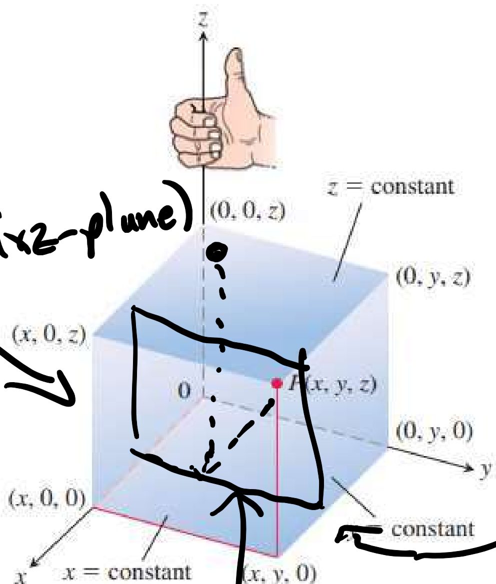
Section 12.1: Three-Dimensional Coordinate Systems

$\mathbb{R}^3$

- right-handed
- (x, y, z)

  $(2, 1, 3)$

$y=0$ (xz-plane)



$z=0$ (xy-plane)

$x=2$ is a plane parallel to yz-plane

Q: What shape is the set of solutions (x,y,z) to the equation $x^2+y^2=1$?

- cylinder (no top/bottom, ∞ height)

In $\mathbb{R}^2$:
solutions to $x^2+y^2=1$ are $\rightarrow$

- unit circle

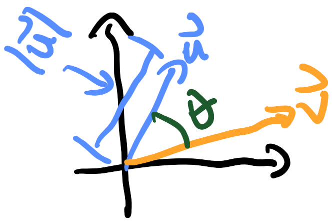
Section 12.3/4: Dot & Cross Products

Definition 1. The **dot product** of two vectors $\mathbf{u} = \langle u_1, u_2, \ldots, u_n \rangle$ and $\mathbf{v} = \langle v_1, v_2, \ldots, v_n \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n \leftarrow \text{a scalar}$$

This product tells us about the angle between two vectors.

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta \qquad |\vec{u}| = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2}$$



In particular, two vectors are **orthogonal** if and only if their dot product is 0.

ex: Are $\vec{u} = \langle 1, 1 \rangle$, $\vec{v} = \langle 2, -1 \rangle$ orthogonal?

$$\vec{u} \cdot \vec{v} = 1 \cdot 2 + 1 \cdot (-1) = 2 - 1 = 1$$

$$|\vec{u}| = \sqrt{1+1} = \sqrt{2}$$

$$|\vec{v}| = \sqrt{4+1} = \sqrt{5}$$

$$1 = \sqrt{2} \cdot \sqrt{5} \cos\theta$$

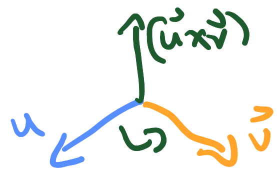
$$\cos\theta = \frac{1}{\sqrt{10}} \qquad \theta = \arccos\left(\frac{1}{\sqrt{10}}\right).$$

Daily Announcements & Reminders:

- HW 12.2, 12.3 due at 8 tonight
- No studio on M
- <u>History</u>: Dot & cross products originated in studying tetrahedra & quaternions (Lagrange & Hamilton)

Goal: Given two vectors, produce a vector orthogonal to both of them in a "nice" way.

1. Right-handed:



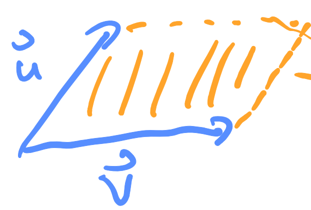
2. Scalars multiplication & addition should be nice: $(2\vec{u})\times\vec{v} = 2(\vec{u}\times\vec{v})$
$(\vec{u}+\vec{w})\times\vec{v} = (\vec{u}\times\vec{v})+(\vec{w}\times\vec{v})$

Definition 2. The **cross product** of two vectors $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ <u>in $\mathbb{R}^3$</u> is

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

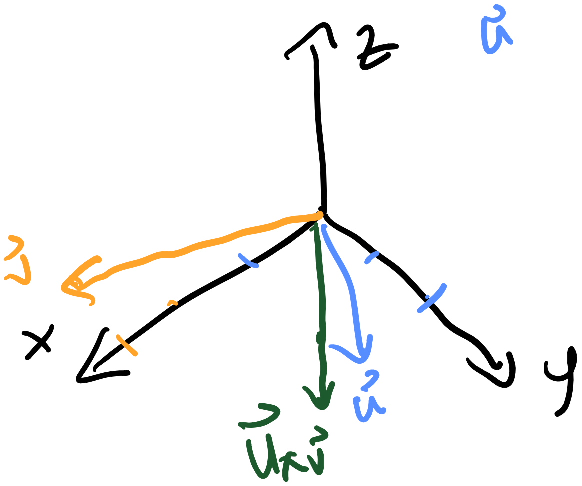
- anticommutative: $\vec{v}\times\vec{u} = -(\vec{u}\times\vec{v})$
- $|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta$

$\hookrightarrow$



Area is $|\vec{u}||\vec{v}|\sin\theta$

<u>ex</u>: Find $\underset{\vec{u}}{\langle 1,2,0\rangle} \times \underset{\vec{v}}{\langle 3,-1,0\rangle}$.



$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ 3 & -1 & 0 \end{vmatrix}$$

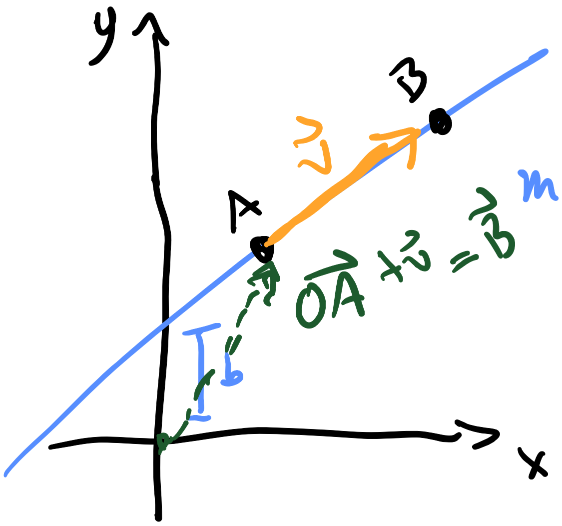
$$= (0-0)\vec{i} - (0-0)\vec{j} + (-1-6)\vec{k}$$

$$= -7\vec{k} = \langle 0,0,-7\rangle$$

$$= \begin{bmatrix} 0 \\ 0 \\ -7 \end{bmatrix}$$

Section 12.5 Lines & Planes

Lines in $\mathbb{R}^2$, a new perspective:



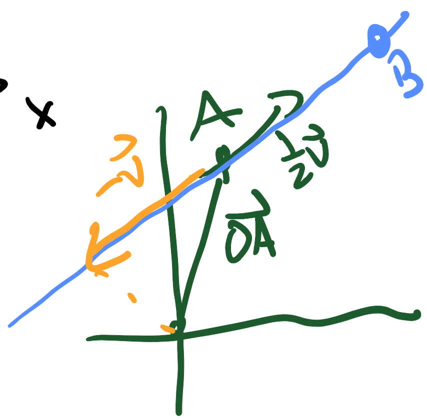
$y = mx + b$

m = slope

• Lines are determined by 1) 2 points or 2) 1 point & direction

$y = x$

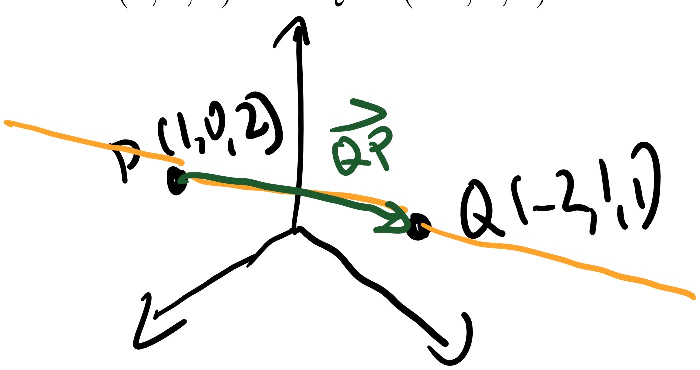
$2y = 2x$



vector equation

$$\vec{r}(t) = \overrightarrow{OP} + \vec{v} \cdot t$$

Example 3. Find a vector equation for the line that goes through the points $P = (1, 0, 2)$ and $Q = (-2, 1, 1)$.



$$\overrightarrow{OP} = \langle 1, 0, 2 \rangle$$

$$\overrightarrow{QP} = \langle -2-1, 1-0, 1-2 \rangle = \langle -3, 1, -1 \rangle = \text{"}\vec{v}\text{"}$$

$$\vec{r}(t) = \langle 1, 0, 2 \rangle + \langle -3, 1, -1 \rangle t$$

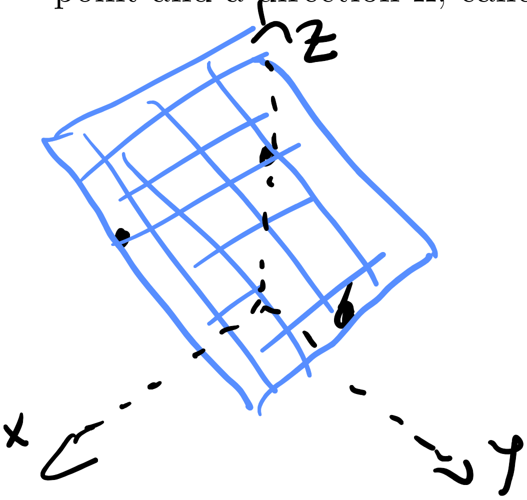
Parametric Equations for a Line:

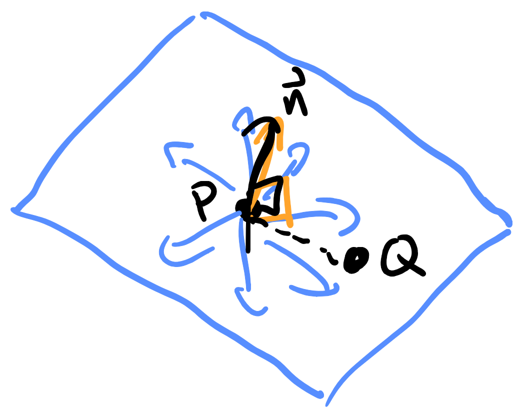
$$\vec{r}(t) = \langle 1, 0, 2\rangle + \langle -3, 1, -1\rangle t$$

$$\iff \quad x(t) = 1 - 3t$$
$$y(t) = 0 + t$$
$$z(t) = 2 - t$$

<u>Planes in $\mathbb{R}^3$</u>

Conceptually: A plane is determined by either three points in $\mathbb{R}^3$ or by a single point and a direction **n**, called the *normal vector*.





Algebraically: A plane in $\mathbb{R}^3$ has a *linear* equation (back to Linear Algebra! imposing a single restriction on a 3D space leaves a 2D linear space, i.e. a plane)

$$ax + by + cz = d$$

$$\vec{PQ} \cdot \vec{n} = 0$$

$$\vec{n} = \langle a, b, c\rangle$$

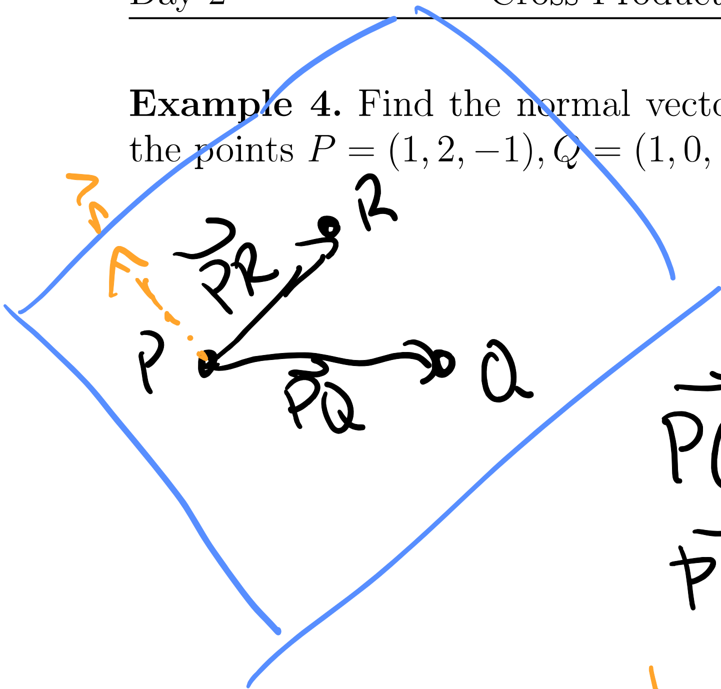
$$\langle x - x_0, y - y_0, z - z_0\rangle \cdot \langle a, b, c\rangle = 0$$

$$P = \langle x_0, y_0, z_0\rangle$$

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

$$Q = \langle x, y, z\rangle$$

Example 4. Find the normal vector and an equation for the plane that contains the points $P = (1, 2, -1), Q = (1, 0, -1)$, and $R = (0, 1, 3)$.



• Use crossproduct to get $\vec{n}$

$\vec{PQ} = \langle 0, -2, 0 \rangle$

$\vec{PR} = \langle -1, -1, 4 \rangle$

$$\vec{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -2 & 0 \\ -1 & -1 & 4 \end{vmatrix} = \langle -8-0, -(0-0), 0-2 \rangle$$

$$= \langle -8, 0, -2 \rangle$$

$$-8(x-1) + 0(y-2) + (-2)(z-(-1)) = 0$$

$$-8x + 8 - 2z - 2 = 0$$

$$8x + 2z = 6$$

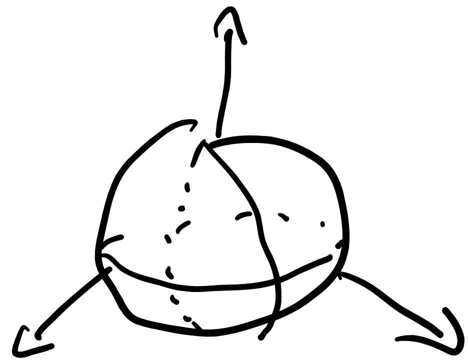
$$4x + z = 3$$

• Can use <u>any</u> point on plane

Section 12.6 Quadric Surfaces

Definition 5. A **quadric surface** in $\mathbb{R}^3$ is the set of points that solve a quadratic equation in x, y, and z.

You know several examples already:

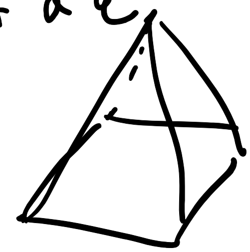


Spheres: $x^2+y^2+z^2=R^2$

- centered at $(0,0,0)$
- radius R

- $(x-1)^2+(y+2)^2+z^2=4$
  - centered at $(1,-2,0)$
  - radius is 2

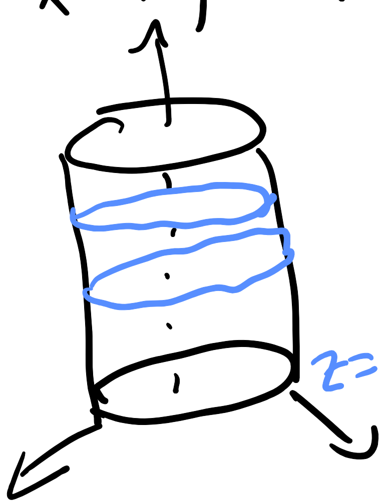
not a quadric surface



- 5 faces, each w/ diff. eqn
- each face is linear

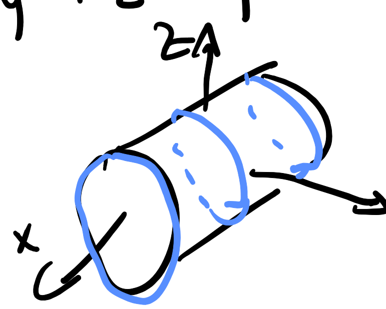
Circular cylinder:

e.g. $x^2+y^2=9$



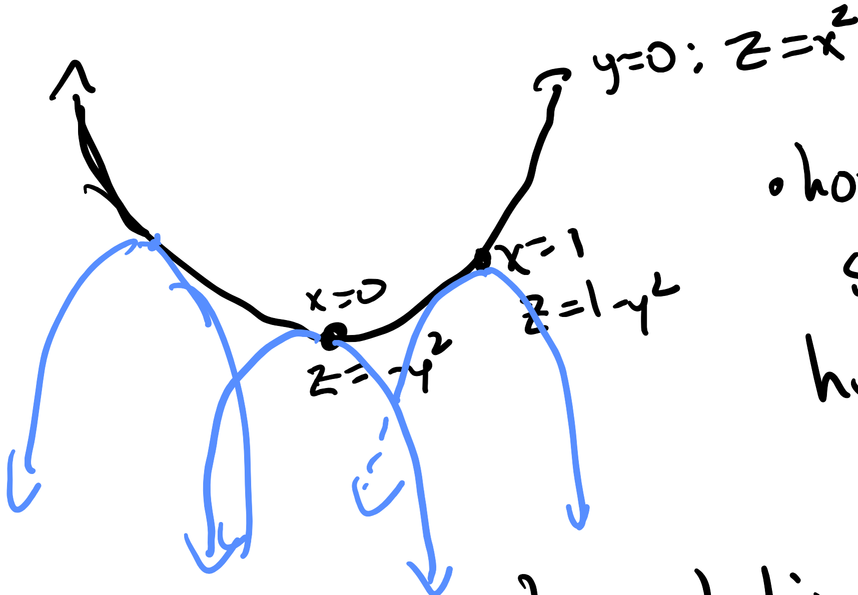
$z=0;\ x^2+y^2=9$

$y^2+z^2=9$



The most useful technique for recognizing and working with quadric surfaces is to examine their cross-sections.

Example 6. Use cross-sections to sketch and identify the quadric surface $z = x^2 - y^2$.



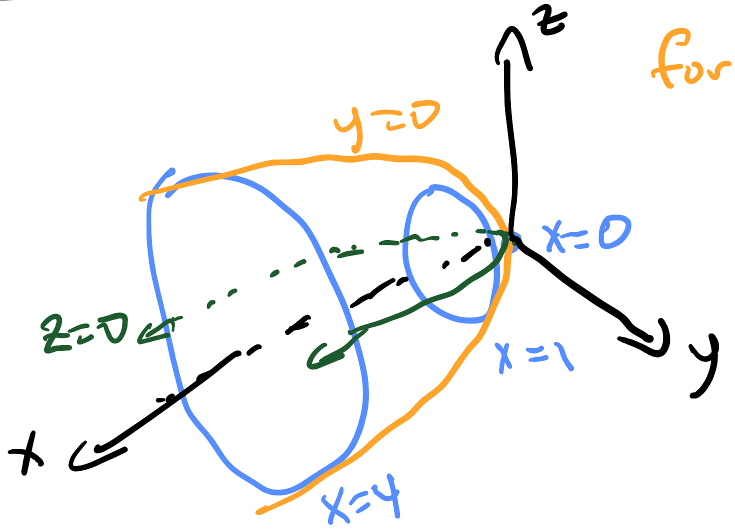
• horizontal slices are hyperbolas

hyperbolic paraboloid

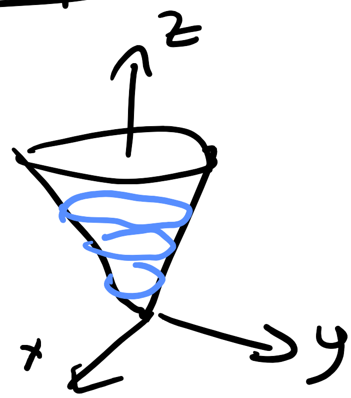
Example 7. Use cross-sections to sketch and identify the quadric surface $x = z^2 + y^2$.

Filled in after: for any $x = k$, we get a circle $k = z^2 + y^2$

for any $y = k$, or $z = k$, get a parabola

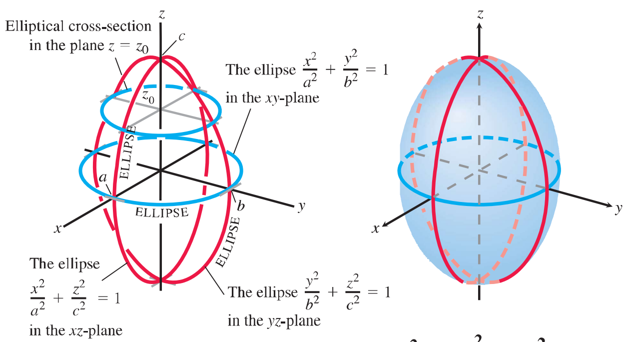


elliptic paraboloid

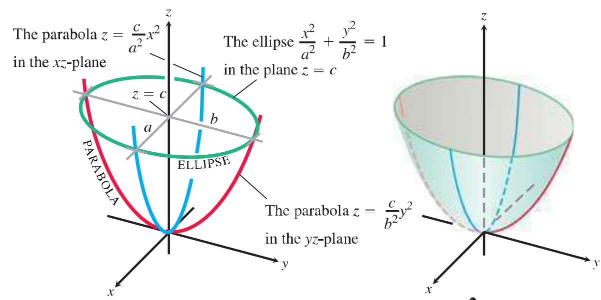


Cones: $z = a\sqrt{x^2 + y^2}$

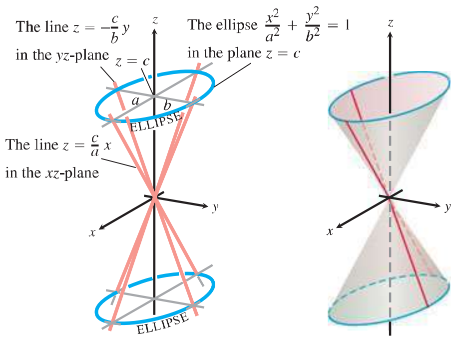
TABLE 12.1 Graphs of Quadric Surfaces



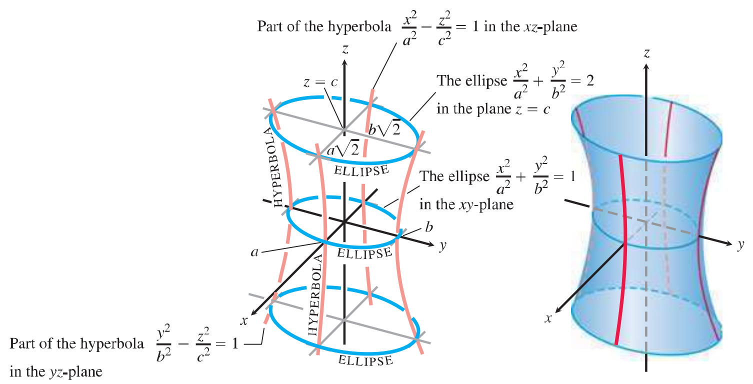
ELLIPSOID $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$



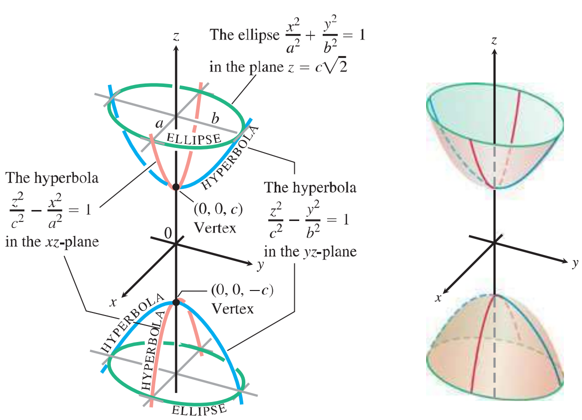
ELLIPTICAL PARABOLOID $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$



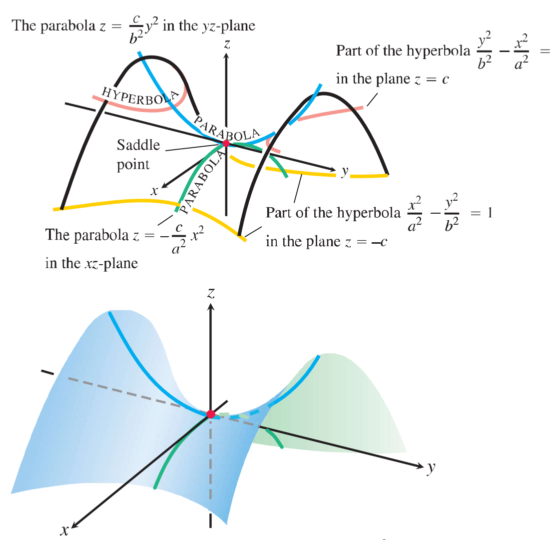
ELLIPTICAL CONE $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$



HYPERBOLOID OF ONE SHEET $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$



HYPERBOLOID OF TWO SHEETS $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



HYPERBOLIC PARABOLOID $\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}, c > 0$

Section 13.1 Curves in Space & Their Tangents

Daily Announcements & Reminders:

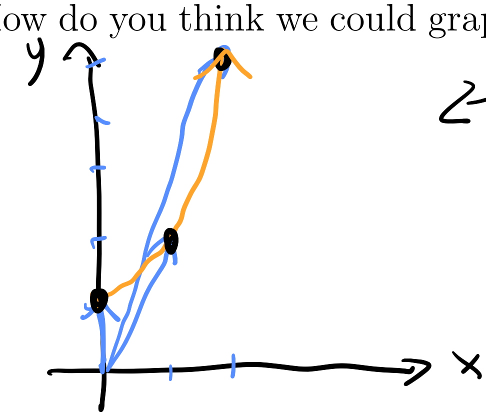
- HW 12.1 due tonight at 8
- Quiz 1 tomorrow: any Ch. 12 topic is fair game

Warm-up problem: Let $\mathbf{r}(t) = \langle t, t^2 + 1 \rangle$, where $t \geq 0$.

a) Find $\mathbf{r}(0), \mathbf{r}(1)$, and $\mathbf{r}(2)$.

$\vec{r}(0) = \langle 0, 1 \rangle$ $\quad \vec{r}(1) = \langle 1, 2 \rangle$ $\quad \vec{r}(2) = \langle 2, 5 \rangle$

b) How do you think we could graph this function? What does the graph look like?



- draw vectors, connect tips
- plot output vectors as points

Graph: looks parabolic

c) Is there a function $y = f(x)$ that has the same graph as $\mathbf{r}(t)$?

$y = x^2 + 1 \; ; \; x \geq 0$

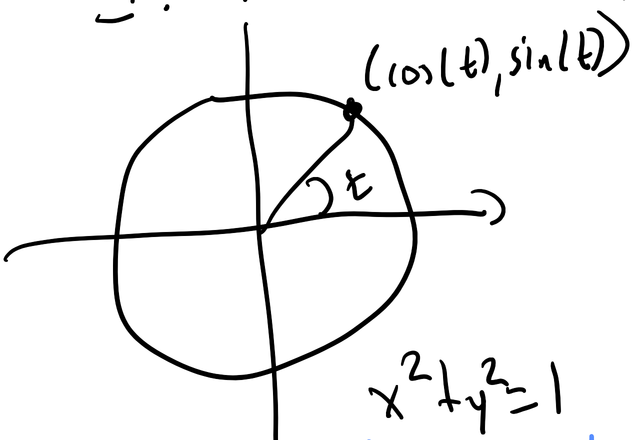
The function $\mathbf{r}(t)$ in the warm-up problem is an example of a **vector-valued function**: its input is a real number t and its output is a vector. We graph a vector-valued function by plotting all of the terminal points of its output vectors, placing their initial points at the origin.

You have seen several examples already:

ex: lines (in $\mathbb{R}^3$)

$$\vec{r}(t) = \langle 1, -2, 1 \rangle t + \langle 0, \pi, e \rangle$$

ex: $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle, \quad t \in [0, 2\pi)$



$$\underbrace{x^2 + y^2 = 1}_{\substack{\text{not a function} \\ y = f(x) \text{ or } x = g(y)}}$$

• If we want to parameterize (write down $\vec{r}(t)$ whose graph matches a given curve)

$$x^2 + y^2 = r^2 \iff \vec{r}(t) = \langle r\cos(t), r\sin(t) \rangle$$
$$= r\langle \cos(t), \sin(t) \rangle$$

ex: If $y = f(x)$ or $x = g(y)$, we can parameterize:

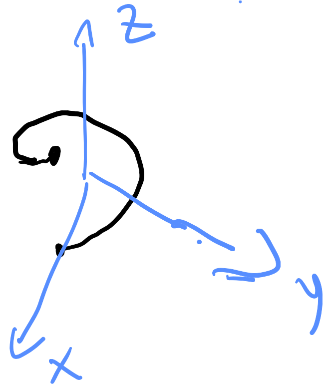
$$\vec{r}(t) = \langle t, f(t) \rangle \qquad \vec{r}(t) = \langle g(t), t \rangle$$

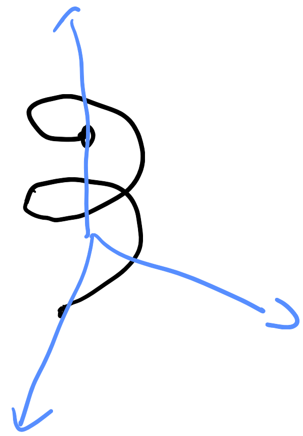
Example 1. Consider $\mathbf{r}_1(t) = \langle \cos(t), \sin(t), t \rangle$ and $\mathbf{r}_2(t) = \langle \cos(2t), \sin(2t), 2t \rangle$, each with domain $[0, 2\pi]$. What do you think the graph of each looks like? How are they similar and how are they different?

Spiral? (x, y-components look like a circle)

- not a cylinder

Technical: $\boxed{\text{helix}}$





- visually looks the same, just moving through points faster

Check your intuition

Calculus of vector-valued functions

Unifying theme: Do what you already know, componentwise.

This works with limits:

Example 2. Compute $\lim_{t\to e}\langle t^2, 2, \ln(t)\rangle$.

$$\lim_{t\to e}\langle t^2, 2, \ln(t)\rangle = \langle \lim_{t\to e} t^2, \lim_{t\to e} 2, \lim_{t\to e} \ln(t)\rangle$$
$$= \langle e^2, 2, 1\rangle$$

- As $t \to e$, the graph of $\langle t^2, 2, \ln(t)\rangle$ approaches $\langle e^2, 2, 1\rangle$

And with continuity:

Example 3. Determine where the function $\mathbf{r}(t) = t\mathbf{i} - \frac{1}{t^2-4}\mathbf{j} + \sin(t)\mathbf{k}$ is continuous.

$\vec{r}(t)$ is cts $\Leftrightarrow$ $x(t)$ is cts AND $y(t)$ is cts AND $z(t)$ is cts

$x(t) = t$: cts on $\mathbb{R}$ $(-\infty, \infty)$

$y(t) = -\frac{1}{t^2-4}$: cts on on $\boxed{(-\infty, -2) \cup (-2, 2) \cup (2, \infty)}$

$z(t) = \sin(t)$: cts on $\mathbb{R}$

- Overall: $\vec{r}(t)$ is cts at all t except $t = \pm 2$.

And with derivatives:

Example 4. If $\mathbf{r}(t) = \langle 2t - \frac{1}{2}t^2 + 1, t - 1\rangle$, find $\mathbf{r}'(t)$.

$$\vec{r}\,'(t) = \langle x'(t), y'(t)\rangle$$
$$= \langle 2 - t, 1\rangle$$
$$\vec{r}\,''(t) = \langle -1, 0\rangle$$

Interpretation: If $\mathbf{r}(t)$ gives the position of an object at time t, then

- $\mathbf{r}'(t)$ gives velocity at time t, $\vec{v}(t)$
- $|\mathbf{r}'(t)|$ gives speed at time t, $|\vec{v}(t)|$
- $\mathbf{r}''(t)$ gives acceleration at time t $\vec{a}(t)$

Let's see this graphically

Example 5. Find an equation of the tangent line to $\mathbf{r}(t) = \langle 2t - \frac{1}{2}t^2 + 1, t - 1\rangle$ at time $t = 2$.

tangent line: at $t=2$ $\vec{\ell}(s) = \underbrace{\vec{r}(2)}_{\text{point}} + \underbrace{\vec{r}\,'(2)}_{\text{direction}} \cdot s$

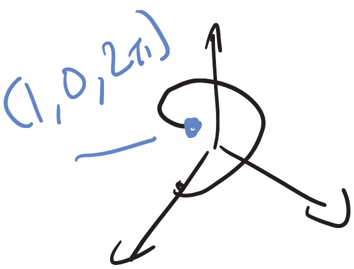
$$\vec{\ell}(s) = \langle 4-2+1, 2-1\rangle + \langle 2-2, 1\rangle s$$
$$= \langle 3, 1\rangle + s\langle 0, 1\rangle$$

Daily Announcements & Reminders:

- HW 12.4 due tonight
- Quiz 2 next W (12.6, 13.1, 13.2)
- Exam 1 on 11/31 (T), more info next week

Let's revisit the Itempool question from the end of last lecture:

Example 13. Find parametric equations for the tangent line to the curve $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + t\mathbf{k}$ at the point $(1, 0, 2\pi)$.



tangent line: – point is $(1, 0, 2\pi)$ at $t = t_0$

– direction $\vec{r}\,'(t_0)$

To get t_0: solve $\vec{r}(t_0) = \langle 1, 0, 2\pi \rangle$

$\cos(t_0) = 1$

$\sin(t_0) = 0$

$\boxed{t_0 = 2\pi}$

So $\vec{r}\,'(2\pi) = \langle -\sin(2\pi), \cos(2\pi), 1 \rangle$

$= \langle 0, 1, 1 \rangle$

so tangent line is: $x(t) = 1 + 0t = 1$

$y(t) = 0 + t = t$

$z(t) = 2\pi + t$

Continuing with integrals:

$u = 2t$
$du = 2\,dt$

Example 14. Find $\int_0^1 \langle t, e^{2t}, \sec^2(t)\rangle\ dt$.

$$\int_0^1 \langle t, e^{2t}, \sec^2(t)\rangle dt = \langle \int_0^1 t\,dt, \int_0^1 e^{2t}\,dt, \int_0^1 \sec^2(t)\,dt \rangle$$
$$= \langle \tfrac{1}{2}t^2\Big|_0^1, \tfrac{1}{2}e^{2t}\Big|_0^1, \tan(t)\Big|_0^1 \rangle$$
$$= \langle \tfrac{1}{2}, \tfrac{1}{2}(e^2-1), \tan(1) \rangle$$

- $\int_a^b \vec{v}(t)\,dt$ = displacement between $t=b$, $t=a$

At this point we can solve initial-value problems like those we did in single-variable calculus:

Example 15. Wallace is testing a rocket to fly to the moon, but he forgot to include instruments to record his position during the flight. He knows that his velocity during the flight was given by

$$\mathbf{v}(t) = \langle -200\sin(2t), 200\cos(t), 400 - \frac{400}{1+t}\rangle\ m/s.$$

If he also knows that he started at the point $\mathbf{r}(0) = \langle 0,0,0\rangle$, use calculus to reconstruct his flight path.

1) Find antiderivative of $\vec{v}(t)$

$$\vec{r}(t) = \int \langle -200\sin(2t), 200\cos(t), 400(1 - \tfrac{1}{1+t})\rangle\,dt$$
$$= \langle 100\cos(2t) + C_1, 200\sin(t) + C_2, 400(t - \ln|1+t|) + C_3 \rangle$$
$$= \langle 100\cos(2t), 200\sin(t), 400(t - \ln|1+t|)\rangle + \vec{C}$$

$\vec{C} = \langle C_1, C_2, C_3\rangle$

$\langle -100, 0, 0\rangle$

2) Apply I.C.

$$\boxed{\langle 0,0,0\rangle} = \vec{r}(0) = \langle 100 + C_1, C_2, \cancel{400(0 - \ln(1))} + C_3\rangle$$

$C_1 = -100$, $C_2 = 0$, $C_3 = 0$

$$\boxed{\vec{r}(t) = \langle 100\cos(2t) - 100, 200\sin(t), 400(t - \ln|1+t|)\rangle}$$

13.3 Arc length of curves

We have discussed motion in space using by equations like $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$.

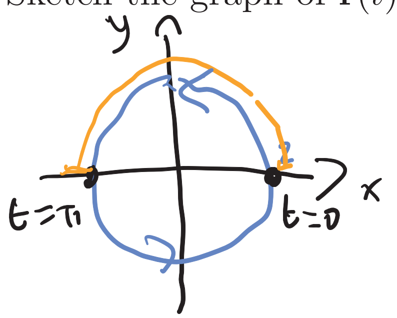
Our next goal is to be able to measure distance traveled or arc length.

Motivating problem: Suppose the position of a fly at time t is

$$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t)\rangle,$$

where $0 \le t \le 2\pi$.

a) Sketch the graph of $\mathbf{r}(t)$. What shape is this?



circle of radius 2

$|\langle -2\rangle|$

$= \sqrt{(-2)^2}$

$= 2$

b) How far does the fly travel between $t = 0$ and $t = \pi$?

4, $\boxed{2\pi} = \frac{1}{2}(2\cdot\pi\cdot 2)$

integral

$\frac{1}{2}$ of circumference

c) What is the speed $|\mathbf{v}(t)|$ of the fly at time t?

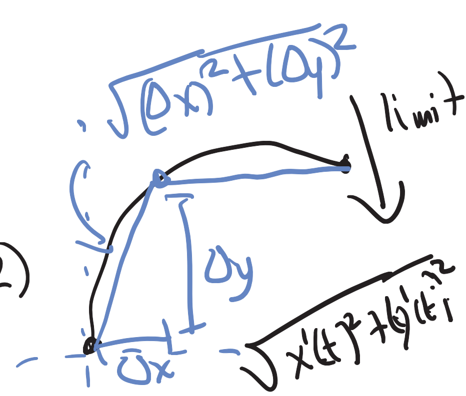
$$|\vec{v}(t)| = |\langle -2\sin(t), 2\cos(t)\rangle|$$
$$= \sqrt{4(\sin^2(t) + \cos^2(t))} = \sqrt{4} = 2$$

d) Compute the integral $\int_0^{\pi} |\mathbf{v}(t)|\, dt$. What do you notice?

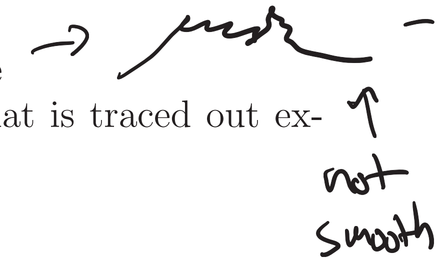
$$\int_0^{\pi} 2\, dt = 2t\Big|_0^{\pi} = 2\pi$$

1) dist=speed·time $\Longleftrightarrow \int_0^{\pi} \underbrace{|v(t)|}_{\text{speed}}\, \underbrace{dt}_{\text{small time}}$

2)



Definition 16. We say that the **arc length** of a smooth curve $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ from $t=a$ to $t=b$ that is traced out exactly once is



$$L = \int_a^b |\vec{r}\,'(t)|\,dt$$

Example 17. Set up an integral for the arc length of the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ from the point $(1, 1, 1)$ to the point $(2, 4, 8)$.

$t=a$: $(1,1,1)$ $a=1$ b/c $\langle 1, 1^2, 1^3 \rangle = \langle 1,1,1 \rangle$

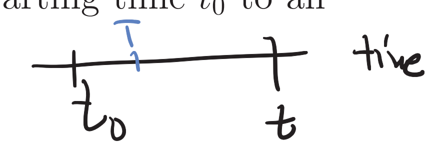
$t=b$: $(2,4,8)$ $b=2$ b/c $\langle 2, 2^2, 2^3 \rangle = \langle 2,4,8 \rangle$

$$\vec{r}\,'(t) = 1\vec{i} + 2t\vec{j} + 3t^2\vec{k}$$

$$|\vec{r}\,'(t)| = \sqrt{1 + 4t^2 + 9t^4}$$

$$\boxed{L = \int_1^2 \sqrt{1 + 4t^2 + 9t^4}\,dt}$$

Sometimes, we care about the distance traveled from a fixed starting time t_0 to an arbitrary time t, which is given by the **arc length function**.



$$s(t) = \int_{t_0}^{t} |\vec{v}(T)|\,dT$$

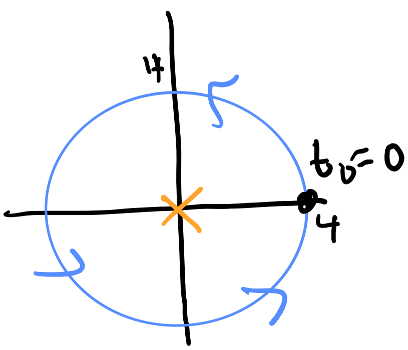
We can use this function to produce parameterizations of curves where the parameter s measures distance along the curve: the points where $s = 0$ and $s = 1$ would be exactly 1 unit of distance apart.

- this is like using mile markers to describe position on a highway

Daily Announcements & Reminders:

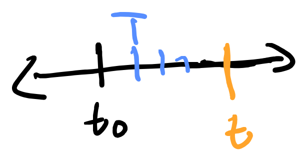
- 12.5, 12.6 HW due tonight
- Quiz 2: 12.6, 13.1, 13.2 in studio tomorrow
- Exam 1 in lecture next T, 1/31 – see Canvas

Example 18. Find an arc length parameterization of the circle of radius 4 about the origin in $\mathbb{R}^2$, $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t)\rangle, 0 \le t \le 2\pi$.



Last time: arc length function

$$s(t) = \int_{t_0}^{t} |\vec{r}\,'(T)|\, dT$$



1) Compute arc length function

$$s(t) = \int_0^t |\langle -4\sin(T), 4\cos(T)\rangle|\, dT$$

$$= \int_0^t \sqrt{16(\sin^2(T) + \cos^2(T))}\, dT$$

$$= \int_0^t 4\, dT = 4t - 0 = 4t$$

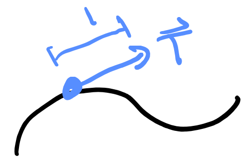
2) Invert & solve for t: • always possible, frequently difficult

$$s = 4t \iff t = \frac{s}{4}$$

3) Plug in: $\vec{r}(s) = \left\langle 4\cos\left(\frac{s}{4}\right), 4\sin\left(\frac{s}{4}\right)\right\rangle, \quad 0 \le s \le 8\pi$

The next idea we are going to explore is the curvature of a curve in space along with two vectors that orient the curve.

Idea: measure how "curvy" a path is



First, we need the **unit tangent vector**, denoted **T**:

- In terms of an arc-length parameter s: $\vec{T}(s) = \vec{r}\,'(s)$
- In terms of any parameter t: $\vec{T}(t) = \vec{r}\,'(t) / |\vec{r}\,'(t)|$

- For every arc-length parameterization, $|\vec{r}\,'(s)| = 1$, a unit-speed parameterization

This lets us define the **curvature**, $\kappa(s) = |\vec{T}\,'(s)|$

- Measures rate of change of direction of motion
- For a line: $\kappa(s) = 0$

Example 19. At the start of class we found an arc length parameterization of the circle of radius 4 centered at $(0,0)$ in $\mathbb{R}^2$:

$$\mathbf{r}(s) = \left\langle 4\cos\left(\frac{s}{4}\right), 4\sin\left(\frac{s}{4}\right)\right\rangle, \qquad 0 \le s \le 8\pi.$$

Use this to find $\mathbf{T}(s)$ and $\kappa(s)$.

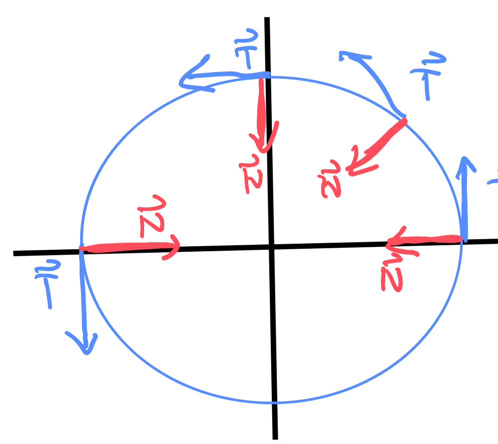
$$\vec{T}(s) = \vec{r}\,'(s) = \left\langle -4\sin\left(\tfrac{s}{4}\right)\cdot\tfrac{1}{4},\ 4\cos\left(\tfrac{s}{4}\right)\cdot\tfrac{1}{4}\right\rangle$$
$$= \left\langle -\sin\left(\tfrac{s}{4}\right),\ \cos\left(\tfrac{s}{4}\right)\right\rangle$$

$$\kappa(s) = |\vec{T}\,'(s)| = \left|\left\langle -\tfrac{1}{4}\cos\left(\tfrac{s}{4}\right),\ -\tfrac{1}{4}\sin\left(\tfrac{s}{4}\right)\right\rangle\right|$$
$$= \frac{1}{4}$$

- <u>Notice</u>: $\kappa(s) = \frac{1}{\text{radius}}$! This is true for all circles.

- Only shapes with constant curvature are
  - lines
  - circles
  - helixes

<u>Picture</u>:



Question: In which direction is $\mathbf{T}$ changing? : this is $\vec{T}\,'(s)/|\vec{T}\,'(s)| = \dfrac{\vec{r}\,''(s)}{|\vec{r}\,''(s)|}$

This is the direction of the **principal unit normal**, $\mathbf{N}(s) =$ ________________

We said that it is often hard to find arc length parameterizations, so what do we do if we have a generic parameterization $\mathbf{r}(t)$?

- $\mathbf{T}(t) =$ $\vec{r}'(t) / |\vec{r}'(t)|$
- $\mathbf{N}(t) =$ $\vec{T}'(t) / |\vec{T}'(t)|$
- $\kappa(t) =$ $|\vec{T}'(t)| / |\vec{r}'(t)|$ or $|\vec{r}'(t) \times \vec{r}''(t)| / |\vec{r}'(t)|^3$

Example 20. Find $\mathbf{T}, \mathbf{N}, \kappa$ for the helix $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), t-1\rangle$.

$\vec{r}'(t) = \langle -2\sin(t), 2\cos(t), 1\rangle$ $\quad |\vec{r}'(t)| = \sqrt{4\sin^2(t)+4\cos^2(t)+1}$

$= \sqrt{4+1} = \sqrt{5}$

So $\vec{T}(t) = \frac{1}{\sqrt{5}}\langle -2\sin(t), 2\cos(t), 1\rangle$

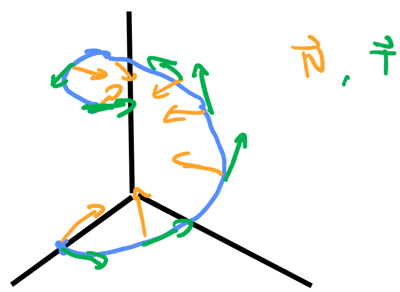
Now $\vec{T}'(t) = \frac{1}{\sqrt{5}}\langle -2\cos(t), -2\sin(t), 0\rangle$ & $|\vec{T}'(t)| = \frac{1}{\sqrt{5}} \cdot \sqrt{4\cos^2(t)+4\sin^2(t)}$

$= \frac{2}{\sqrt{5}}$

so $\vec{N}(t) = \frac{1}{2/\sqrt{5}} \cdot \frac{1}{\sqrt{5}}\langle -2\cos(t), -2\sin(t), 0\rangle$

$= \langle -\cos(t), -\sin(t), 0\rangle$

& $\kappa(t) = \dfrac{\frac{2}{\sqrt{5}}}{\sqrt{5}} = \frac{2}{5}$



Daily Announcements & Reminders:

- 13.1, 13.2 HW due tonight, 13.3, 13.4 due T night
- Quiz 1 regrade requests open until T morning
- See Canvas for exam 1 info: T, 1/31, 9:30 in lecture

Topics for Today: Functions of Multiple Variables

- examples
- domains
- contours
- graphs
- traces

Definition 1. A function of two variables is a rule that assigns to each pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$.

domain of function

function named f

$$f : D \to \mathbb{R}, \text{ where } D \subseteq \mathbb{R}^2$$

D is a subset of $\mathbb{R}^2$

domain is the set D

output (codomain) is a real #

- but onto! (every real # is a possible output)
- is not 1-to-1

Example 2. Three examples are

- not 1-to-1

$(\pm 1, 0) \to 1$

- range is $[0, \infty)$

$$f(x, y) = x^2 + y^2, \qquad g(x, y) = \ln(x + y), \qquad h(x, y) = \frac{1}{\sqrt{x + y}}.$$

> 0

$z = x^2 + y^2$

- graph is an elliptic paraboloid

$g(1, 0) = \ln(1 + 0)$
$= \ln(1)$
$= 0$

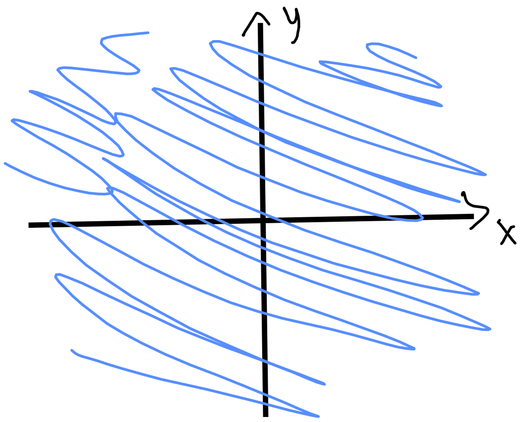
- not 1-to-1
- not onto

range:

Example 3. Find the domains of f, g, and h.

$f(x,y) = x^2 + y^2$

Domain is all of $\mathbb{R}^2 = \{(x,y) \mid x,y \in \mathbb{R}\}$



$g(x,y) = \ln(x+y)$

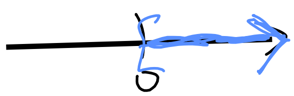
• need $x+y > 0$

Domain: $\{(x,y) \mid \overset{x+y>0}{y > -x}\}$

$h(x,y) = \frac{1}{\sqrt{x+y}}$

Domain: $\{(x,y) \mid x+y>0\}$

Range: $[0, \infty)$



Definition 4. If f is a function of two variables with domain D, then the graph of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

Here are the graphs of the three functions above.

Example 5. Suppose a small hill has height $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$ m at each point (x, y). How could we draw a picture that represents the hill in 2D?

$4 = h(x,y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$

$\Rightarrow x^2+y^2=0$

$3 = h(x,y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$

$-1 = -\frac{1}{4}x^2 - \frac{1}{4}y^2$

$x^2+y^2=4$

$2 = h(x,y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$

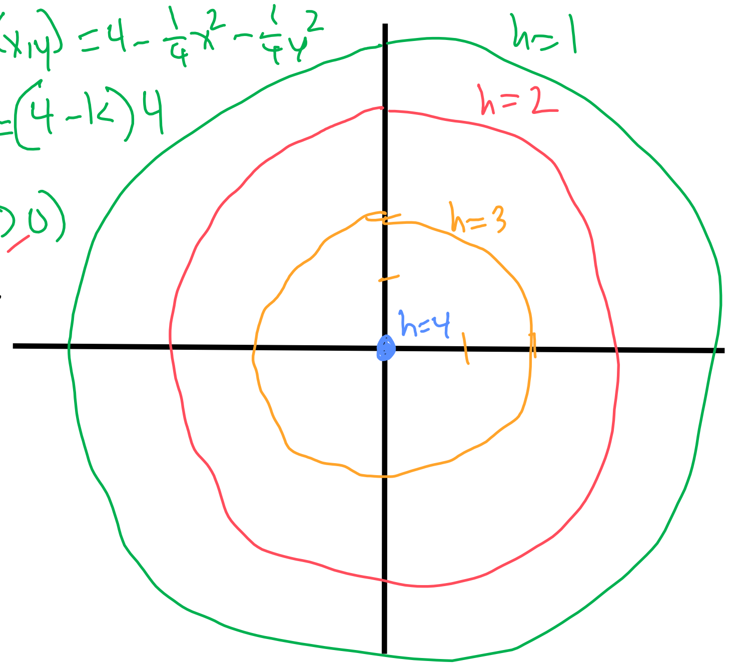
$8 = x^2+y^2$

$k = h(x,y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$

$x^2+y^2 = (4-k)4$

$(4 > k > 0)$

$0 < k < 4$



In 3D, it looks like this.

Definition 6. The contours (also called level curves) of a function f of two variables are the curves with equations $f(x,y)=k$, where k is a constant (in the range of f). A plot of contours for various values of ~~z~~ k is a contour map (or level curve map).

Some common examples of these are:

- topographical maps
- equipotential lines
- weather maps

Example 7. Create a contour diagram of $f(x, y) = x^2 - y^2$

$0 = x^2 - y^2$: $x^2 = y^2 \iff x = \pm y$

$1 = x^2 - y^2$ hyperbola, horiz.

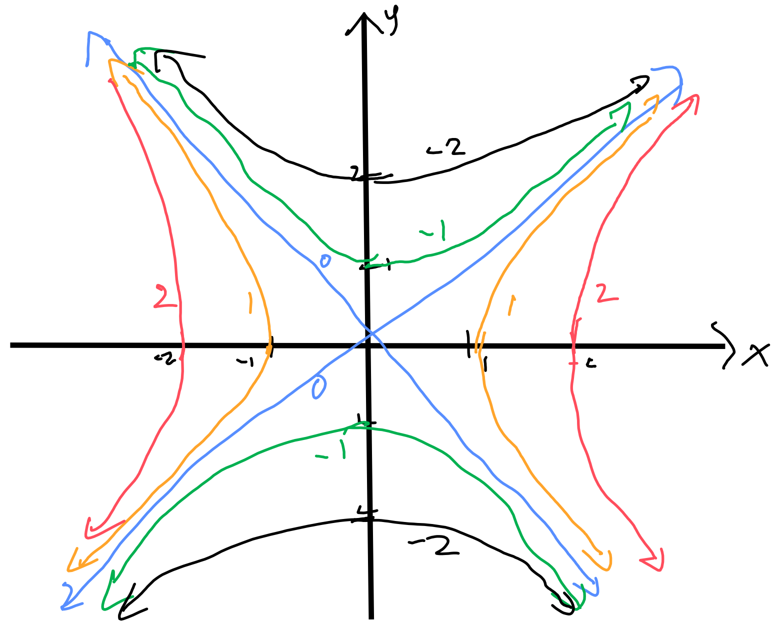
$4 = x^2 - y^2$ " "

$-1 = x^2 - y^2$ hyperbola, vert.

$1 = y^2 - x^2$

$-4 = x^2 - y^2$ hyperbola, vert.

$4 = y^2 - x^2$



Example 8. Create a contour diagram of $g(x, y) = y\sin(x)$

$0 = y\sin(x) \iff y = 0$ or $\sin(x) = 0$

$x = \pi k$

$1 = y\sin(x)$

$y = \csc(x)$

$\frac{1}{2} = y\sin(x)$

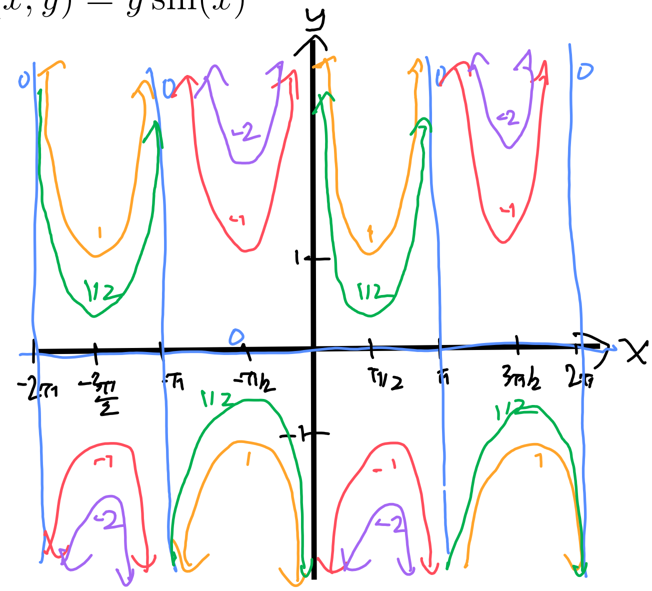
$y = \frac{1}{2}\csc(x)$

$-1 = y\sin(x)$

$y = -1\csc(x)$

$-2 = y\sin(x)$

$y = -2\csc(x)$



$z = f(x, y)$

Definition 9. The <u>traces</u> of a surface are the curves of <u>intersection</u> of the surface with planes parallel to the <u>xz & yz planes</u>.

Example 10. Use the traces and contours of $z = f(x, y) = 4 - 2x - y^2$ to sketch the portion of its graph in the first octant.

$x, y, z \geq 0$

Set $x = k$ or $y = k$

$x = 0: \quad z = 4 - y^2$

$x = k: \quad z = 4 - 2k - y^2$

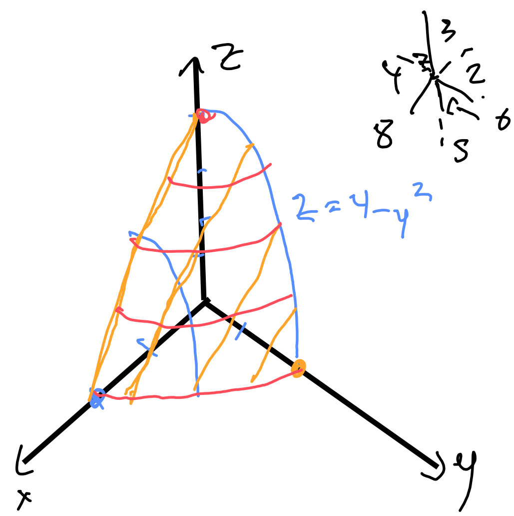
$y = 0: \quad z = 4 - 2x$

$y = k: \quad z = 4 - k^2 - 2x$

$z = 0: \quad 4 - 2x - y^2 = 0$

$x = 2 - \frac{1}{2}y^2$

$z = k: \quad 4 - 2x - y^2 = k \quad \} \quad x = 2 - \frac{1}{2}k - \frac{1}{2}y^2$



Let's check our work: https://tinyurl.com/math2551-2var-graph

Daily Announcements & Reminders:

- no HW due tonight
- Exam 1 will be graded by F, 2/10

Goals for Today:

- Evaluate limits of functions of two variables
- Show that a limit does not exist using the two-path test
- Determine the set of points where a function is continuous
- Start to understand how we can measure how a function of two variables is changing

Definition 34. What is a limit of a function of two variables?

DEFINITION We say that a function $f(x, y)$ approaches the **limit** L as (x, y) approaches (x_0, y_0), and write

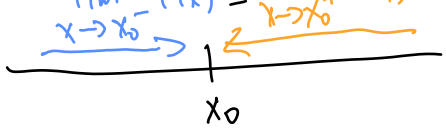
$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,

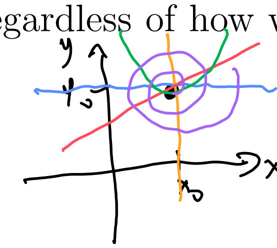
$$|f(x, y) - L| < \epsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

We won't use this definition much: the big idea is that $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$ if and only if $f(x, y)$ approaches L regardless of how we approach (x_0, y_0).

1 var: $\lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x)$ if limit exists



2 vars:



Limits of functions of two variables work like limits you are used to in many ways:

THEOREM 1—Properties of Limits of Functions of Two Variables The following rules hold if L, M, and k are real numbers and

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M.$$

1. *Sum Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) + g(x,y)) = L + M$
2. *Difference Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) - g(x,y)) = L - M$
3. *Constant Multiple Rule:* $\lim_{(x,y)\to(x_0,y_0)} kf(x,y) = kL$ (any number k)
4. *Product Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) \cdot g(x,y)) = L \cdot M$
5. *Quotient Rule:* $\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}$, $M \neq 0$
6. *Power Rule:* $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer, and if n is even, we assume that $L > 0$.

- $\lim_{(x,y)\to(x_0,y_0)} x = x_0$
- $\lim_{(x,y)\to(x_0,y_0)} y = y_0$
- $\lim_{(x,y)\to(x_0,y_0)} k = k$

Example 35. Evaluate $\lim_{(x,y)\to(2,0)} \frac{\sqrt{2x-y}-2}{2x-y-4}$, if it exists.

1) Plug in: $\frac{\sqrt{2(2)-0}-2}{2(2)-0-4} = \frac{\sqrt{4}-2}{4-4} = \frac{0}{0}$ ⟹ work harder

2) Try algebraic manipulations
- $\sqrt{\ } \to$ difference of squares
- ~~L'Hospital~~ X
- multiply by conjugate

$$\lim_{(x,y)\to(2,0)} \frac{\sqrt{2x-y}-2}{(\sqrt{2x-y})^2-2^2} = \lim_{(x,y)\to(2,0)} \frac{\cancel{(\sqrt{2x-y}-2)}^{\,1}}{\cancel{(\sqrt{2x-y}-2)}_{\,1}(\sqrt{2x-y}+2)}$$

$$= \lim_{(x,y)\to(2,0)} \frac{1}{\sqrt{2x-y}+2}$$

$$= \frac{1}{\sqrt{4-0}+2} = \boxed{\frac{1}{4}}$$

Sometimes, life is harder in $\mathbb{R}^2$ and limits can fail to exist in ways that are very different from what we've seen before.

Big Idea: Limits can behave differently along different paths of approach

Example 36. Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$, if it exists. Here is its graph.

• Goal: Show that the limit along the x-axis & the limit along the y-axis are different.

Along x-axis: all points have the form $(x, 0)$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=0}} \frac{x^2}{x^2+y^2} = \lim_{(x,0)\to(0,0)} \frac{x^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{x^2+0^2} = \lim_{x\to 0} 1 = 1$$

Along y-axis: all points have the form $(0, y)$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } x=0}} \frac{x^2}{x^2+y^2} = \lim_{(0,y)\to(0,0)} \frac{x^2}{x^2+y^2} = \lim_{y\to 0} \frac{0^2}{0^2+y^2} = \lim_{y\to 0} 0 = 0$$

Because these two limits along different paths through (0,0) are different, the limit does not exist.

This idea is called the **two-path test:**

If we can find any two paths to (x_0, y_0) along which the limit of $f(x,y)$ takes on two different values, then the limit does not exist.

CAUTION: If you find two paths where the limit agrees, this is not enough to show the limit exists.

Definition 37. A function $f(x, y)$ is **continuous** at (x_0, y_0) if

1. $f(x_0, y_0)$ exists
2. $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists
3. $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$

- f is continuous if it's continuous on its entire domain.
- f is continuous on $\mathbb{R}^2$ if it is cts & its domain is $\mathbb{R}^2$

Key Fact: Adding, subtracting, multiplying, dividing, or composing two continuous functions results in another continuous function.

Example 38. For each function below, determine the domain on which it is continuous:

- $f(x, y) = x$ all of $\mathbb{R}^2$ b/c $\lim_{(x,y)\to(x_0,y_0)} x = x_0 = f(x_0, y_0)$ for all (x_0, y_0) in $\mathbb{R}^2$

- $\boxed{g(x, y) = \dfrac{1}{xy}}$ all of $\mathbb{R}^2$ except $x=0$ and $y=0$

  $\hookrightarrow \frac{1}{u} \circ xy$

  $(x,y) \to xy \to \frac{1}{xy}$

  $u \to \frac{1}{u}$

  ↑ fn of 2 vars ↑ cts on all of $\mathbb{R}^2$

  fn of 1 var ↑ cts if $u \neq 0$

  $f(x) = \frac{1}{x^2-1}$

  $x \to x^2-1 \to \frac{1}{\sqrt{x^2-1}}$

  $u \to \frac{1}{u}$

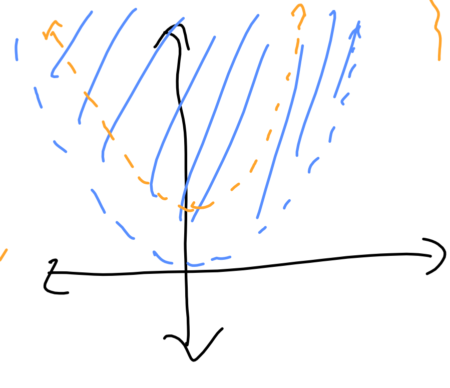
- $h(x, y) = \dfrac{1}{\ln(y - x^2)}$

  $y - x^2 > 0$

  $\ln(y-x^2) \neq 0$

  $y - x^2 \neq 1$

  $\{(x,y) \mid y > x^2,\ y \neq x^2+1\}$



14.3: Partial Derivatives

Goal: Describe how a function of two (or three, later) variables is changing at a point (a, b).

Example 39. Let's go back to our example of the small hill that has height

$$h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$$

meters at each point (x, y). If we are standing on the hill at the point with $(2, 1, 11/4)$, and walk due north (the positive y-direction), at what rate will our height change? What if we walk due east (the positive x-direction)?

Let's investigate graphically.

Due north: Graphically, we see our path is a parabola.

- x is fixed at 2

Our height at each new y-value is:

$$h(2, y) = 4 - \frac{1}{4} \cdot 2^2 - \frac{1}{4}y^2 = 3 - \frac{1}{4}y^2$$

To find rate of change of height as y increases:

$$\frac{d}{dy}(h(2,y)) = \frac{d}{dy}\left(3 - \frac{1}{4}y^2\right) = -\frac{1}{2}y$$

At $(2, 1, 11/4)$, the rate of change is $-\frac{1}{2}(1) = \boxed{-\frac{1}{2} \frac{m}{m}}$

- This is the slope of the hill in positive y-direction at $(2,1)$.

Partial derivative of f w.r.t. y: $\frac{\partial f}{\partial y}(2,1) = \frac{d}{dy}(f(2,y))\Big|_{y=1}$

Daily Announcements & Reminders:

- HW 14.1 due tonight, 14.2 due Th
- Quiz 3 tomorrow on 14.1 & 14.2
- Studio worksheets 9, 10, 11 updated yesterday
  - download individually

Goals for Today:

- Learn how to compute partial derivatives of functions of multiple variables
- Learn how to compute higher-order partial derivatives
- Understand Clairaut's theorem
- Connect partial derivatives and approximation via linear algebra and the total derivative

Last time: we started to think about rates of changes of functions of two variables in the context of walking on this hill.

Definition 40. If f is a function of two variables x and y, its partial derivatives are the functions f_x and f_y defined by

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

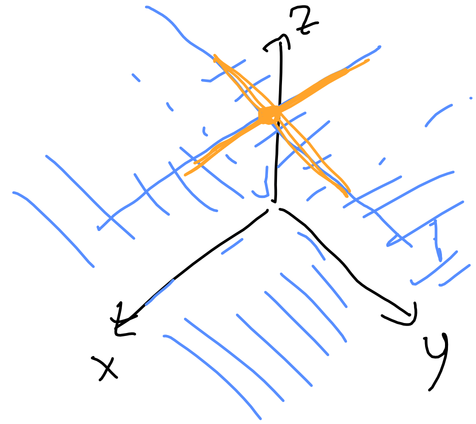
$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

- Use derivative rules to actually compute

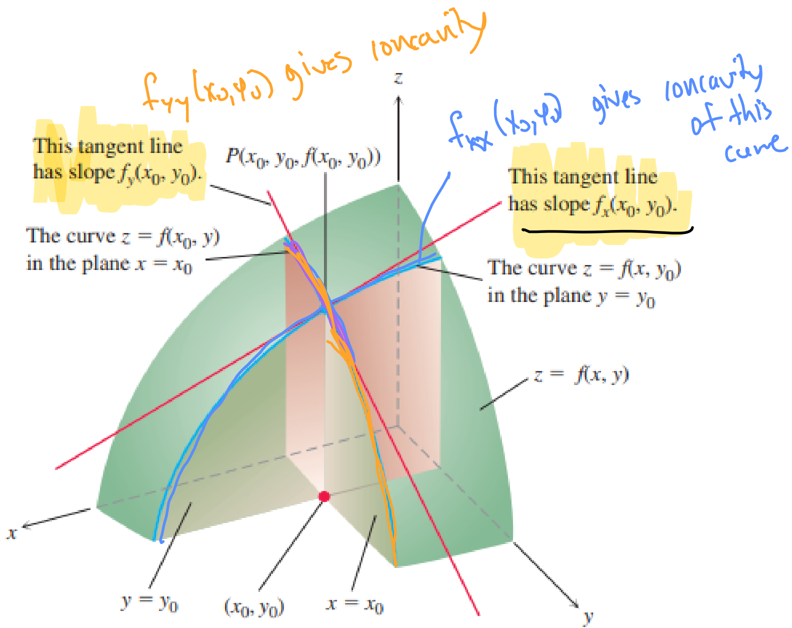
Notations:

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(f)$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(f)$$



Interpretations:



$$f(x,y) = \begin{cases} 1 & xy = 0 \\ 0 & \text{else} \end{cases}$$

- $f_x(0,0) = f_y(0,0) = 0$

Example 41. Find $f_x(1, 2)$ and $f_y(1, 2)$ of the functions below.

a) $f(x, y) = \sqrt{5x - y}$

$$f_y(x,y) = \frac{\partial}{\partial y}\left(\sqrt{5x-y}\right) = \frac{1}{2\sqrt{5x-y}} \cdot (0-1)$$

$$f_x(x,y) = \frac{\partial}{\partial x}\left(\sqrt{5x-y}\right)$$

$$= \frac{1}{2\sqrt{5x-y}} \cdot \frac{\partial}{\partial x}(5x-y) = \frac{5}{2\sqrt{5x-y}}$$

$$\boxed{f_x(1,2) = \frac{5}{2\sqrt{5-2}} = \frac{5}{2\sqrt{3}} \qquad f_y(1,2) = \frac{-1}{2\sqrt{3}}}$$

b) $f(x, y) = \tan(xy)$

$$f_x(x,y) = \sec^2(xy) \cdot y$$

$$f_y(x,y) = x\sec^2(xy)$$

$$\boxed{f_x(1,2) = 2\sec^2(2) \qquad f_y(1,2) = \sec^2(2)}$$

Question: How would you define the second partial derivatives?

- Take partial derivatives of the partial derivatives

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \qquad f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} \qquad f_{yy} = (f_y)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

mixed partials

Example 42. Find f_{xx}, f_{xy}, f_{yx}, and f_{yy} of the functions below.

"pure" partials

a) $f(x, y) = \sqrt{5x - y}$

$$f_x = \frac{5}{2\sqrt{5x-y}} \qquad f_y = \frac{-1}{2\sqrt{5x-y}}$$

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{5}{2\sqrt{5x-y}}\right) = \frac{\partial}{\partial x}\left(\frac{5}{2}(5x-y)^{-1/2}\right) = -\frac{5}{4}(5x-y)^{-3/2}\cdot 5$$

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{-1}{2\sqrt{5x-y}}\right) = \frac{1}{4}(5x-y)^{-3/2}\cdot 5$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{5}{2\sqrt{5x-y}}\right) = -\frac{5}{4}(5x-y)^{-3/2}\cdot(-1)$$

=

$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{-1}{2\sqrt{5x-y}}\right) = \frac{1}{4}(5x-y)^{-3/2}\cdot(-1)$$

b) $f(x, y) = \tan(xy)$

$$f_x = y\sec^2(xy) \qquad f_y = x\sec^2(xy)$$

$$\frac{d}{du}(\sec(u)) = \sec(u)\tan(u)$$

$$f_{xx} = \frac{\partial}{\partial x}(y\sec^2(xy)) = y\cdot 2\sec(xy)\sec(xy)\tan(xy)\cdot y = 2y^2\sec^2(xy)\tan(xy)$$

$$f_{xy} = \frac{\partial}{\partial y}(y\sec^2(xy)) = 1\cdot\sec^2(xy) + y\,(2\sec^2(xy)\tan(xy)\,x)$$

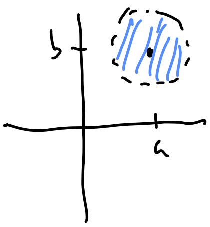
$$\parallel$$

$$f_{yx} = \frac{\partial}{\partial x}(x\sec^2(xy)) = 1\sec^2(xy) + x\,(2\sec^2(xy)\tan(xy)\,y)$$

$$f_{yy} = \frac{\partial}{\partial y}(x\sec^2(xy)) = x\cdot 2\sec^2(xy)\tan(xy)\,x = 2x^2\sec^2(xy)\tan(xy)$$

What do you notice about f_{xy} and f_{yx} in the previous examples?

Theorem 43 (Clairaut's Theorem)**.** *Suppose f is defined on a disk D that contains the point (a, b). If the functions $f, f_x, f_y, f_{xy}, f_{yx}$ are all continuous on D, then*



$f_{xy} = f_{yx}$.

Also $f_{xxy} = f_{xyx} = f_{yxx}$ if all partials are cts

Example 44. What about functions of three variables? How many partial derivatives should $f(x, y, z) = 2xyz - z^2y$ have? Compute them.

$$f_x = \frac{\partial}{\partial x}\left(2xyz - z^2y\right) = 2yz - 0$$

$$f_y = \frac{\partial}{\partial y}\left(2xyz - z^2y\right) = 2xz - z^2$$

$$f_z = \frac{\partial}{\partial z}\left(2xyz - z^2y\right) = 2xy - 2zy$$

Example 45. How many rates of change should the function $f(s, t) = \begin{bmatrix} s^2 + t \\ 2s - t \\ st \end{bmatrix}$ $\begin{bmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{bmatrix}$ have? Compute them.

$\begin{bmatrix} f_1(s,t) \\ f_2(s,t) \\ f_3(s,t) \end{bmatrix}$

$$\frac{\partial f_1}{\partial s} = 2s + 0 \qquad \frac{\partial f_1}{\partial t} = 0 + 1$$

$$\frac{\partial f_2}{\partial s} = 2 \qquad \frac{\partial f_2}{\partial t} = -1$$

$$\frac{\partial f_3}{\partial s} = t \qquad \frac{\partial f_3}{\partial t} = s$$

In the previous example, we computed 6 partial derivatives. How might we **organize** this information?

For any function $f : \mathbb{R}^n \to \mathbb{R}^m$ having the form $f(x_1, \ldots, x_n) = \begin{bmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_m(x_1, \ldots, x_n) \end{bmatrix}$,

we have n inputs, m output, and $n \cdot m$ partial derivatives, which we can use to form the **total derivative**.

This is a linear map from $\mathbb{R}^n \to \mathbb{R}^m$, denoted Df, and we can represent it with an $m \times n$, with one column per input and one row per output.

It has the formula $Df_{ij} = \dfrac{\partial f_i}{\partial x_j}$ $\begin{matrix} \leftarrow i^{th} \\ \leftarrow j^{th} \end{matrix}$

Example 46. Find the total derivatives of each function:

a) $f(x) = x^2 + 1$ $\quad Df(x) = [\,2x\,] = \dfrac{df}{\partial x}$

outputs $m = 1$

inputs $n = 1$

b) $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$ $\quad D\vec{r}(t) = \begin{bmatrix} -\sin(t) \\ \cos(t) \\ 1 \end{bmatrix} = \vec{r}\,'(t)$

outputs $m = 3$

inputs $n = 1$

c) $f(x, y) = \sqrt{5x - y}$ $\quad Df(x,y) = \left[\dfrac{\partial f}{\partial x} \quad \dfrac{\partial f}{\partial y} \right] = \left[\dfrac{5}{2\sqrt{5x-y}} \quad \dfrac{-1}{2\sqrt{5x-y}} \right]$

inputs $n = 2$

outputs $m = 1$

d) $f(x, y, z) = 2xyz - z^2 y$ $\quad Df(x,y,z) = \left[\dfrac{\partial f}{\partial x} \quad \dfrac{\partial f}{\partial y} \quad \dfrac{\partial f}{\partial z} \right]$

$n = 3$

$m = 1$

$= \left[2yz \quad 2xz - z^2 \quad 2xy - 2zy \right]$

e) $\mathbf{f}(s, t) = \langle s^2 + t, 2s - t, st \rangle$

$$D\vec{f}(s,t) = \begin{array}{cc} \frac{\partial}{\partial s} & \frac{\partial}{\partial t} \\ \end{array}$$

$$\begin{array}{l} \\ \end{array} D\vec{f}(s,t) = \begin{bmatrix} 2s & 1 \\ 2 & -1 \\ t & s \end{bmatrix} \begin{array}{l} \leftarrow \frac{\partial f_1}{\partial \_} \\ \leftarrow \frac{\partial f_2}{\partial \_} \\ \leftarrow \partial f_3 / \partial \_ \end{array}$$

What does it mean? In differential calculus, you learned that one interpretation of the derivative is as a slope. Another interpretation is that the derivative measures how a function transforms a neighborhood around a given point.

Let's see that with an app (credit to samuel.gagnon.nepton, who was inspired by 3Blue1Brown.)

Next time

In particular, the (total) derivative of **any** function $f : \mathbb{R}^n \to \mathbb{R}^m$, evaluated at $\mathbf{a} = (a_1, \ldots, a_n)$, is the linear function that best approximates $f(\mathbf{x}) - f(\mathbf{a})$ at $\mathbf{a}$.

This leads to the familiar linear approximation formula for functions of one variable: $f(x) = f(a) + f'(a)(x - a)$.

Definition 47. The **linearization** or **linear approximation** of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$ at the point $\mathbf{a} = (a_1, \ldots, a_n)$ is

$$L(\mathbf{x}) =$$

Example 48. Find the linearization of the function $f(x, y) = \sqrt{5x - y}$ at the point $(1, 1)$. Use it to approximate $f(1.1, 1.1)$.

Question: What do you notice about the equation of the linearization?

Daily Announcements & Reminders:

- Exam 1 grades released, regrades open until W
- 14.2 HW due tonight
- Clairaut's thm for n variables:
  need the partials to be cts in an n-dimensional ball.
  $\Rightarrow$ mixed partials are equal

Goals for Today:

- Understand derivatives as transformations
- Find the best linear approximation of a differential function
- Learn the Chain Rule for derivatives of functions of multiple variables
- Be able to compute implicit partial derivatives

What does it mean? In differential calculus, you learned that one interpretation of the derivative is as a slope. Another interpretation is that the derivative measures how a function transforms a neighborhood around a given point.

Let's see that with an app (credit to samuel.gagnon.nepton, who was inspired by 3Blue1Brown.)

- f maps $x=a$ to $f(a)$

  Near a: $f(x) \approx f(a) + \underset{\text{linear function}}{f'(a)}\underbrace{(x-a)}_{\text{dist. from } a}$

In particular, the (total) derivative of **any** function $f : \mathbb{R}^n \to \mathbb{R}^m$, evaluated at $\mathbf{a} = (a_1, \ldots, a_n)$, is the linear function that best approximates $f(\mathbf{x}) - f(\mathbf{a})$ at $\mathbf{a}$.

This leads to the familiar linear approximation formula for functions of one variable: $f(x) = f(a) + f'(a)(x - a)$.

Definition 47. The **linearization** or **linear approximation** of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$ at the point $\mathbf{a} = (a_1, \ldots, a_n)$ is

$$\underbrace{L(\mathbf{x})}_{\text{near } \vec{a}} = f(\vec{a}) + \underbrace{Df(\vec{a})}_{\substack{\text{matrix of partial derivs} \\ \text{evaluated } \vec{a}}} \underbrace{(\vec{x} - \vec{a})}_{\text{vector (cols.)}}$$

Example 48. Find the linearization of the function $f(x, y) = \sqrt{5x - y}$ at the point $(1, 1)$. Use it to approximate $f(1.1, 1.1)$.

$$f_x = \frac{5}{2\sqrt{5x-y}} \qquad f_y = \frac{-1}{2\sqrt{5x-y}} \qquad \vec{a} = \langle 1, 1 \rangle$$

$$f(\vec{a}) = \sqrt{5-1} = 2$$

$$Df(\vec{a}) = \left[\frac{5}{2\sqrt{5x-y}} \quad \frac{-1}{2\sqrt{5x-y}}\right]\Bigg|_{(1,1)} = \left[\frac{5}{4} \quad \frac{-1}{4}\right]$$

1 row per output
1 col per input

$$L(x,y) = 2 + \left[\frac{5}{4} \quad \frac{-1}{4}\right]\left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

$$= 2 + \left[\frac{5}{4} \quad -\frac{1}{4}\right]\begin{bmatrix} x-1 \\ y-1 \end{bmatrix} = \boxed{2 + \frac{5}{4}(x-1) - \frac{1}{4}(y-1)}$$

$f(1.1, 1.1) = 2.048\ldots$

$\approx L(1.1, 1.1)$

$= 2 + \frac{5}{4}(.1) - \frac{1}{4}(.1) = 2 + .125 - .025 = 2.1$

Question: What do you notice about the equation of the linearization?

This is the equation of a plane!

(Tangent plane to $z = \sqrt{5x-y}$ at $(1,1)$)

If $y=f(x)$: $df = f'(x)\,dx$

Example 49. The **differential** of a function $f(\mathbf{x})$ is

$$Df \cdot \begin{bmatrix} dx_1 \\ \vdots \\ dx_n \end{bmatrix} \leftarrow \quad df = f_{x_1}\, dx_1 + f_{x_2}\, dx_2 + \dots f_{x_n}\, dx_n.$$

Find the differential df for $f(x, y, z) = x^2 + y^2 + z^2$ and use it to estimate the change in f between $(1, 1, 1)$ and $(1.1, 1, 0.9)$.

Idea: $\Delta f \approx df$ with $dx_1 \approx \Delta x_1, \dots$

$$\Delta f \approx f_x(1,1,1)\overset{\Delta x}{(1.1-1)} + f_y(1,1,1)\overset{\Delta y}{(1-1)} + f_z(1,1,1)\overset{\Delta z}{(.9-1)}$$

$$\approx 2x\big|_{(1,1,1)} (.1) + 2y\big|_{(1,1,1)} \cdot 0 + 2z\big|_{(1,1,1)} \cdot (-.1)$$

$$= 0.2 + 0 - .2$$

$$= 0$$

$Df(1,1,1) = [2\ \ 2\ \ 2]$

$f(1.1, 1, 0.9) - f(1,1,1) = 3.02 - 3 = 0.02$

The Chain Rule

Example 50. If $f(t) = \ln(t^2)$, then $\dfrac{df}{dt} = \dfrac{1}{t^2} \cdot 2t$

$$\big(f(g(x))\big)' = f'(g(x)) \cdot g'(x)$$

Similarly, the **Chain Rule** for functions of multiple variables says that if $f : \mathbb{R}^p \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ are both differentiable functions then

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "g"] \arrow[rr, bend right, "f \circ g"'] & \mathbb{R}^p \arrow[r, "f"] & \mathbb{R}^m \end{tikzcd}$$

$$D(f(g(\mathbf{x}))) = Df(g(\mathbf{x}))Dg(\mathbf{x}).$$

Example 51. Suppose we are walking on our hill with height $h(x,y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$ along the curve $\mathbf{r}(t) = \langle t+1, 2-t^2 \rangle$ in the plane. How fast is our height changing at time $t = 1$ if the positions are measured in meters and time is measured in minutes?

- height on the path at time t:

$$h(\vec{r}(t))$$

- $t=1 \Rightarrow (x,y) = (2,1)$

$$1 \text{ input} \xrightarrow{\vec{r}} 2 \text{ outputs} \xrightarrow{h} 1 \text{ output}$$

Want: $D[h(\vec{r}(t))] = \frac{dh}{dt}$

evaluate not mult.

1×2 $\quad$ 2×1

by Chain Rule: $(D[h(\vec{r}(t))])(1) = (Dh)(\vec{r}(1)) \cdot (D\vec{r})(1)$

$$= \begin{bmatrix} -\frac{x}{2} & -\frac{y}{2} \end{bmatrix}_{(2,1)} \begin{bmatrix} 1 \\ -2t \end{bmatrix}_{t=1}$$

$$= \begin{bmatrix} -1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$= -1 + 1 = 0$$

$$\begin{bmatrix} -\frac{t+1}{2} & -\frac{(2-t^2)}{2} \end{bmatrix} \begin{bmatrix} 1 \\ -2t \end{bmatrix}$$

$$= \boxed{-\frac{t+1}{2} + t(2-t^2)}$$

Post class

Tree Diagram:

$$\begin{array}{ccccc} & & h & & \\ & \frac{\partial h}{\partial x} \swarrow & & \searrow \frac{\partial h}{\partial y} & \\ x & & & & y \\ & \frac{dx}{dt} \searrow & & \swarrow \frac{dy}{dt} & \\ & & t & & \end{array}$$

so $\frac{dh}{dt} = \frac{\partial h}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial h}{\partial y} \cdot \frac{dy}{dt}$

$$= \left(-\frac{x}{2}\right)_{x=2} \cdot (1)_{t=1} + \left(-\frac{y}{2}\right)_{y=1} \cdot (-2t)_{t=1}$$

$$= (-1)(1) + \left(-\frac{1}{2}\right)(-2)$$

$$= 0$$

Example 52. Suppose that $W(s,t) = F(u(s,t), v(s,t))$, where F, u, v are differentiable functions and we know the following information.

$$\begin{aligned} u(1,0) &= 2 \quad v(1,0) &= 3 \\ u_s(1,0) &= -2 \quad v_s(1,0) &= 5 \\ u_t(1,0) &= 6 \quad v_t(1,0) &= 4 \\ F_u(2,3) &= -1 \quad F_v(2,3) &= 10 \end{aligned}$$

$W = F \circ g$

$F(u,v)$

$g(s,t) = \begin{bmatrix} u(s,t) \\ v(s,t) \end{bmatrix}$

Find $W_s(1,0)$ and $W_t(1,0)$.

$$\begin{bmatrix} W_s(1,0) & W_t(1,0) \end{bmatrix} = DW(1,0)$$

(Chain Rule) $= DF(\underset{=2}{u(1,0)}, \underset{=3}{v(1,0)}) \cdot Dg(1,0)$

(defn of total deriv.) $= \begin{bmatrix} F_u(2,3) & F_v(2,3) \end{bmatrix} \begin{bmatrix} u_s(1,0) & u_t(1,0) \\ v_s(1,0) & v_t(1,0) \end{bmatrix}$

$= \begin{bmatrix} -1 & 10 \end{bmatrix} \begin{bmatrix} -2 & 6 \\ 5 & 4 \end{bmatrix}$

$(-1)(-2) + 10(5)$

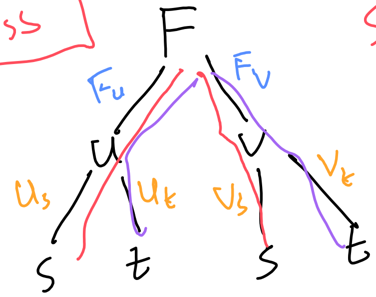
$(-1)(6) + 10(4)$

$= \begin{bmatrix} 52 & 34 \end{bmatrix}$

$W_s(1,0) = 52$

$W_t(1,0) = 34$

Post class

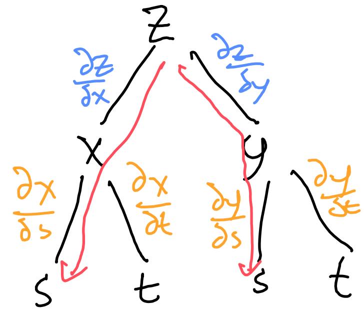


So $W_s(1,0) = F_u(2,3)\, u_s(1,0) + F_v(2,3)\, v_s(1,0)$
$= (-1)(-2) + (10)(5) = 52$

$W_t(1,0) = F_u(2,3)\, u_t(1,0) + F_v(2,3)\, v_t(1,0)$
$= (-1)(6) + (10)(4) = 34$

An alternate perspective to organize the Chain Rule: tree diagrams

If z is a function of x, y & x, y are each functions of s, t, we have the following dependencies



To compute $\frac{\partial z}{\partial s}$, say, follow all paths from z to s, multiply derivatives & add:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

- This is just another way of organizing the matrix mult.

Let's go back to the last two examples and apply this idea to compute the same rates of change.

Application to Implicit Differentiation: If $F(x, y, z) = c$ is used to *implicitly* define z as a function of x and y, then the chain rule says:

$$W = F(x, y, z(x,y)) = c$$

$$\frac{\partial W}{\partial x} = F_x \cdot \cancel{\frac{\partial x}{\partial x}}^{1} + F_y \cdot \cancel{\frac{\partial y}{\partial x}}^{0} + F_z \cdot \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(c) = 0$$

$$\text{so} \quad F_x + F_z \cdot \frac{\partial z}{\partial x} = 0$$

$$\text{so} \quad \boxed{\frac{\partial z}{\partial x} = \frac{-F_x}{F_z}}$$

$$\text{similarly,} \quad \boxed{\frac{\partial z}{\partial y} = \frac{-F_y}{F_z}}$$

Daily Announcements & Reminders:

- 14.3 HW due tonight
- 14.4 HW due Th
- Quiz 4 tomorrow on partial derivatives ← 14.3
  & linearization ← 14.6?

Goals for Today:

- Learn to compute the rate of change of a multivariable function in any direction
- Investigate the connection between the gradient vector and level curves/surfaces
- Discuss tangent planes to surfaces, how to find them, and when they exist

Example 53. Recall that if $z = f(x, y)$, then f_x represents the rate of change of z in the x-direction and f_y represents the rate of change of z in the y-direction. What about other directions? How can we compute the rate of change of z in other directions?

Let's go back to our hill example again, $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$. How could we figure out the rate of change of our height from the point $(2, 1)$ if we move in the direction $\langle -1, 1\rangle$?

Parameterize line through $(2,1)$ in direction $\langle -1,1\rangle$

$\vec{r}(t) = (2,1) + t\langle -1,1\rangle$

Want: $\frac{dh}{dt} = Dh(\vec{r}(t))\Big|_{t=0} \cdot D\vec{r}(t)\Big|_{t=0}$

$= \begin{bmatrix} -\frac{1}{2}x & -\frac{1}{2}y \end{bmatrix}_{(2,1)} \begin{bmatrix} -1 \\ 1 \end{bmatrix}_{t=0}$

$= \begin{bmatrix} -1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 1 - \frac{1}{2} = \boxed{\frac{1}{2}}$

Interpret: If we move away from $(2,1)$ with velocity $\langle -1,1\rangle$, then height is inc. at a rate of $\frac{1}{2}$ m / $\sqrt{2}$ m

w/1 ↓

verb.

in $\langle -1, 1 \rangle$ direction

Definition 54. The directional derivative of $f : \mathbb{R}^n \to \mathbb{R}$ at the point $\mathbf{p}$ in the direction of a unit vector $\mathbf{u}$ is

direction of change

$$D_{\mathbf{u}} f(\mathbf{p}) = \lim_{h \to 0} \frac{f(\vec{p} + h\vec{u}) - f(\vec{p})}{h}$$

if this limit exists.

E.g. for our hill example from the last page we have:

$$D_{\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle} h(2,1) = \frac{1}{2\sqrt{2}} \quad m/m$$

Note that $D_{\mathbf{i}} f = f_x \qquad D_{\mathbf{j}} f = f_y \qquad D_{\mathbf{k}} f = f_z$

$$\text{b/c} \quad D_{\vec{i}} f(x,y) = \lim_{h \to 0} \frac{f(\overbrace{(x,y) + h(1,0)}^{(x+h,\ y)}) - f(x,y)}{h} = f_x(x,y)$$

Definition 55. If $f : \mathbb{R}^n \to \mathbb{R}$, then the gradient of f at $\mathbf{p} \in \mathbb{R}^n$ is the vector function ∇f (or grad f) defined by

$$\nabla f(\mathbf{p}) = \langle f_{x_1}(\vec{p}), f_{x_2}(\vec{p}), \ldots, f_{x_n}(\vec{p}) \rangle$$

$$= (Df(\vec{p}))^T \qquad \text{e.g. } \nabla h = \begin{bmatrix} -\frac{1}{2}x \\ -\frac{1}{2}y \end{bmatrix}$$

Note: If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at a point $\mathbf{p}$, then f has a directional derivative at $\mathbf{p}$ in the direction of any unit vector $\mathbf{u}$ and

$$D_{\mathbf{u}} f(\mathbf{p}) = Df(\vec{p})\,\vec{u} = \nabla f(\vec{p}) \cdot \vec{u}$$

Example 56. Find the gradient vector and the directional derivative of each function at the given point **p** in the direction of the given vector **u**.

$D_{\vec{u}} f(\vec{p}) = \nabla f(\vec{p}) \cdot \vec{u}$

a) $f(x, y, z) = z \ln(x^2 + y^2), \mathbf{p} = (-1, 1, 0), \mathbf{u} = \left\langle \frac{1}{3}, \frac{-2}{3}, \frac{2}{3} \right\rangle$ – check unit vector

$\sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9}} = \sqrt{1} = 1$

$\nabla f = \langle f_x, f_y, f_z \rangle = \left\langle \frac{z}{x^2+y^2} \cdot 2x, \frac{z}{x^2+y^2} \cdot 2y, \ln(x^2+y^2) \right\rangle$

$\nabla f(-1,1,0) = \langle 0, 0, \ln(1+1) \rangle$

$D_{\langle \frac{1}{3}, -\frac{2}{3}, \frac{2}{3} \rangle} f(-1,1,0) = \langle 0, 0, \ln(2) \rangle \cdot \langle \frac{1}{3}, -\frac{2}{3}, \frac{2}{3} \rangle = \boxed{\frac{2}{3} \ln(2)}$

b) $g(x, y, z) = x^2 + 4xy^2 + z^2$, $\mathbf{p} = (1, 2, 1)$, **u** the unit vector in the direction of $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

$\nabla g = \langle 2x + 4y^2, 8xy, 2z \rangle$

$\vec{u} = \frac{\langle 1, 2, -1 \rangle}{|\langle 1, 2, -1 \rangle|} = \frac{1}{\sqrt{6}} \langle 1, 2, -1 \rangle$

$\nabla g(\underbrace{1,2,1}_{p}) = \langle 2 + 16, 16, 2 \rangle$

$18 + 32 - 2$

$D_{\vec{u}} g(1,2,1) = \langle 18, 16, 2 \rangle \cdot \frac{1}{\sqrt{6}} \langle 1, 2, -1 \rangle = \frac{1}{\sqrt{6}} \left(\langle 18, 16, 2 \rangle \cdot \langle 1, 2, -1 \rangle \right)$

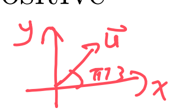
$= \frac{48}{\sqrt{6}} = 8\sqrt{6}$

c) $h(x, y) = e^{xy} - x^2, \mathbf{p} = (1, 1)$, **u** which makes an angle of $\pi/3$ with the positive x-axis.

Post-class:

$\nabla h = \langle y e^{xy} - 2x, x e^{xy} \rangle$

$\vec{u} = \langle \cos(\pi/3), \sin(\pi/3) \rangle = \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$



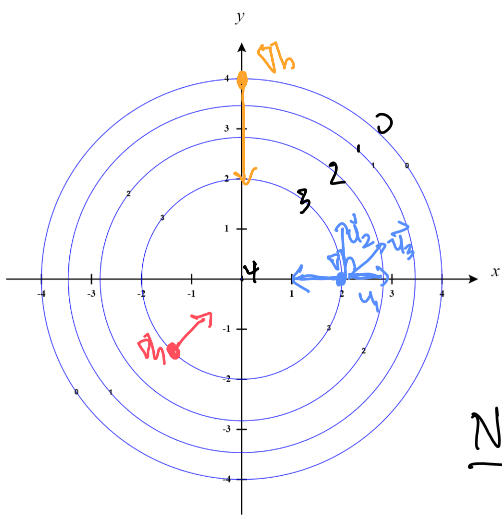
$\nabla h(1,1) = \langle e-2, e \rangle$

$D_{\vec{u}} h(1,1) = \langle e-2, e \rangle \cdot \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$

$= \frac{1}{2}e - 1 + \frac{\sqrt{3}}{2} e$

$= \boxed{\frac{1+\sqrt{3}}{2} e - 1}$

Example 57. If $h(x,y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$, the contour map is given below. Find and draw ∇h on the diagram at the points $(2,0), (0,4)$, and $(-\sqrt{2}, -\sqrt{2})$. At the point $(2,0)$, compute $D_{\mathbf{u}}h$ for the vectors $\mathbf{u}_1 = \mathbf{i}, \mathbf{u}_2 = \mathbf{j}, \mathbf{u}_3 = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$.



$\nabla h(x,y) = \langle -\frac{1}{2}x, -\frac{1}{2}y \rangle$

- $\nabla h(2,0) = \langle -1, 0 \rangle$
- $\nabla h(0,4) = \langle 0, -2 \rangle$
- $\nabla h(-\sqrt{2}, -\sqrt{2}) = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$

$D_{u_1}h(2,0) = \langle -1,0 \rangle \cdot \langle 1,0 \rangle = -1$

$D_{u_2}h(2,0) = \langle -1,0 \rangle \cdot \langle 0,1 \rangle = 0$

$D_{u_3}h(2,0) = \langle -1,0 \rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = -\frac{1}{\sqrt{2}}$

Notice:
- all pointing towards $(0,0)$
- all orthogonal to contour of h passing through the point used
- $\vec{u}_2$ tangent to level curve at $(2,0)$ and $D_{\vec{u}_2}h(2,0) = 0$

Q: Which direction should we move to maximize $D_{\vec{u}}f$?

A: $D_{\vec{u}}f(\vec{p}) = \nabla f(\vec{p}) \cdot \vec{u} = |\nabla f(p)| \cancel{|\vec{u}|} \cos\theta$

$\hookrightarrow$ biggest if $\cos\theta = 1 \Rightarrow \theta = 0$ $\Rightarrow \vec{u}$ is in direction of ∇f

$\hookrightarrow 0$ if $\cos\theta = 0$, $\theta = \pi/2$

$\hookrightarrow$ smallest if $\cos\theta = -1 \Rightarrow \theta = \pi$, $\vec{u}$ is exactly opposite ∇f

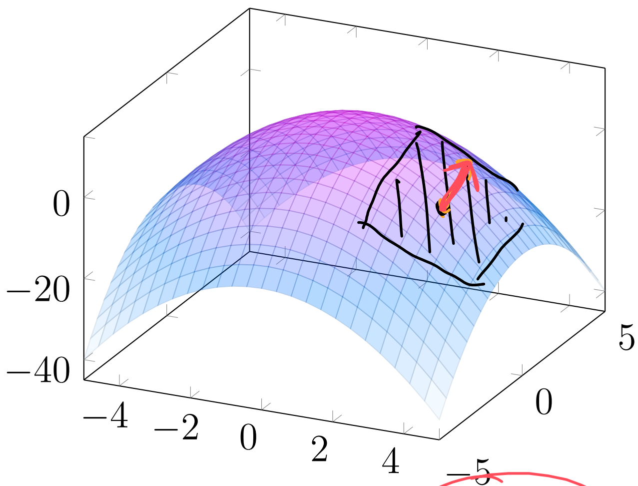
- Max rate of change $D_{\vec{u}}f(\vec{p}) = |\nabla f(p)|$

Note that the gradient vector is orthogonal to level curves.

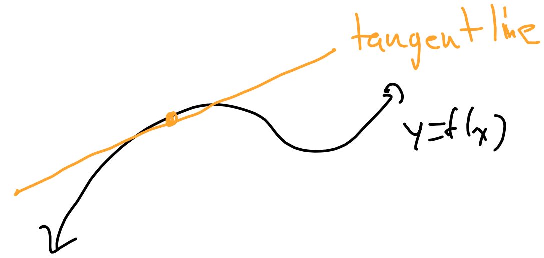
Similarly, for $f(x,y,z)$, $\nabla f(a,b,c)$ is orthogonal to level surfaces

Tangent planes to level surfaces

Suppose S is a surface with equation $F(x, y, z) = k$. How can we find an equation of the tangent plane of S at $P(x_0, y_0, z_0)$?



$x^2 + y^2 + z = 10, P = (-1, 3, 0)$



To find eqn of a plane, need

1) normal vector to plane ∇F at $\vec{p}$

2) point on plane ← given point of tangency

$F(x,y,z) = x^2+y^2+z = 10$

$\nabla F = \langle 2x, 2y, 1\rangle$

$\nabla F(-1,3,0) = \langle -2, 6, 1\rangle$

$P = (-1,3,0)$

$\rightarrow \boxed{-2(x+1)+6(y-3)+1(z-0)=0}$

tangent to surface at $(-1,3,0)$

This is a good place to actually define differentiable! We say $f : \mathbb{R}^n \to \mathbb{R}$ is **differentiable** at $\mathbf{a}$ if its linearization is a good approximation of f near $\mathbf{a}$. (Technical definition in textbook).

In particular, if f is a function $f(x, y)$ of two variables, it is differentiable at (a, b) if it has a unique tangent plane at (a, b).

Example 58. Find the equation of the tangent plane at the point $(-2, 1, -1)$ to the surface given by

$$z = 4 - x^2 - y$$

Rearange: $0 = 4 - x^2 - y - z$

$0 = F(x, y, z)$, where $F(x,y,z) = 4 - x^2 - y - z$

Tangent plane: need $\nabla F(-2,1,-1) = \langle -2x, -1, -1 \rangle \big|_{(-2,1,-1)}$

$= \langle 4, -1, -1 \rangle$

$4(x+2) - (y-1) - (z+1) = 0$

Special case: if we have $z = f(x, y)$ and a point $(a, b, f(a, b))$, the equation of the tangent plane is

$0 = f(x,y) - z$

$\downarrow$ same work

$f_x(a,b)(x-a) + f_y(a,b)(y-b) - (z - f(a,b)) = 0$

This should look familiar: it's the linearization

Daily Announcements & Reminders:

- 14.4 HW due tonight
- 14.5/6 HW due T
- Exam 2 is on 2/28, through next T material
- If you earned a U midterm grade, now is the best time to work on study habits, talk to me, come to office hours, etc.

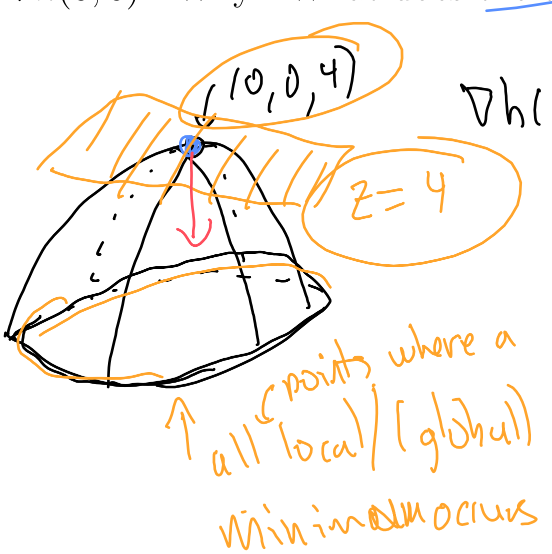
Goals for Today:

- Define local & global extreme values for functions of two variables
- Learn how to find local extreme values for functions of two variables
- Learn how to classify critical points for functions of two variables

Last time: If $f(x, y)$ is a function of two variables, we said $\nabla f(a, b)$ points in the direction of greatest change of f.

$\nabla h \perp$ to contours of h

Back to the hill $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$! What should we expect to get if we compute $\nabla h(0, 0)$? Why? What does the tangent plane to $z = h(x, y)$ at $(0, 0, 4)$ look like?



$\nabla h(0,0) = \langle 0, 0 \rangle$ b/c $\nabla h = \langle -\frac{1}{2}x, -\frac{1}{2}y \rangle$

b/c h has a local maximum at $(0,0)$

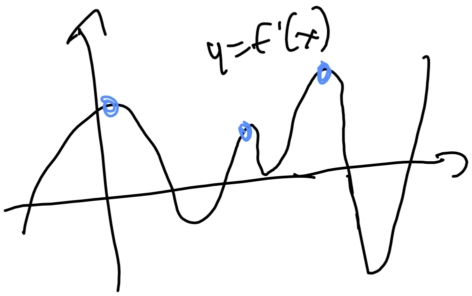
$0 = \underbrace{4 - \frac{1}{4}x^2 - \frac{1}{4}y^2 - z}_{F(x,y,z)}$

$\nabla F \perp$ to plane

$\boxed{\nabla F(0,0,4) = \langle 0, 0, -1 \rangle}$

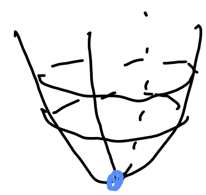
Definition 59. Let $f(x, y)$ be defined on a region containing the point (a, b). We say

- $f(a, b)$ is a local maximum value of f if $f(a, b)$ $\geq$ $f(x, y)$ for all domain points (x, y) in a disk centered at (a, b)
- $f(a, b)$ is a local minimum value of f if $f(a, b)$ $\leq$ $f(x, y)$ for all domain points (x, y) in a disk centered at (a, b)



• local max

local min of f(x,y)



local min

$-\nabla f$ points in direction of greatest

In $\mathbb{R}^3$, another interesting thing can happen. Let's look at $z = x^2 - y^2$ (a hyperbolic paraboloid!) near $(0, 0)$.

This is called a saddle point

Notice that in all of these examples, we have a horizontal tangent plane at the point in question, i.e.

$$\nabla f(a,b) = \langle 0, 0 \rangle$$

$$Df(a,b) = [0 \;\; 0]$$

Definition 60. If $f(x, y)$ is a function of two variables, a point (a, b) in the domain of f with $Df(a, b) =$ $[0, 0]$ or where $Df(a, b)$ does not exist is called a critical point of f.

• every local extreme value occurs at crit. pt.

<u>but</u> not all crit. pts are local extrema

Example 61. Find the critical points of the functions $f(x,y) = x^3 + y^3 - 3xy$ and $g(x,y) = -(x^2-1)^2 - (x^2y - x - 1)^2 + 2$.

1) a) $Df(x,y) = \begin{bmatrix} 3x^2 - 3y & 3y^2 - 3x \end{bmatrix}$

b) set $= [0\ 0]$

$3x^2 - 3y = 0$

$3y^2 - 3x = 0$

$x^2 - y = 0 \rightarrow y = x^2$

$y^2 - x = 0$ substitute

$x^4 - x = 0$

$x(x^3 - 1) = 0$

$x = 0$ OR $x^3 - 1 = 0$

$y = 0^2 = 0$

$x^3 = 1$

$x = 1$

$y = 1^2 = 1$

Crit pts: $(0,0)$, $(1,1)$

2) $Dg(x,y) =$

$$\Big[\underbrace{-2(x^2-1)2x - 2(x^2y - x - 1)(2xy - 1)}_{g_x} \quad \underbrace{-2(x^2y - x - 1)x^2}_{g_y}\Big]$$

Set $Dg = [0\ 0]$

① $-4x(x^2-1) - 2(x^2y - x - 1)(2xy - 1) = 0$ $\rightarrow -4x(x^2-1) - 0 = 0$

② $-2x^2(x^2y - x - 1) = 0$

$x \neq 0$

$x^2 - 1 = 0$

$(x-1)(x+1) = 0$

$x^2 = 0$ not possible OR ③ $x^2y - x - 1 = 0$ plug into ①

$x = 0$

$-2(0 - 0 - 1)(0 - 1) = 0$

$-2 = 0$ ✗

Plug into ③ $x = 1$ OR $x = -1$

$y - 1 - 1 = 0$

$y = 2$

$y + 1 - 1 = 0$

$y = 0$

Crit pts: $\boxed{(1,2)\ \&\ (-1,0)}$

To classify critical points, we turn to the **second derivative test** and the **Hessian matrix**.

The **Hessian matrix** of $f(x, y)$ at (a, b) is

$$Hf(a,b) = \begin{bmatrix} f_{xx}(a,b) & f_{yx}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{bmatrix}$$

$\det(Hf(a,b))$
$= f_{xx}f_{yy} - f_{xy}f_{yx}$ $\quad f_{xy}^2$

Theorem 62 (2nd derivative test)**.** *Suppose (a, b) is a critical point of $f(x, y)$ and $Hf(a, b)$ exists. Then we have:*

f behaves same way in x, y directions

- *If* $\det(Hf(a, b)) > 0$ *and* $f_{xx}(a, b) > 0$, *$f(a, b)$ is a local minimum*
- *If* $\det(Hf(a, b)) > 0$ *and* $f_{xx}(a, b) < 0$, *$f(a, b)$ is a local maximum*
- *If* $\det(Hf(a, b)) < 0$, *f has a saddle point at (a, b)* $\Leftarrow$ f behaves differently in x & y directions
- *If* $\det(Hf(a, b)) = 0$, *the test is inconclusive.*

Example 63. Classify the critical points of $f(x, y) = x^3 + y^3 - 3xy$ and $g(x, y) = -(x^2 - 1)^2 - (x^2y - x - 1)^2 + 2$ from Example 61.

1) f has crit pts $(0,0)$ & $(1,1)$

$f_x = 3x^2 - 3y \qquad f_y = 3y^2 - 3x$

- $Hf(x,y) = \begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6x & -3 \\ -3 & 6y \end{bmatrix}$

- Apply test at each crit pt.

Example 63 (cont.)

At $(0,0)$: $Hf(0,0) = \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix}$, so $\det(Hf(0,0)) = 0 - (-3)(-3)$
$= -9 < 0$

so by 2^{nd} derivative test, f has a saddle point at $(0,0)$

At $(1,1)$: $Hf(1,1) = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}$, so $\det(Hf(1,1)) = 36 - (-3)(-3)$
$= 27 > 0$

and $f_{xx}(1,1) > 0$

so by 2^{nd} derivative test, f has a local min at $(1,1)$

$f(1,1) = 1^3 + 1^3 - 3(1)(1) = -1$

2) g: With some algebra, we see ~~is~~ that <u>both</u> crit. pts are local maxima.

- This is <u>not</u> a thing that can happen for functions of 1 variable.

Appendix: Computations for classifying crit pts of g

Crit pts: $(1,2)$ & $(-1,0)$

$$g_x = -4x(x^2-1) - 2(x^2y-x-1)(2xy-1)$$

$$g_y = -2x^2(x^2y-x-1)$$

Compute 2nd derivatives:

$$g_{xx} = -4(x^2-1) - 4x(2x) - 2(2xy-1)(2xy-1) - 2(x^2y-x-1)(2y)$$

$$g_{xy} = 0 - 2(x^2)(2xy-1) - 2(x^2y-x-1)(2x)$$

$$g_{yx} = -4x(x^2y-x-1) - 2x^2(2xy-1)$$

$$g_{yy} = -2x^2(x^2)$$

Evaluate $Hg(1,2)$, $Hg(-1,0)$:

$$Hg(1,2) = \begin{bmatrix} 0 - 8 - 2(3)^2 - 2\overset{0}{(2-1-1)}(4) & -4(1)(0) - 2(1)(3) \\ -2(1)(3) - 2(0)(2) & -2(1)^4 \end{bmatrix} = \begin{bmatrix} -26 & -6 \\ -6 & -2 \end{bmatrix}$$

$$Hg(-1,0) = \begin{bmatrix} 0 - 4(-1)(-2) - 2(0-1)^2 - \overset{0}{(0+1-1)}(0) & -4(-1)(0) - 2(1)(-1) \\ -2(1)(-1) - 2(0)(-2) & -2(-1)^4 \end{bmatrix} = \begin{bmatrix} -10 & 2 \\ 2 & -2 \end{bmatrix}$$

Apply 2nd derivative test:

$\det(Hg(1,2)) = 52 - 36 = 16 > 0$ and $g_{xx}(1,2) = -26 < 0$

so g has a local max at $(1,2)$

$\det(Hg(-1,0)) = 20 - 4 = 16 > 0$ and $g_{xx}(-1,0) = -10 < 0$

so g has a local max at $(-1,0)$

Global extreme values

Start here on Tues

A global maximum of $f(x, y)$ is like a local maximum, except we must have $f(a, b) \geq f(x, y)$ for **all** (x, y) in the domain of f.

Theorem 64. *On a closed & bounded domain, any continuous function $f(x, y)$ attains a global minimum & maximum.*

Closed:

Bounded:

Strategy for finding global min/max of $f(x, y)$ on a closed & bounded domain R

1. Find all critical points of f inside R.
2. Find all critical points of f on the boundary of R
3. Evaluate f at each critical point as well as at any endpoints on the boundary.
4. The smallest value found is the global minimum; the largest value found is the global maximum.

Example 65. Find the global minimum and maximum of $f(x, y) = 4x^2 - 4xy + 2y$ on the region R bounded by $y = x^2$ and $y = 4$.

Daily Announcements & Reminders:

- HW 14.5/6 due tonight, 14.7 Th, 14.8 next T
- Quiz 5 tomorrow: Chain Rule, Gradient, Dir. Deriv
- Exam 2 next T, announcement on Canvas this afternoon

  ↑ 14.1 - 14.8

Goals for Today:

- Find global extreme values of continuous functions of two variables on closed & bounded domains
- Apply the method of Lagrange multipliers to find extreme values of functions of two or more variables subject to one or more constraints

• $h(0,0)=4$ is the global max for hill function

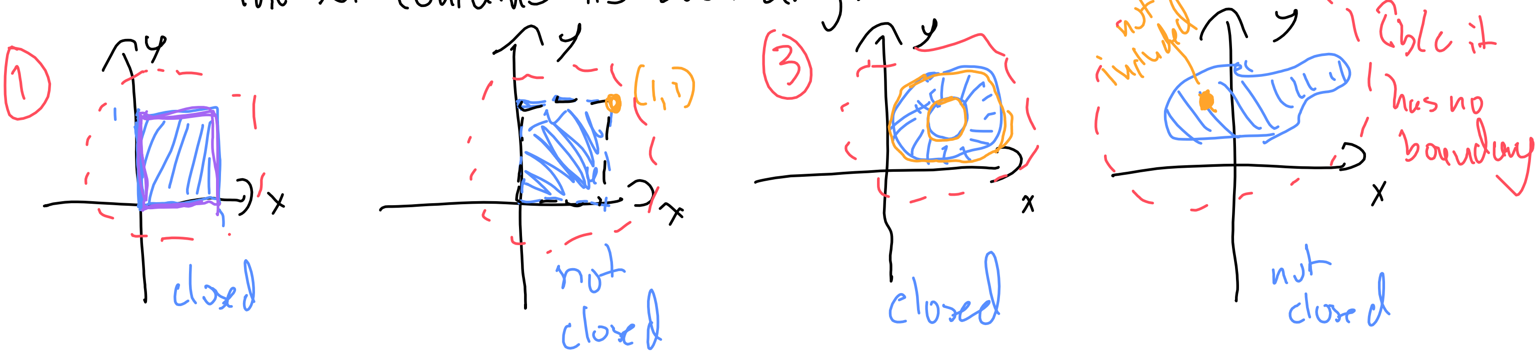
A global maximum of $f(x,y)$ is like a local maximum, except we must have $f(a,b) \geq f(x,y)$ for **all** (x,y) in the domain of f. A global minimum is defined similarly.

Theorem 64. *On a closed & bounded domain, any continuous function $f(x,y)$ attains a global minimum & maximum.*

EVT: If $f(x)$ is cts on $[a,b]$, f attains a global min/max on $[a,b]$.

Closed: The set contains its boundary.

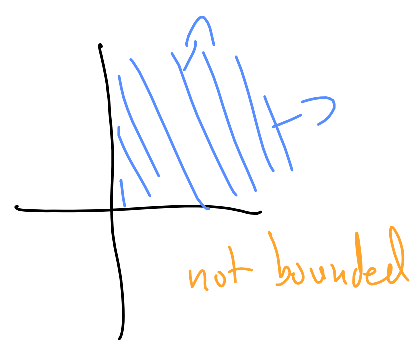
• $\mathbb{R}^2$ is closed b/c it has no boundary



Bounded:

The set fits in a big enough circle.

- All 4 pictures above are bounded.
- $\mathbb{R}^2$ is not bounded, line/parabola in $\mathbb{R}^2$

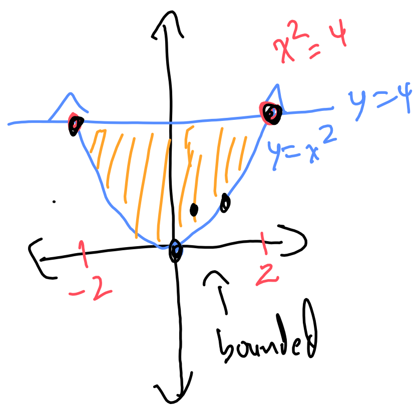


Strategy for finding global min/max of cts $f(x,y)$ **on a closed & bounded domain** R

1. Find all critical points of f inside R.
2. Find all critical points of f on the boundary of R
3. Evaluate f at each critical point as well as at any endpoints on the boundary.
4. The smallest value found is the global minimum; the largest value found is the global maximum.

Example 65. Find the global minimum and maximum of $f(x,y) = 4x^2 - 4xy + 2y$ on the closed region R bounded by $y = x^2$ and $y = 4$.

{polynomial → cts



Domain of optimization

1) Find crit pts of f inside R — not on boundary

- $Df = [0\ 0]$, solve

$$\begin{bmatrix} 8x-4y & -4x+2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

① $8x - 4y = 0$

② $-4x + 2 = 0 \to x = \frac{-2}{-4} = \frac{1}{2} \xrightarrow{\text{plug in ①}} 4 - 4y = 0 \Rightarrow y = 1$

• $(\frac{1}{2}, 1)$ is a crit. pt.

2) Find crit. pts of f on boundary

– Use boundary eqns to simplify to 1-var, optimize

a) On $y=4$: $f(x,4) = 4x^2 - 4x(4) + 2(4)$

$g(x) = 4x^2 - 16x + 8$

for $-2 \le x \le 2$

$\to g'(x) = 8x - 16 = 0$

$x = 2, y = 4$

• Add endpoints: $x=-2, y=4$, $x=2, y=4$

③ Evaluate

Test pts	f
$(\frac{1}{2}, 1)$	1
$(-2,4)$	~~5~~ 6
$(2,4)$	-8
$(0,0)$	0
$(1,1)$	2

Global max $= 56$ at $(-2,4)$

Global min $= -8$ at $(2,4)$

b) On $y=x^2$: Parameterize: $\vec{r}(t) = \langle t, t^2 \rangle$

$-2 \le t \le 2$

$f(\vec{r}(t))$

$f(t, t^2) = 4t^2 - 4 \cdot t \cdot t^2 + 2t^2$

$h(t) = -4t^3 + 6t^2$, $-2 \le t \le 2$

$\rightarrow h'(t) = -12t^2 + 12t = 0$

$12t(-t+1) = 0$

$t=0$ $\qquad$ $t=1$ $\downarrow$ Plug in $\vec{r}(t)$

$(x,y) = (0,0)$ $\qquad$ $(x,y) = (1,1)$

- check endpoints: $(-2,4)$, $(2,4)$

Constrained Optimization

Goal: Maximize or minimize $f(x,y)$ or $f(x,y,z)$ subject to a *constraint*, $g(x,y) = c$.

Example 66. A new hiking trail has been constructed on the hill with height $h(x,y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$, above the points $y = \cancel{-0.6x^2 + x} + 3$ in the xy-plane. What is the highest point on the hill on this path?

$y = -.5x^2 + 3$

Objective function: Thing we are optimizing

- $h(x,y)$

Constraint equation: Restriction on objectives

- $\underbrace{y + 0.5x^2}_{g(x,y) = y + 0.5x^2} = 3$

Idea: Think about contours

- Find all points where $\nabla h = \lambda \nabla g$ & $g(x,y) = c$

(x,y)	$h(x,y)$
$(0,3)$	$4 - \frac{9}{4} = \frac{7}{4}$
$(2,1)$	$4 - 1 - \frac{1}{4} = \frac{11}{4}$
$(-2,1)$	$4 - 1 - \frac{1}{4} = \frac{11}{4}$

$\nabla h = \langle -\frac{1}{2}x, -\frac{1}{2}y \rangle$

$\nabla g = \langle x, 1 \rangle$

$\nabla h = \lambda \nabla g, \quad g = c$

$\rightarrow$ Solve this

$$\begin{cases} \langle -\frac{1}{2}x, -\frac{1}{2}y \rangle = \lambda \langle x, 1 \rangle \\ y + 0.5x^2 = 3 \quad (3) \end{cases}$$

(1) $-\frac{1}{2}x = \lambda x$, (2) $-\frac{1}{2}y = \lambda$

$x = 0$ or $-\frac{1}{2} = \lambda$ $\xrightarrow{\text{Plug into (2)}}$ $-\frac{1}{2}y = -\frac{1}{2} \rightarrow \underline{y = 1}$

$x = 0$: Plug into (3) $\leftarrow$ $y + 0 = 3$

$y = 1$: Plug into (3) $\leftarrow$ $1 + 0.5x^2 = 3$ $\leftarrow$ $x = \pm 2$

- Highest elevation on the path is $\frac{11}{4}$, attained at $(2,1), (-2,1)$

Method of Lagrange Multipliers: To find the maximum and minimum values attained by a function $f(x, y, z)$ subject to a constraint $g(x, y, z) = c$, find all points where $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = c$ and compute the value of f at these points.

If we have more than one constraint $g(x, y, z) = c_1, h(x, y, z) = c_2$, then find all points where $\nabla f(x, y, z) = \lambda g(x, y, z) + \mu h(x, y, z)$ and $g(x, y, z) = c_1, h(x, y, z) = c_2$.

Example 67. Find the points on the surface $z^2 = xy + 4$ that are closest to the origin.

This example on Th before we start new material

Daily Announcements & Reminders:

- 14.7 HW due tonight, 14.8 on T
- Exam 2 on T, see Canvas announcement

Goals for Today:

- Review Lagrange Multipliers
- Define a double integral
- Compute simple double integrals

Method of Lagrange Multipliers: To find the maximum and minimum values attained by a function $f(x, y, z)$ subject to a constraint $g(x, y, z) = c$, find all points where $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = c$ and compute the value of f at these points.

If we have more than one constraint $g(x, y, z) = c_1, h(x, y, z) = c_2$, then find all points where $\nabla f(x, y, z) = \cancel{\lambda g(x, y, z) + \mu h(x, y, z)}$ and $g(x, y, z) = c_1, h(x, y, z) = c_2$.

$\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z)$

- ∇f is in Span$(\nabla g, \nabla h, \ldots)$

Example 68. Find the points on the surface $z^2 = xy + 4$ that are closest to the origin.

Objective: $d = \text{distance to origin} = \sqrt{x^2+y^2+z^2}$

dist (x,y,z) to $(1,1,1) = \sqrt{(x-1)^2+(y-1)^2+(z-1)^2}$

Constraint: $\underbrace{0 = xy - z^2+4}_{g(x,y,z)}$ OR $z^2 - xy = 4$

Useful trick: d is smallest/largest when d^2 is smallest or largest

$D = d^2 = x^2+y^2+z^2$

Apply Lagrange: $\nabla D = \lambda \nabla g$

$\Rightarrow \quad \langle 2x, 2y, 2z\rangle = \lambda\langle y, x, -2z\rangle$

$0 = xy - z^2 + 4$

$$\begin{cases} 2x = \lambda y & (1) \\ 2y = \lambda x & (2) \\ 2z = -2\lambda z & (3) \\ 0 = xy - z^2+4 & (4) \end{cases}$$

(3): $2z(1+\lambda) = 0$

$z = 0$ OR $1+\lambda = 0$

$\lambda = -1$

$z=0$:

Plug into (4): $0 = xy+4$

$\boxed{y = \frac{-4}{x}}$

$\lambda = -1$:

Plug into (1) / (2): $2x = -y$

$2y = -x$

$\downarrow$

Plug into ① / ②

$$2x = \lambda\left(-\frac{4}{x}\right)$$

$$2\left(-\frac{4}{x}\right) = \lambda x$$

$\downarrow$

a) $x^2 = -2\lambda \Rightarrow \lambda = \frac{x^2}{-2}$

b) $-8 = \lambda x^2$

$$-8 = \frac{x^4}{-2}$$

Plug a) into b)

$$x^4 = 16$$

$$x = \pm 2$$

$$y = \frac{-4}{x}$$

$x = 2 \Rightarrow y = -2$

$x = -2 \Rightarrow y = 2$

$(2, -2, 0)$

$(-2, 2, 0)$

$$2(-2x) = -x$$

$$-4x = -x$$

$$x = 0$$

so $y = -2(0) = 0$

Plug $x=0, y=0$ into ④

$$0 = 0 \cdot 0 - z^2 + 4$$

$$z^2 = 4$$

$$z = \pm 2$$

$(0, 0, 2)$, $(0, 0, -2)$

Plug all points into d:

$$d(0, 0, \pm 2) = 2$$

$$d(2, -2, 0) = d(-2, 2, 0)$$

$$= \sqrt{4 + 4 + 0}$$

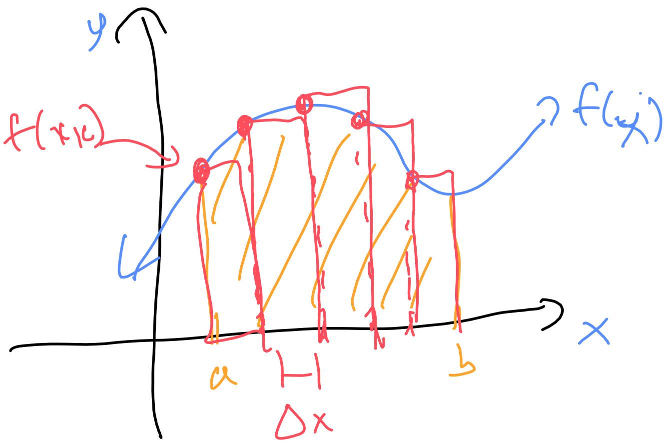
$$= \sqrt{8}$$

The points on $z^2 = xy + 4$ closest to origin are $(0,0,2)$ & $(0,0,-2)$.

Meaning of λ: λ is the rate of change of the extreme value w.r.t. to c

Double Integrals

Recall: Riemann sum and the definite integral from single-variable calculus.



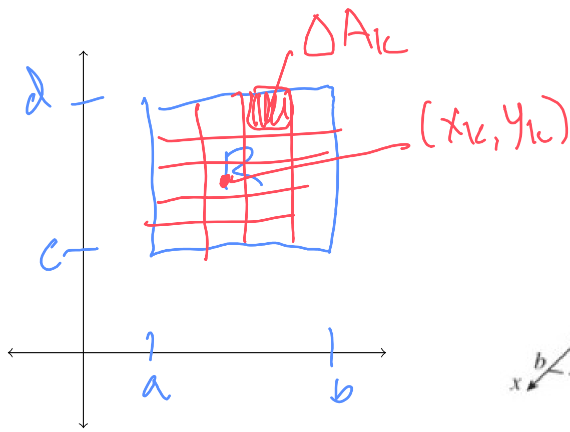
$$\text{Area} \approx \sum_{k=1}^{n} f(x_k)\Delta x$$

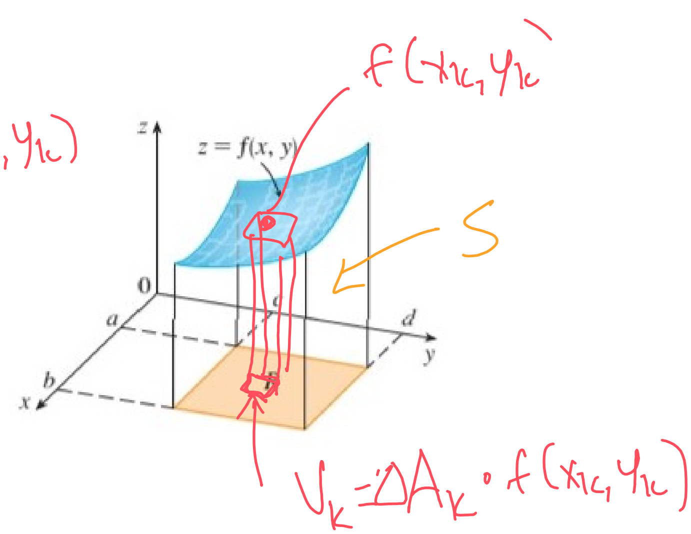
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$$

Volumes and Double integrals Let R be the closed rectangle defined below:

$$R = \underbrace{[a,b]}_{x} \times \underbrace{[c,d]}_{y} = \{(x,y) \in \mathbb{R}^2 | a \le x \le b, c \le y \le d\}$$

Let $f(x,y)$ be a function defined on R such that $f(x,y) \ge 0$. Let S be the solid that lies above R and under the graph f.





Question: How can we estimate the volume of S?

$$\text{Volume} \approx \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k$$

Definition 69. The double integral of $f(x, y)$ over a rectangle R is

$$\iint_R f(x,y)\ dA = \lim_{|P|\to 0} \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k$$

if this limit exists.

$|P|$ = biggest area of a rectangle in your division of R

- $\iint_R f(x,y)\,dA$ = volume between $z=f(x,y)$ & $z=0$ above R

- If f is cts on R, the limit exists
- Some discontinuous f are integrable.

Question: How can we compute a double integral?

Answer: Iterated Integrals

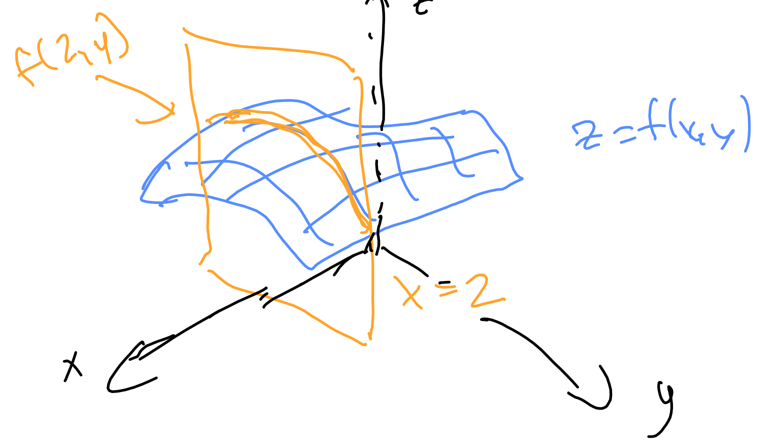
Suppose that f is a function of two variables that is integrable on the rectangle $R = [a, b] \times [c, d]$.

e.g. $f = 5 - y^2 - x$

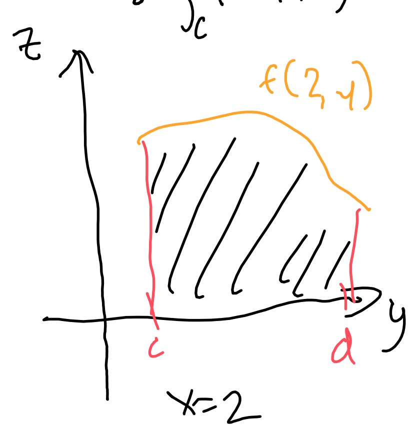
$[0,3]\times[1,2]$

What does $\int_c^d f(2, y)dy$ represent?

$f(2,y)$ is a function of 1-var.



- $\int_c^d f(2,y)dy$ = cross-sectional area of S at $x=2$



What about $\underbrace{\int_c^d f(x,y)dy}$? : area of cross-section of solid S for each different x

Let $\boxed{A(x) = \int_c^d f(x,y)dy.}$ Then,

Volume of S $= \displaystyle\int_a^b A(x)dx = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$

This is called an iterated integral.

Example 70. Evaluate $\displaystyle\int_1^2 \int_3^4 6x^2y\;dy\;dx$. $\overset{?}{=} \displaystyle\int_3^4 \int_1^2 6x^2y\,dx\,dy$

$3 \le y \le 4$ $\quad 1 \le x \le 2$

$$= \int_1^2 \left(6x^2 \cdot \frac{y^2}{2}\right)\Big|_{y=3}^{y=4} dx = \int_1^2 6x^2 \cdot 8 - 6x^2 \cdot \frac{9}{2}\, dx$$

$$= \int_1^2 21x^2\, dx$$

$$= 7x^3\Big|_1^2 = 7\cdot 8 - 7\cdot 1$$

$$= \boxed{49}$$

Theorem 71 (Fubini's Theorem)**.** *If f is continuous on the rectangle $R = [a,b] \times [c,d]$, then*

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy = \iint_R f(x,y)\,dA$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integral exist.

Caution: Some functions don't work.

Try $f(x,y) = \dfrac{x^2 - y^2}{(x^2+y^2)^2}$ on $[0,1] \times [0,1]$

Daily Announcements & Reminders:

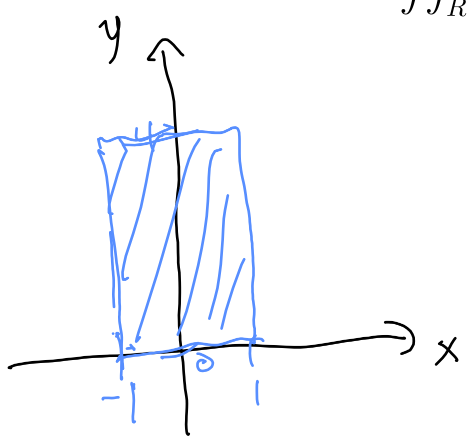
- 15.1 HW due tonight
- 15.2 HW due T
- Quiz next W on 15.1–15.3
- Exam 2 graded by next F

Goals for Today:

- Be able to set up & evaluate a double integral over a general region
- Change the order of integration for general regions
- Compute areas of general regions in the plane
- Compute the average value of a function of two variables

Example 72. Compute $\iint_R xe^{e^{e^y}}\, dA$, where R is the rectangle $[-1,1]\times[0,4]$.

R:



const. wrt x

$$\int_0^4 \int_{-1}^{1} x e^{e^{e^y}}\, dx\, dy$$

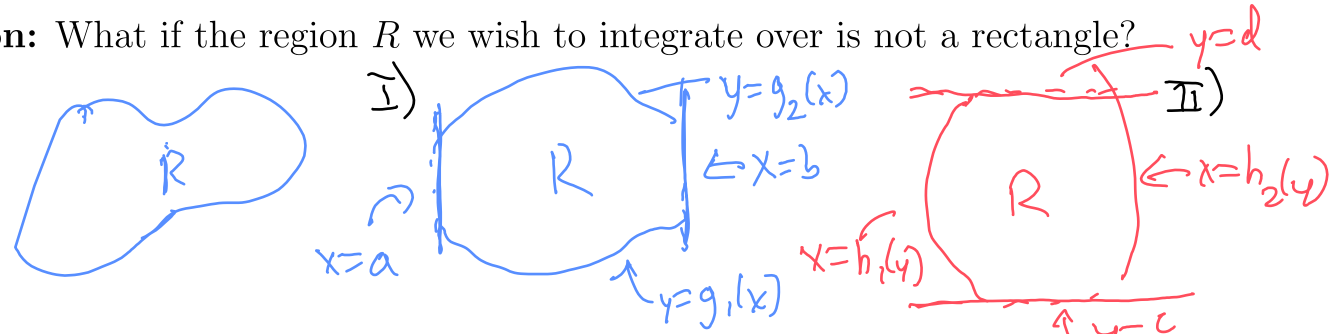
$$= \int_0^4 \frac{x^2}{2}\cdot e^{e^{e^y}} \Big|_{x=-1}^{x=1} dy$$

$$= \int_0^4 e^{e^{e^y}} \cancelto{0}{\left(\frac{1}{2} - \frac{(-1)^2}{2}\right)} dy$$

$$= \int_0^4 0\, dy = 0$$

- Other order $dy\,dx$ very hard!

Question: What if the region R we wish to integrate over is not a rectangle?

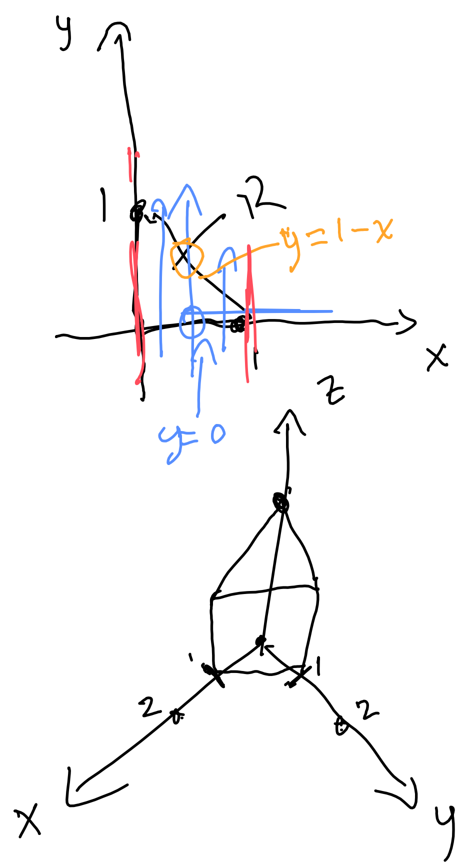


Answer: Repeat same procedure - it will work if the boundary of R is smooth and f is continuous.

Example 73. Compute the volume of the solid whose base is the triangle with vertices $(0,0), (0,1), (1,0)$ in the xy-plane and whose top is $z = 2 - x - y$.

Type I region: Use Fubini: $\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$

Sketch:



$$V = \iint_R f(x,y)\,dA = \iint_R 2 - x - y\,dA$$

$g_2(x) = 1 - x$

$g_1(x) = 0$

$$= \int_0^1 \underbrace{\int_0^{1-x} 2 - x - y\,dy}\,dx$$

$y = -x$

$y = 1 - x$

$\boxed{\text{ALWAYS CONSTANTS}}$

$$= \int_0^1 2y - xy - \frac{y^2}{2}\Big|_{y=0}^{y=1-x}\,dx$$

$$= \int_0^1 2(1-x) - x(1-x) - \frac{(1-x)^2}{2} - (0)\,dx$$

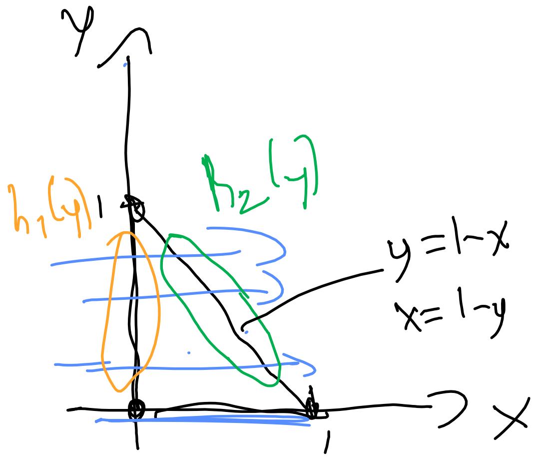
$$= \int_0^1 2 - 2x - x + x^2 - \frac{1}{2}(1 - 2x + x^2)\,dx$$

$$= \int_0^1 \frac{3}{2} - 2x + \frac{1}{2}x^2\,dx$$

$$= \frac{3}{2}x - x^2 + \frac{1}{6}x^3\Big|_0^1 = \frac{3}{2} - 1 + \frac{1}{6}$$

$$= \frac{2}{3}$$

Type II region:



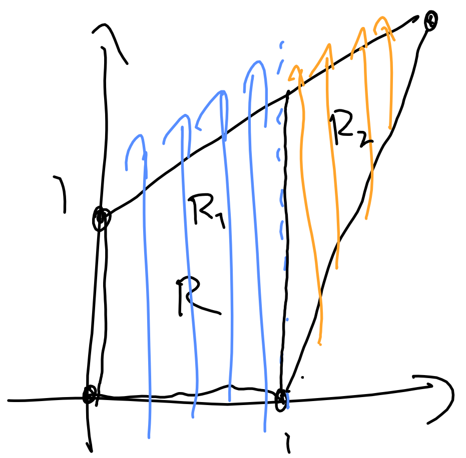
$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

$f(x,y)=2-x-y$

$$V=\int_0^1 \int_0^{1-y} 2-x-y\,dx\,dy$$

top: 1 $\times$ $1-y$ ✓

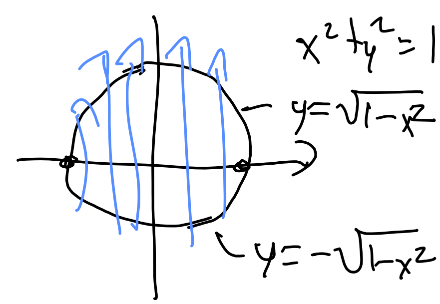
$y-1$ $\times$



$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$

– split region where bounds are consistent on small region

Example 74. Write the two iterated integrals for $\iint_R 1\ dA$ for the region R which is bounded by $y=\sqrt{x}, y=0$, and $x=9$.



$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\,dy\,dx$$

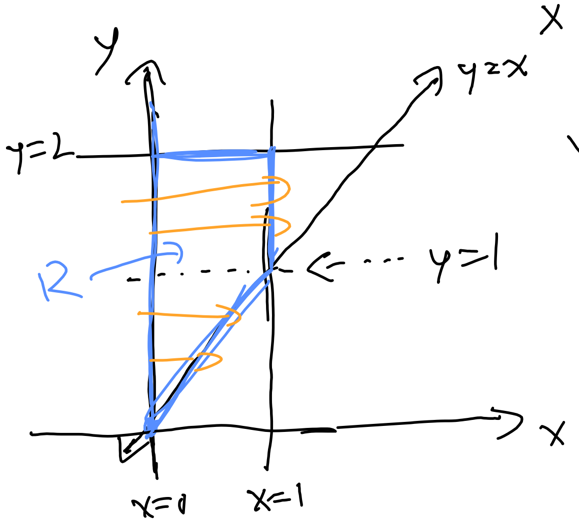
next time!

Example 75. Set up an iterated integral to evaluate the double integral $\iint_R 6x^2y\, dA$, where R is the region bounded by $y = 1, x = 0, x = 2$, and $y = x$.

$x=1 \quad x=0 \quad y=2 \quad y=x$

Sketch region: $0 \le x \le 1$

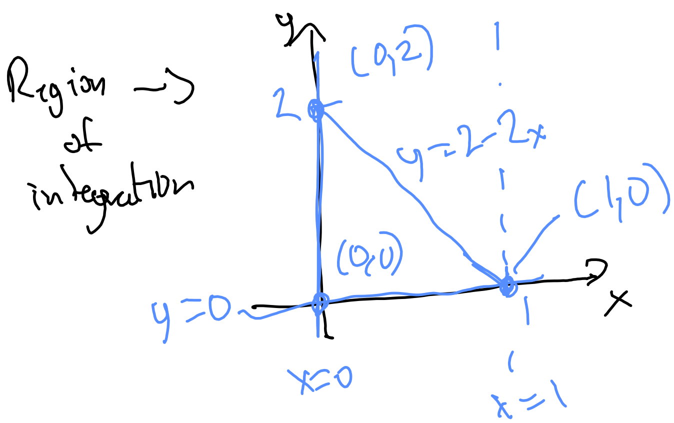
$x \le y \le 2$



vertical: $\iint_R 6x^2y\, dA = \int_0^1 \int_x^2 6x^2y\, dy\, dx$

horiz: $\int_0^1 \int_0^y 6x^2y\, dx\, dy + \int_1^2 \int_0^1 6x^2y\, dx\, dy$

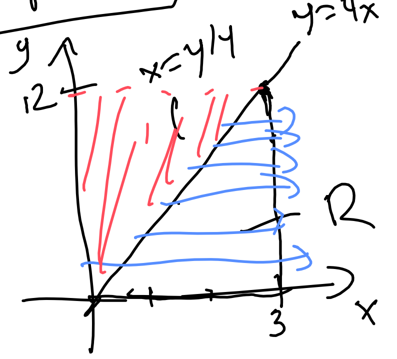
Itempool 1) $\int_0^1 \int_0^{2-2x} f(x,y)\, dy\, dx$



$0 \le y \le 2-2x$

$0 \le x \le 1$

Itempool 2)



$0 \le y \le 4x$

$0 \le x \le 3$

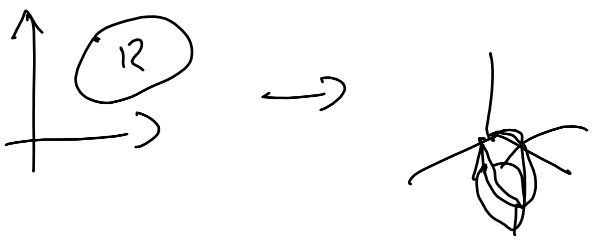
$\int_0^{12} \int_{y/4}^3 f(x,y)\, dx\, dy$

Area & Average Value

Two other applications of double integrals are computing the area of a region in the plane and finding the average value of a function over some domain.

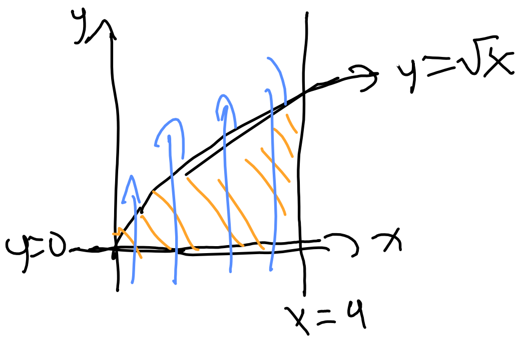
Area: If R is a region bounded by smooth curves, then

$$\text{Area}(R) = \iint_R 1 \cdot dA$$



$$V = \iint_R 1 \cdot dA = \text{Area } R \cdot 1$$

Example 76. Find the area of the region R bounded by $y = \sqrt{x}$, $y = 0$, and $x = 9$.



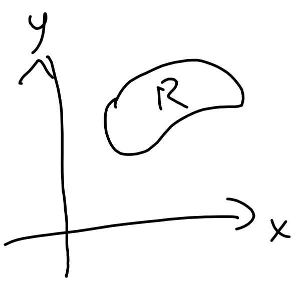
$$\text{Area} = \iint_R dA = \int_0^9 \int_0^{\sqrt{x}} dy\, dx$$

$$= \int_0^9 y\Big|_0^{\sqrt{x}} dx = \int_0^9 \sqrt{x}\, dx$$

$$= \frac{2}{3} x^{3/2}\Big|_0^9$$

$$= 18$$

Average Value: The average value of $f(x, y)$ on a region R contained in $\mathbb{R}^2$ is



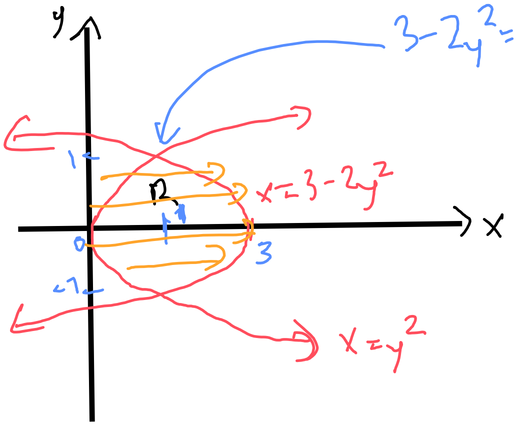
$$f_{avg} = \frac{\iint_R f(x,y)\, dA}{\text{Area }(R)}$$

$$f_{avg} = \frac{1}{b-a} \int_a^b f(x)\, dx$$

Example 77. The temperature at any point on a metal plate in the xy-plane is given by $T(x,y) = 100 - 4x^2 - y^2$, where x and y are measured in inches and T in degrees Celsius. Consider the portion of the plate that lies on the region that is the finite region that lies between the parabolas $x = y^2$ and $x = 3 - 2y^2$. Find the average temperature of this portion of the plate.

• Post lecture:

1) Sketch R $\boxed{\text{Type II Region}}$



$3-2y^2 = y^2 \iff 3 = 3y^2$

$y = \pm 1$

$x = 1$

2) Compute Area(R):

$$\iint_R 1\, dA = \int_{-1}^{1}\int_{y^2}^{3-2y^2} dx\, dy$$
$$= \int_{-1}^{1} 3 - 2y^2 - y^2\, dy$$
$$= \int_{-1}^{1} 3 - 3y^2\, dy$$
$$= 3y - y^3 \Big|_{-1}^{1}$$
$$= (3-1) - (-3-(-1)^3)$$
$$= 4$$

3) Compute $\iint_R T(x,y)\, dA$: (same setup of integral)

$$= \int_{-1}^{1}\int_{y^2}^{3-2y^2} 100 - 4x^2 - y^2\, dx\, dy$$
$$= \int_{-1}^{1} 100x - \frac{4}{3}x^3 - y^2 x \Big|_{y^2}^{3-2y^2} dy$$
$$= \int_{-1}^{1} \left[300 - 200y^2 - \frac{4}{3}(3-2y^2)^3 - y^2(3-2y^2)\right] - \left[100y^2 - \frac{4}{3}y^6 - y^4\right] dy$$
$$= \int_{-1}^{1} 300 - 303y^2 + 3y^4 + \frac{4}{3}y^6 - \frac{4}{3}(27 - 54y^2 + 36y^4 - 8y^6)\, dy$$
$$= \int_{-1}^{1} 264 - 231y^2 - 45y^4 - \frac{28}{3}y^6\, dy$$
$$= 264y - 77y^3 - 9y^5 - \frac{4}{3}y^7 \Big|_{-1}^{1}$$
$$= 2\left(264 - 77 - 9 - \frac{4}{3}\right) = \frac{1060}{3}$$

303

72

4) $$T_{avg} = \frac{\iint_R T\, dA}{\iint_R dA} = \frac{1060/3}{4} = \frac{265}{3}\,^\circ C = 83.\overline{3}\,^\circ C$$

Daily Announcements & Reminders:

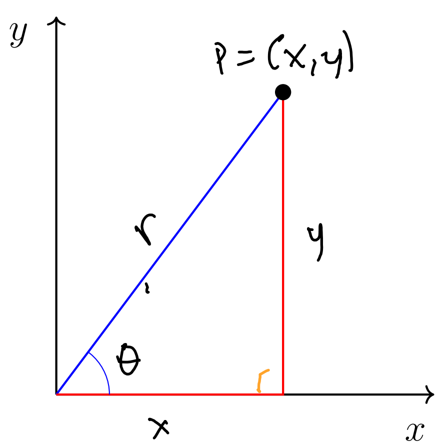
- Quiz 6 tomorrow, 15.1-15.3
- 15.2 HW due tonight
- 15.3 HW due R
- Deadline to withdraw/change grade mode is 3/15 (next W)

- By end of semester, need 200 HW points – you will need to do some of the "Practice Problems"

Goals for Today: Sections 15.4, 15.5

- Introduce the polar coordinate system
- Convert double integrals to iterated polar integrals
- Define triple integrals and compute basic triple integrals

Polar Coordinates:



Cartesian coordinates: Give the distances in $\vec{i}$ and $\vec{j}$ directions from (0,0)

Polar coordinates:

- r = distance from (0,0) to P
- θ = angle between the ray $\overrightarrow{OP}$ and the positive x-axis

We can use trigonometry to go back and forth.

Polar to Cartesian:

$$x = r\cos(\theta) \qquad y = r\sin(\theta)$$

- unique

Cartesian to Polar:

$$r^2 = x^2 + y^2 \qquad \tan(\theta) = \frac{y}{x}$$

- not unique

Example 78. a) Find a set of polar coordinates for the point $(x, y) = (1, 1)$.

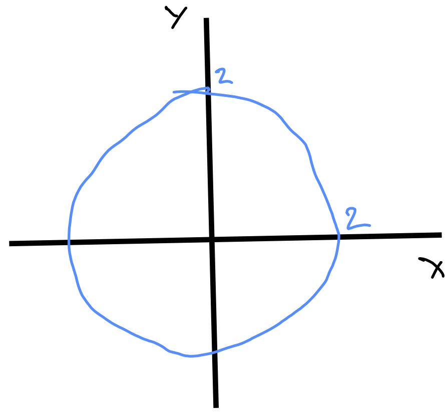
$$r^2 = 1^2 + 1^2 = 2 \qquad \tan(\theta) = \frac{1}{1} = 1$$

$$\boxed{r = \sqrt{2} \qquad \theta = \pi/4}$$

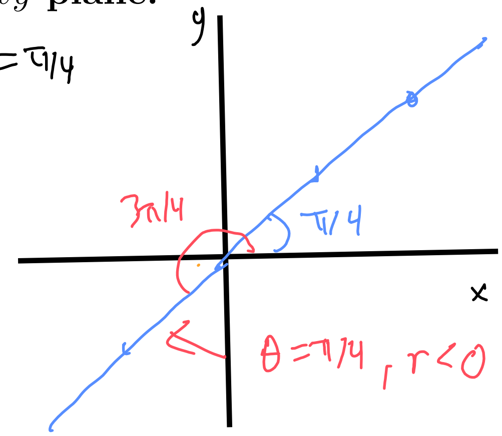
dist from $(0,0) = 2$

b) Graph the set of points (x, y) that satisfy the equation $r = 2$ and the set of points that satisfy the equation $\theta = \pi/4$ **in the** xy**-plane.**

$r=2$



$\theta = \pi/4$



c) Write the function $f(x, y) = \sqrt{x^2 + y^2}$ in polar coordinates.

$$f(r, \theta) = \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2} = \sqrt{r^2(\cos^2\theta + \sin^2\theta)}$$

$$= \sqrt{r^2} = |r|$$

d) [Itempool] Write a Cartesian equation describing the points that satisfy $r = 2\sin(\theta)$.

$$r = 2\sin\theta$$

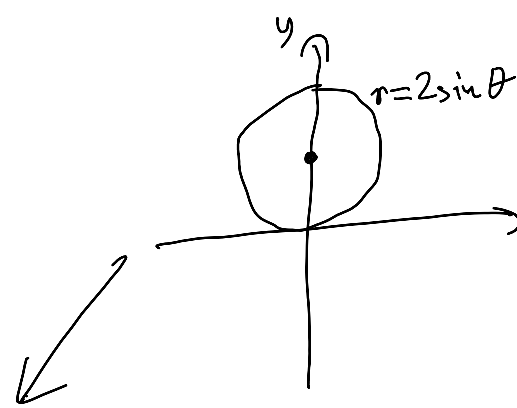
$$r^2 = 2r\sin\theta$$

$$x^2 + y^2 = 2y$$

$$x^2 + y^2 - 2y = 0$$

$$x^2 + y^2 - 2y + 1 = 1$$

$$x^2 + (y-1)^2 = 1$$



$r = 2\cos\theta \Rightarrow$ circle shifted along x-axis

15.4: Double Integrals in Polar Coordinates

Goal: Given a region R in the xy-plane described in polar coordinates and a function $f(r, \theta)$ on R, compute $\iint_R f(r, \theta)\, dA$.

Example 79. Compute the area of the disk of radius 5 centered at $(0, 0)$.

Area $= 25\pi$

Area $= \iint_R 1 \boxed{dA}$ WRONG

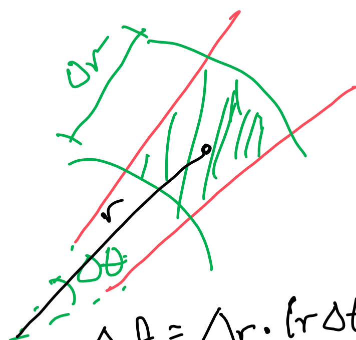
$= \int_0^{2\pi} \int_0^5 1 \boxed{dr\, d\theta}$

$= \int_0^{2\pi} r \Big|_0^5 d\theta$

$= \int_0^{2\pi} 5\, d\theta$

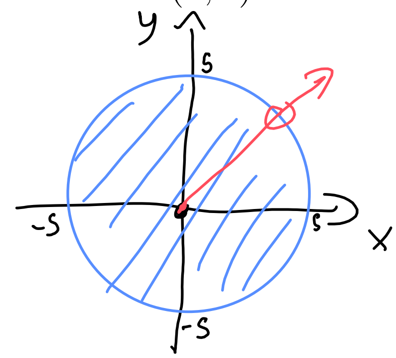
$= 5\theta \Big|_0^{2\pi}$

$= 10\pi$



$\Delta A = \Delta r \cdot (r \Delta\theta)$

↳ $\boxed{dA = r\, dr\, d\theta}$



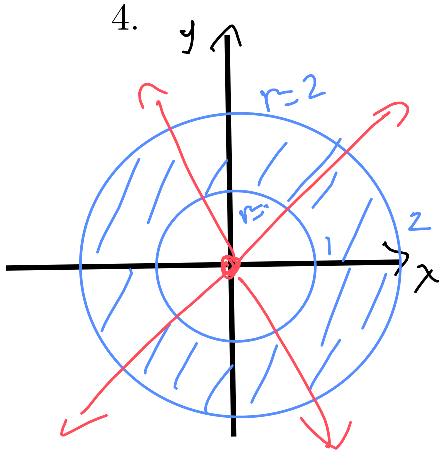
$0 \le r \le 5$

$0 \le \theta \le 2\pi$

Area $= \iint_R 1\, dA = \int_0^{2\pi} \int_0^5 1 \cdot r\, dr\, d\theta = 25\pi$ ✓

Remember: In polar coordinates, the area form $dA =$ $r\, dr\, d\theta$

Example 80. Compute $\iint_D e^{-(x^2+y^2)}\, dA$ on the washer-shaped region $1 \le x^2+y^2 \le 4$.



$1 \le r \le 2$

$0 \le \theta \le 2\pi$

$$\iint_D e^{-(x^2+y^2)}\, dA$$

no antiderivative wrt x, y

$$= \int_0^{2\pi} \int_1^2 e^{-r^2}\, r\, dr\, d\theta = \int_0^{2\pi} -\tfrac{1}{2} e^{-r^2} \Big|_1^2 \, d\theta$$

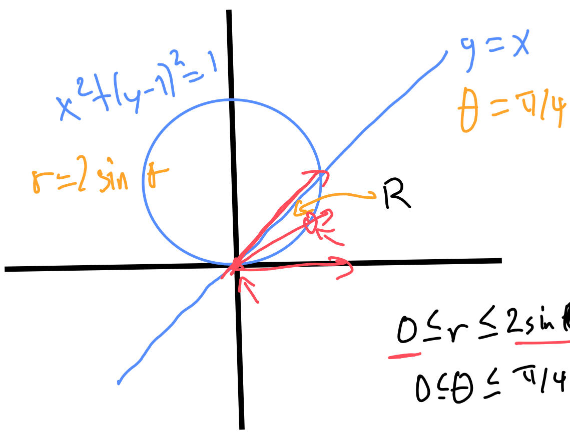
$u = -r^2$

$du = -2r\,dr$

$$= \int_0^{2\pi} -\tfrac{1}{2}\left(e^{-4} - e^{-1}\right) d\theta$$

$$= \boxed{\pi\left(e^{-1} - e^{-4}\right)}$$

Example 81. Compute the area of the smaller region bounded by the circle $x^2 + (y-1)^2 = 1$ and the line $y = x$.



$0 \le r \le 2\sin\theta$

$0 \le \theta \le \pi/4$

$$\text{Area} = \iint_R 1\, dA$$

$$= \int_0^{\pi/4} \int_0^{2\sin\theta} r\, dr\, d\theta$$

MUST BE CONSTANTS

$$= \int_0^{\pi/4} \tfrac{1}{2}(2\sin\theta)^2\, d\theta$$

$$= \int_0^{\pi/4} 2\sin^2\theta\, d\theta$$

$\sin^2\theta = \frac{1}{2}(1-\cos(2\theta))$

$\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$

$$= \int_0^{\pi/4} 1 - \cos(2\theta)\, d\theta$$

$$= \theta - \tfrac{1}{2}\sin(2\theta) \Big|_0^{\pi/4}$$

$$= \tfrac{\pi}{4} - \tfrac{1}{2}(1) - [0-0]$$

$$= \tfrac{\pi}{4} - \tfrac{1}{2}$$

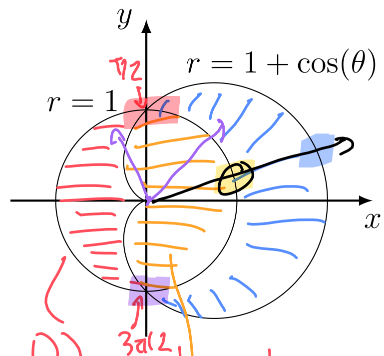
$f(r,\theta) = r^2$

Can define partial derivatives wrt r, θ too!

$f_r = 2r$

$\Leftrightarrow$ rate of change of f in the radial direction is 2 times dist from $(0,0)$

Example 82 (Itempool)**.** Write an integral for the volume under $z = x$ on the region between the cardioid $r = 1 + \cos(\theta)$ and the circle $r = 1$, where $x \geq 0$.



$$V = \int_{3\pi/2}^{5\pi/2} \int_{1}^{1+\cos\theta} r^2 \cos\theta \, dr \, d\theta$$

- θ should be increasing
- $r > 0$ for integrals

D) computes volume over this region

c) dA needs $r\,dr\,d\theta$

$$\int_{-\pi/2}^{\pi/2} \int_0^1 r^2 \cos\theta \, dr \, d\theta + \int_{\pi/2}^{3\pi/2} \int_0^{1+\cos\theta} r^2 \cos\theta \, dr \, d\theta$$

15.5 Triple Integrals

Start here Th

Idea: Suppose D is a solid region in $\mathbb{R}^3$. If $f(x, y, z)$ is a function on D, e.g. mass density, electric charge density, temperature, etc., we can approximate the total value of f on D with a Riemann sum.

$$\sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta V_k,$$

by breaking D into small rectangular prisms ΔV_k. Taking the limit gives a

$$\underline{\hspace{8cm}} : \iiint_D f(x, y, z)\ dV$$

Important special case:

$$\iiint_D 1\ dV = \underline{\hspace{6cm}}$$

Again, we have Fubini's theorem to evaluate these triple integrals as iterated integrals.

Computationally, this is straightforward.

Example 83. Compute $\displaystyle\int_0^1 \int_0^2 \int_0^x \ dz\ dy\ dx$ and interpret your answer.

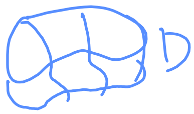
Daily Announcements & Reminders:

- 15.3 HW due tonight
- 15.4–6 HW due next week
- Exam 2 grades released
  - regrade request open until W at 5pm, read solution first
  - median score 86%, mean 82.14%

Goals for Today: Sections 15.5, 15.6

- Learn how to write triple integrals as iterated integrals.
- Change the order of integration in a triple iterated integral.
- Apply our work to find the mass and center of mass of objects in $\mathbb{R}^2$ and $\mathbb{R}^3$

15.5 Triple Integrals



Idea: Suppose D is a solid region in $\mathbb{R}^3$. If $f(x, y, z)$ is a function on D, e.g. mass density, electric charge density, temperature, etc., we can approximate the total value of f on D with a Riemann sum.

$$\sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta V_k,$$

by breaking D into small rectangular prisms ΔV_k. Taking the limit gives a

triple integral : $\iiint_D f(x, y, z)\, dV$

Important special case:

$$\iiint_D 1\, dV = \text{volume of } D$$

$$f_{avg} = \frac{\iiint_D f(x,y,z)\,dV}{\text{volume of } D}$$

Again, we have Fubini's theorem to evaluate these triple integrals as iterated integrals.

Computationally, this is straightforward.

Example 83. Compute $\displaystyle\underbrace{\int_0^1 \underbrace{\int_0^2 \underbrace{\int_0^3 dz}_{z}\, dy}_{y}\, dx}_{x}$ and interpret your answer.

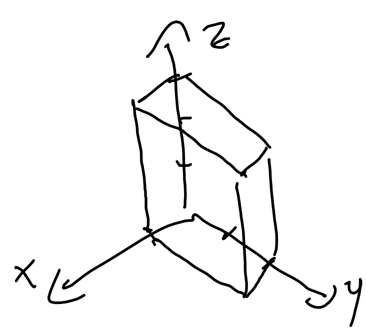
$$= \int_0^1 \int_0^2 z\Big|_0^3\, dy\, dx$$

$$= \int_0^1 \int_0^2 3\, dy\, dx$$

$$= \int_0^1 3y\Big|_0^2\, dx$$

$$= \int_0^1 6\, dx = 6x\Big|_0^1 = 6$$

Volume of region $0 \le z \le 3$, $0 \le y \le 2$, $0 \le x \le 1$



CONSTANTS — only outer variables — any variable except

Example 84. 1. **Mechanics:** Compute $\int_0^1 \int_0^{2-x} \int_0^{2-x-y} dz\, dy\, dx$.

$$= \int_0^1 \int_0^{2-x} z \Big|_0^{2-x-y} dy\, dx$$

$$= \int_0^1 \int_0^{2-x} 2-x-y \; dy\, dx$$

$$= \int_0^1 (2-x)y - \frac{1}{2}y^2 \Big|_0^{2-x} dx$$

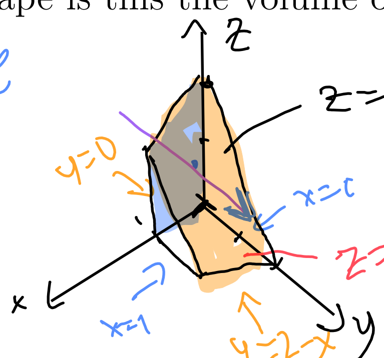
$$= \int_0^1 \frac{1}{2}(2-x)^2\, dx = -\frac{1}{6}(2-x)^3 \Big|_0^1 = -\frac{1}{6}(1)^3 + \frac{1}{6}(2)^3 = \frac{7}{6}$$

2. **Interpretation:** What shape is this the volume of?

$0 \le z \le 2-x-y$ plane

$0 \le y \le 2-x$

$0 \le x \le 1$

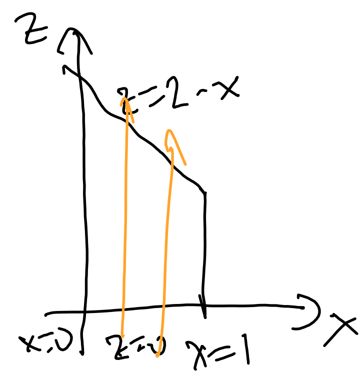


$z = 2-x-y \iff y = 2-x-z$

3. **Rearrange:** Write an equivalent iterated integral in the order $dy\, dz\, dx$.

$$\int_0^1 \int_0^{2-x} \int_0^{2-x-z} 1 \; dy\, dz\, dx$$

1) Find inner bounds by drawing arrow and seeing enter/leave



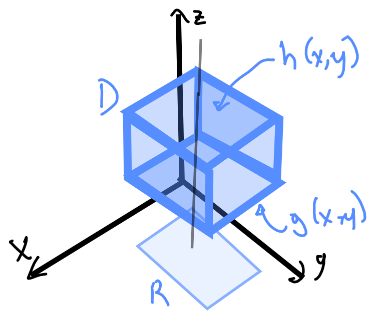
2) Sketch shadow and setup double integral

We will think about converting triple integrals to iterated integrals in terms of the shadow of D on one of the coordinate planes.

Case 1: z**-simple**) region. If R is the shadow of D on the xy-plane and D is bounded above and below by the surfaces $z = h(x,y)$ and $z = g(x,y)$, then

$$\iiint_D f(x,y,z)\, dV = \iint_R \left(\int_{g(x,y)}^{h(x,y)} f(x,y,z)\, dz \right) dy\, dx$$

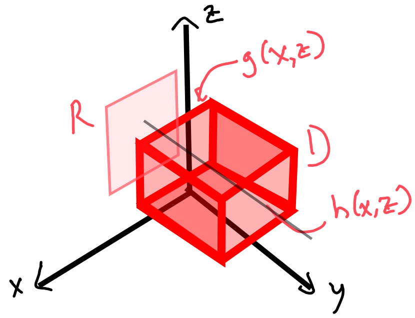
OR $dx\, dy$



Case 2: y**-simple**) region. If R is the shadow of D on the xz-plane and D is bounded right and left by the surfaces $y = h(x,z)$ and $y = g(x,z)$, then

$$\iiint_D f(x,y,z)\, dV = \iint_R \left(\int_{g(x,z)}^{h(x,z)} f(x,y,z)\, dy \right) dz\, dx$$

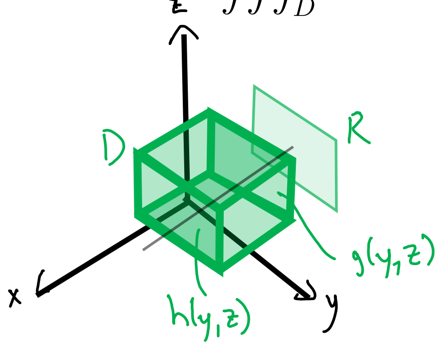
OR $dx\, dz$



Case 3: x**-simple**) region. If R is the shadow of D on the yz-plane and D is bounded front and back by the surfaces $x = h(y,z)$ and $x = g(y,z)$, then

$$\iiint_D f(x,y,z)\, dV = \iint_R \left(\int_{g(y,z)}^{h(y,z)} f(x,y,z)\, dx \right) dz\, dy$$

OR $dy\, dz$

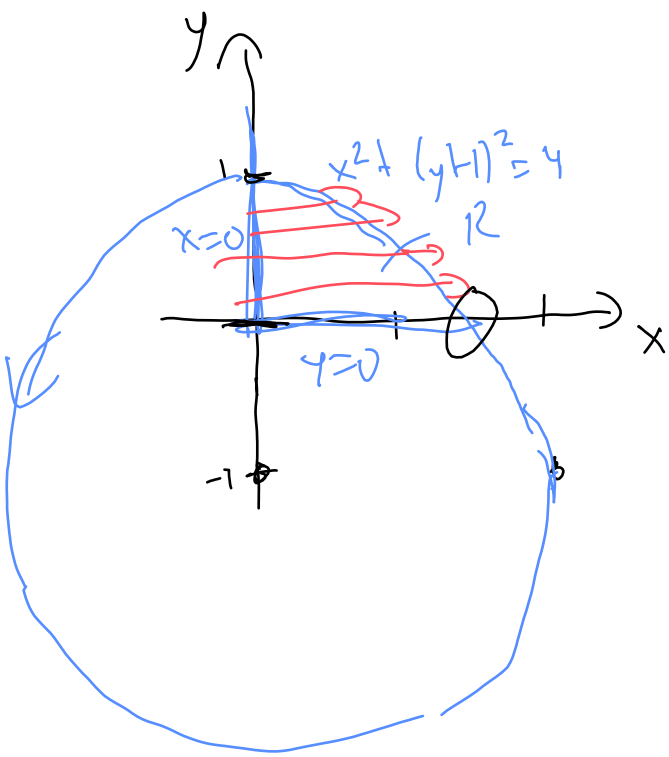


Example 85. Write an integral for the volume of the solid in the first octant bounded by $z = 3 - x^2 - y^2$ and $z = 2y$ treating the solid as a) z-simple and b) x-simple. Is the solid also y-simple?

a) z-simple : $\int_0^1 \int_0^{\sqrt{4-(y+1)^2}} \int_{2y}^{3-x^2-y^2} 1 \, dz \, \underbrace{dx\, dy}$

- z-bounds: enter : plane $z=2y$
  exit : paraboloid $z=3-x^2-y^2$

- Sketch shadow



set z-bounds equal:

$$3-x^2-y^2=2y$$

$$x^2+y^2+2y+1=3+1$$

$$x^2+(y+1)^2=4$$

Solve for x to get RH bound:

$$x^2=4-(y+1)^2$$

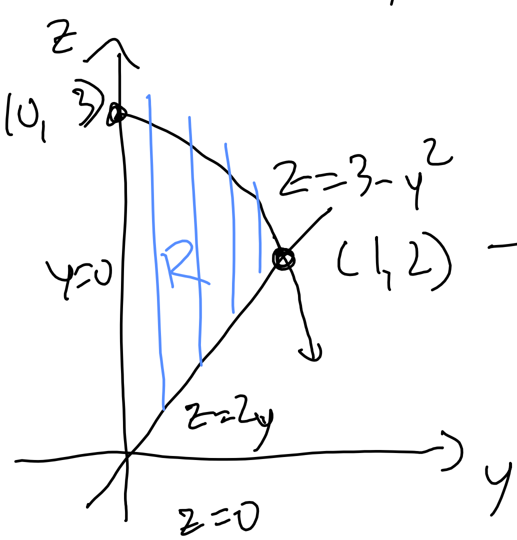
$$x=\sqrt{4-(y+1)^2}$$

Example 85 (cont.)

b) x-simple. $V = \int_0^1 \int_{2y}^{3-y^2} \int_0^{\sqrt{3-y^2-z}} 1 \, dx \, \underbrace{dz\, dy}$

- x-bounds: enter at plane $x=0$
  exit at paraboloid $z = 3 - x^2 - y^2$
  $\hookrightarrow x = \sqrt{3-y^2-z}$

- Sketch shadow:
  when $x=0$ and $x = \sqrt{3-y^2-z} \iff z = 3-y^2$



— Set $3 - y^2 = 2y$ & solve

15.6: Applications

Suppose $\delta(x, y, z)$ is the mass density (mass/unit volume in $\mathbb{R}^3$, mass/unit area in $\mathbb{R}^2$). Then one could approximate the mass of a volume D by breaking D into small rectangular prisms of volume ΔV_k and computing

$$\text{mass}(D) \approx \sum_{k=1}^{n} \delta(x_k, y_k, z_k)\Delta V_k \quad \Rightarrow \quad \text{mass}(D) = \iiint_D \delta(x, y, z)\, dV,$$

by taking the limit. Similarly, one can find formulas for the moment of D around each different coordinate plane and therefore a formula for the center of mass of D.

TABLE 15.1 Mass and first moment formulas

THREE-DIMENSIONAL SOLID

Mass: $M = \iiint\limits_D \delta\, dV$ $\quad\delta = \delta(x, y, z)$ is the density at (x, y, z).

First moments about the coordinate planes:

$$M_{yz} = \iiint\limits_D \underbrace{x\,\delta}\, dV, \qquad M_{xz} = \iiint\limits_D y\,\delta\, dV, \qquad M_{xy} = \iiint\limits_D z\,\delta\, dV$$

Center of mass:

$$\bar{x} = \frac{M_{yz}}{M}, \qquad \bar{y} = \frac{M_{xz}}{M}, \qquad \bar{z} = \frac{M_{xy}}{M}$$

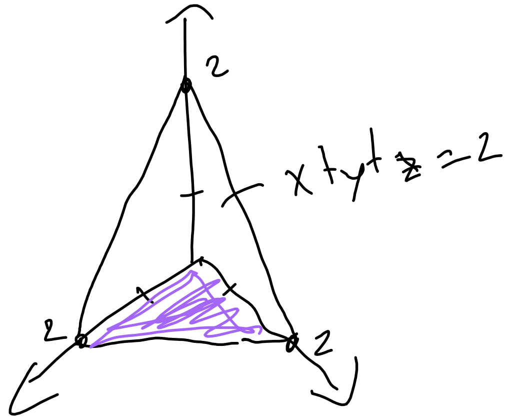
TWO-DIMENSIONAL PLATE

Mass: $M = \iint\limits_R \delta\, dA$ $\quad\delta = \delta(x, y)$ is the density at (x, y).

First moments: $M_y = \iint\limits_R x\,\delta\, dA, \qquad M_x = \iint\limits_R y\,\delta\, dA$

Center of mass: $\bar{x} = \frac{M_y}{M}, \qquad \bar{y} = \frac{M_x}{M}$

Example 86 (Itempool). A solid region in the first octant is bounded by the plane $x + y + z = 2$. The density of the solid is $\delta(x, y, z) = 2x$. Sketch the solid, then compute its mass and give integral expressions for the coordinates $\bar{x}, \bar{y}, \bar{z}$ of the center of mass.



- This region is x, y, and z simple.

$$M = \iiint_D \delta \cdot dV = \int_0^2 \int_0^{2-x} \int_0^{2-x-y} 2x \, \underline{dz} \, \underline{dy} \, \underline{dx}$$

$$= 4/3$$

$$\bar{x} = \frac{M_{yz}}{M} = \frac{3}{4} \cdot \int_0^2 \int_0^{2-x} \int_0^{2-x-y} x \cdot 2x \, dz \, dy \, dx$$

$$\bar{y} = \frac{M_{xz}}{M} = \frac{3}{4} \cdot \int_0^2 \int_0^{2-x} \int_0^{2-x-y} y \cdot 2x \, dz \, dy \, dx$$

$$\bar{z} = \frac{M_{xy}}{M} = \frac{3}{4} \cdot \int_0^2 \int_0^{2-x} \int_0^{2-x-y} z \cdot 2x \, dz \, dy \, dx$$

Daily Announcements & Reminders:

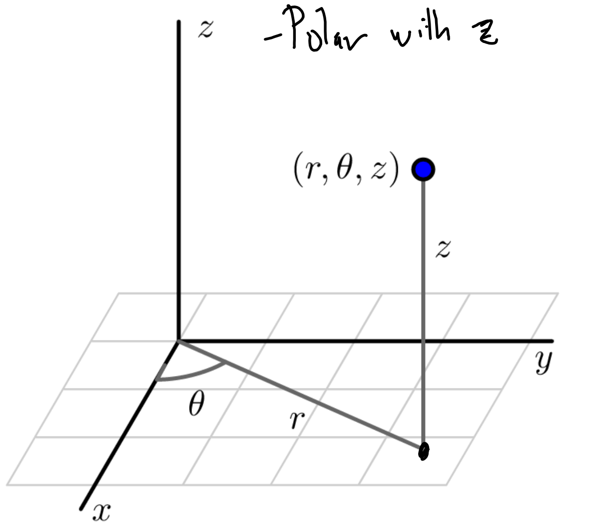
- HW 15.4 due tonight, 15.5/6 due Th
- Quiz 7 tomorrow over 15.4–15.6
  - do not need to know moment formulas (mass = integral of density may be useful)
- Exam 3 is two weeks from today

Goals for Today: Section 15.7

- Be able to convert between Cartesian, cylindrical, and spherical coordinate systems in $\mathbb{R}^3$
- Compute triple integrals expressed in cylindrical coordinates
- Compute triple integrals expressed in spherical coordinates

Cylindrical Coordinate System

- Polar with z



Cylindrical to Cartesian:

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad z = z$$

Cartesian to Cylindrical:

$$r^2 = x^2 + y^2, \quad \tan(\theta) = \frac{y}{x}, \quad z = z$$

For uniqueness:
- $r \geq 0$
- θ is in an interval of length 2π e.g. $[0, 2\pi)$

Example 87. a) Find cylindrical coordinates for the point with Cartesian coordinates $(-1, \sqrt{3}, 3)$.

$x \quad y \quad z$

$z = 3$

$r = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$

$\tan\theta = \sqrt{3}/-1 \Rightarrow \theta = \frac{2\pi}{3}$ QII

$(r, \theta, z) = (2, \frac{2\pi}{3}, 3)$

b) Find Cartesian coordinates for the point with cylindrical coordinates $(2, 5\pi/4, 1)$.

$r \quad \theta \quad z$

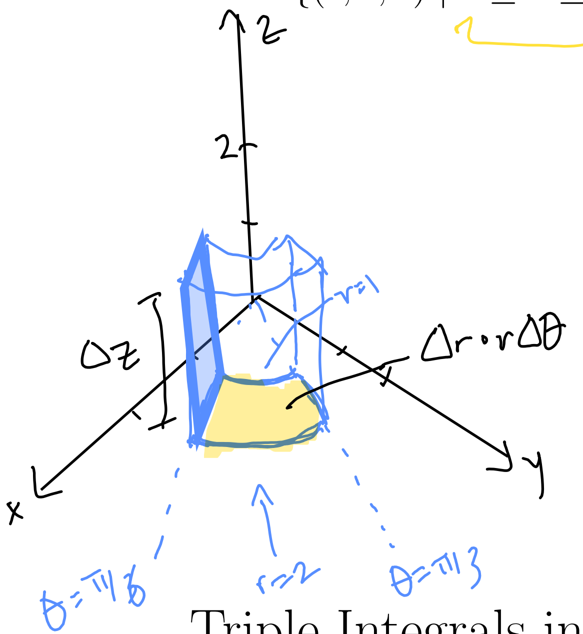
$x = 2\cos(5\pi/4) = -\sqrt{2}$

$y = 2\sin(5\pi/4) = -\sqrt{2}$

$z = 1$

Example 88. In xyz-space sketch the *cylindrical box*

$$B = \{(r, \theta, z) \mid 1 \le r \le 2,\ \pi/6 \le \theta \le \pi/3,\ 0 \le z \le 2\}.$$

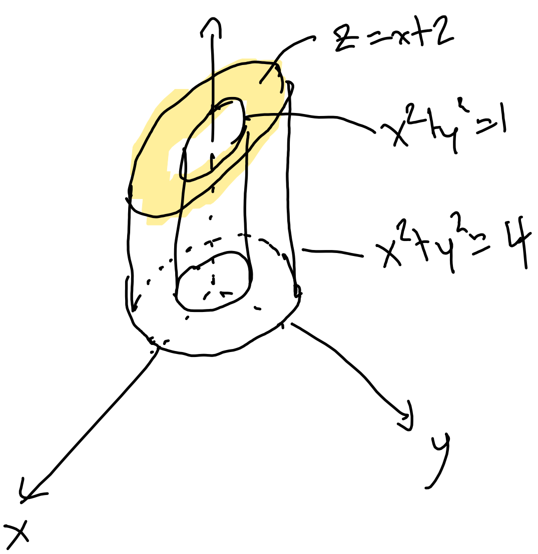


- $r = c \iff$ cylinder of radius c
- $\theta = c \iff$ vert. plane $\perp$ xy plane through origin
- $z = c \iff$ plane $\parallel$ to xy-plane

Triple Integrals in Cylindrical Coordinates

We have $dV = $ $r\,dz\,dr\,d\theta$

Example 89. Set up a iterated integral in cylindrical coordinates for the volume of the region D lying below $z = x + 2$, above the xy-plane, and between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.



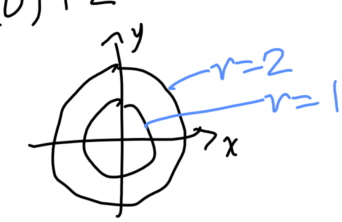
- Need many regions of integration in Cartesian coordinates
- In cylindrical coords:

$0 \le z \le r\cos(\theta) + 2$

In xy-plane:

$1 \le r \le 2$

$0 \le \theta \le 2\pi$



$$V = \int_0^{2\pi} \int_1^2 \int_0^{r\cos\theta + 2} 1\, r\,dz\,dr\,d\theta$$

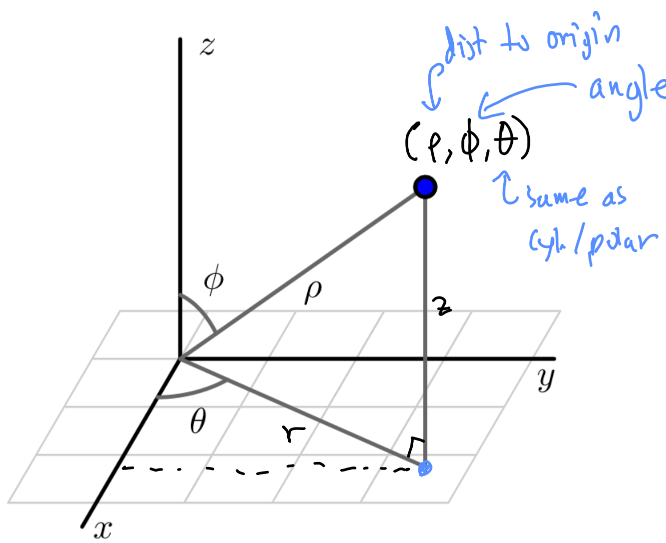
Example 90 (Itempool)**.** Suppose the density of the cone defined by $r = 1 - z$ with $z \geq 0$ is given by $\delta(r, \theta, z) = z$. Set up an iterated integral in cylindrical coordinates that gives the mass of the cone.

$$\text{mass} = \int_0^{2\pi}\int_0^1\int_0^{1-z} z\, r \, dr\, dz\, d\theta \qquad z = 1 - r$$

$$= \int_0^{2\pi}\int_0^1\int_0^{1-r} z\, r\, dz\, dr\, d\theta$$

$$= \int_0^1\int_0^{1-z} zr\left(\int_0^{2\pi} d\theta\right) dr\, dz = \left(\int_0^{2\pi} d\theta\right)\cdot\left(\int_0^1\int_0^{1-z} zr\, dr\, dz\right)$$

Spherical Coordinate System



$\phi = \varphi$

For uniqueness:

- $\rho \geq 0$, $0 \leq \varphi \leq \pi$, $0 \leq \theta \leq 2\pi$

Spherical to Cartesian:

- unique

$$x = \rho \sin(\varphi)\cos(\theta)$$
$$y = \rho \sin(\varphi)\sin(\theta)$$
$$z = \rho\cos(\varphi)$$

Cartesian to Spherical:

- not unique

$$\rho^2 = x^2 + y^2 + z^2$$
$$\tan(\theta) = \frac{y}{x}$$
$$\tan(\varphi) = \frac{\sqrt{x^2+y^2}}{z}$$

Example 91. a) Find spherical coordinates for the point with Cartesian coordinates $(-2, 2, \sqrt{8})$.

$$\rho^2 = (-2)^2 + 2^2 + (\sqrt{8})^2 = 16$$

$\rho = 4$

$$\tan\theta = \frac{2}{-2} = -1 \qquad \tan\varphi = \frac{\sqrt{(-2)^2+2^2}}{\sqrt{8}} = 1$$

$\theta = 3\pi/4$ $\qquad$ $\varphi = \pi/4$

b) Find Cartesian coordinates for the point with spherical coordinates $(2, \pi/2, \pi/3)$.

$\rho \quad \varphi \quad \theta$

$$x = 2\sin\pi/2 \cos\pi/3 = \sqrt{3}$$
$$y = 2\sin\pi/2\sin\pi/3 = 1$$
$$z = 2\cos\pi/2 = 0$$

Example 92. In xyz-space sketch the *spherical box*

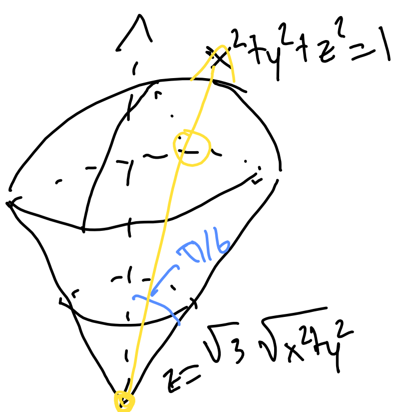
$$B = \{(\rho, \varphi, \theta) \mid 1 \le \rho \le 2,\ 0 \le \varphi \le \pi/4,\ \pi/6 \le \theta \le \pi/3\}.$$

- $\rho = c \iff$ sphere of radius c
- $\varphi = c \iff$ cone with angle c to z-axis
- $\theta = c \iff$ vert plane $\perp$ ~~xy~~-plane through origin

Triple Integrals in Spherical Coordinates

We have $dV =$ $\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$

Example 93. Write an iterated integral for the volume of the "ice cream cone" D bounded above by the sphere $x^2+y^2+z^2 = 1$ and below by the cone $z = \sqrt{3}\sqrt{x^2+y^2}$.



In spherical coords:

$1 = x^2+y^2+z^2 = \rho^2 \iff \boxed{\rho = 1}$

$z = \sqrt{3}\sqrt{x^2+y^2} = \sqrt{3}\sqrt{(\rho\sin\varphi\cos\theta)^2 + (\rho\sin\varphi\sin\theta)^2}$

$z = \sqrt{3}\sqrt{\rho^2\sin^2\varphi}$

$\cancel{\rho}\cos\varphi = \sqrt{3}\,\cancel{\rho}\sin\varphi$

$\tan\varphi = \frac{1}{\sqrt{3}}$ $\boxed{\varphi = \pi/6}$

$$V = \int_0^{2\pi}\int_0^{\pi/6}\int_0^1 1 \cdot \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

Example 94 (Itempool)**.** Write an iterated integral for the volume of the region that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and outside the cylinder $x^2 + y^2 = 1$.

- Note: Either cylindrical or spherical coords work, Itempool uses spherical coord.

sphere: $\rho^2 = 2 \iff \rho = \sqrt{2}$

cyl: $x^2 + y^2 = 1 \iff \rho^2 \sin^2 \varphi = 1$

$\rho = \csc \varphi$

set equal to get φ-bounds

$\sqrt{2} = \csc \varphi$

$\sin \varphi = \frac{1}{\sqrt{2}}$

$\varphi = \pi/4, \frac{3\pi}{4}$

$$V = \int_0^{2\pi} \int_{\pi/4}^{3\pi/4} \int_{\csc \varphi}^{\sqrt{2}} 1 \cdot \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta$$

Daily Announcements & Reminders:

- HW 15.5/6 due tonight, 15.7/8 due after break on T
- Exam 3 on T, 3/28 in lecture
  - 15.1–15.8 (through today's material)
  - announcement on Canvas later today with formula sheet
- No class next week – Spring Break

Goals for Today: Section 15.8

- Change variables in multiple integrals
- Identify choices for changing variables in a given integration problem

Thinking about single variable calculus: Compute $\displaystyle\int_{1/2}^{\sqrt{3}/2} \frac{1}{\sqrt{1-x^2}}\,dx$

Trig Substitution: $x = \sin\theta$ Identify sub.

$dx = \cos\theta\, d\theta$ Find differential

Convert integrand:

$$\frac{1}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-\sin^2\theta}} = \frac{1}{\sqrt{\cos^2\theta}} = \frac{1}{\cos\theta}$$

Convert bounds:

$x = 1/2$, $\sin\theta = 1/2$, $\theta = \pi/6$

$x = \sqrt{3}/2$, $\sin\theta = \sqrt{3}/2$, $\theta = \pi/3$

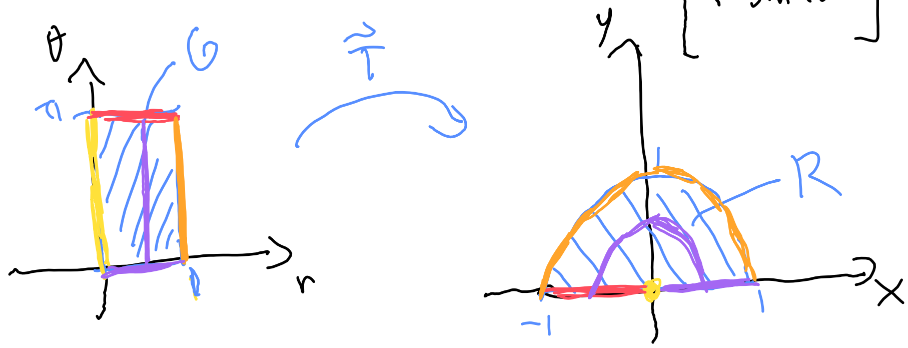
Put it together:

$$\int_{1/2}^{\sqrt{3}/2} \frac{1}{\sqrt{1-x^2}}\,dx = \int_{\pi/6}^{\pi/3} \frac{1}{\cancel{\cos\theta}} \cdot \cancel{\cos\theta}\, d\theta = \theta\Big|_{\pi/6}^{\pi/3} = \boxed{\frac{\pi}{6}}$$

Polar coordinates again, new perspective: Use the substitution $x = r\cos(\theta), y = r\sin(\theta)$ to compute the integral

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} \sqrt{x^2+y^2}\,dy\,dx.$$

1) Identify substitution: $\vec{T}(r,\theta) = \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \end{bmatrix}$ $\quad T:\mathbb{R}^2 \to \mathbb{R}^2$



2) Convert region

Original (xy):

$0 \le y \le \sqrt{1-x^2}$

$-1 \le x \le 1$

In $r\theta$-plane:

$0 \le r \le 1$

$0 \le \theta \le \pi$

3) Differentials: $dr\,d\theta \neq dx\,dy$ (because area of $G \neq$ area of R)

$$D\vec{T}(r,\theta) = \begin{bmatrix} x_r & x_\theta \\ y_r & y_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

Need $|\det(D\vec{T}(r,\theta))| = |r\cos^2\theta + r\sin^2\theta| = |r| = r$

4) Convert integrand: $\sqrt{x^2+y^2} = \sqrt{r^2} = r$

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} \sqrt{x^2+y^2}\,dx\,dy = \underbrace{\int_0^\pi\int_0^1}_{\iint_G} \underbrace{r}_{f(\vec{T}(r,\theta))} \cdot \underbrace{r\,dr\,d\theta}_{|\det(D\vec{T}(r,\theta))|\,dr\,d\theta}$$

Theorem 95. *Suppose* $\mathbf{T}(u,v)$ *is a one-to-one, differentiable transformation that maps the region* G *in the* uv*-plane to the region* R *in the* xy*-plane. Then*

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(\mathbf{T}(u,v))|\det(D\mathbf{T}(u,v))|\,du\,dv.$$ or $dv\,du$

Example 96. Evaluate $\int_0^4 \int_{y/2}^{y/2+1} \frac{2x-y}{2}\,dx\,dy$ via the transformation $u = \frac{2x-y}{2}$, $v = \frac{y}{2}$.

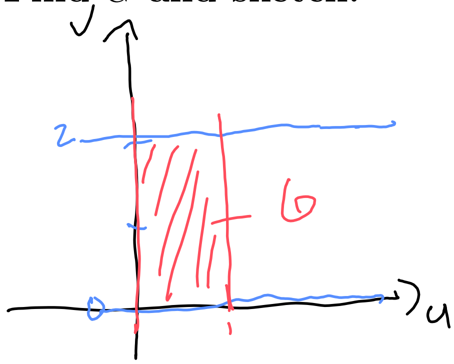
1. **Find T:** $\vec{T}(u,v) = \langle x(u,v), y(u,v)\rangle$

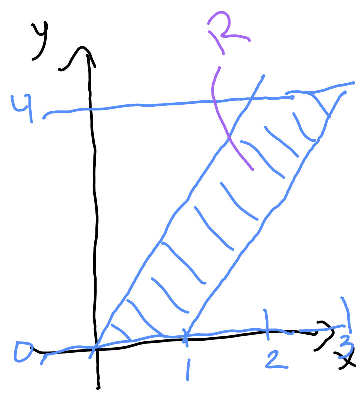
Solve for x, y

$u = x - \frac{y}{2}$ $\qquad u+v = x$ $\qquad \vec{T}(u,v) = \langle u+v, 2v\rangle$

$v = y/2$ $\qquad 2v = y$

2. **Find** G **and sketch:**





$x = y/2$ $\qquad x - \frac{y}{2} = 0$ $\qquad u = 0$

$x = y/2 + 1$ $\qquad x - \frac{y}{2} = 1$ $\qquad u = 1$

$y = 0$ $\qquad 2v = 0$ $\qquad v = 0$

$y = 4$ $\qquad 2v = 4$ $\qquad v = 2$

3. **Find Jacobian:** $= |\det(D\vec{T}(u,v))| = J(u,v) = |J(u,v)| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|$

$$|\det(D\vec{T}(u,v))| = \left|\det \begin{array}{cc} x_u & x_v \\ \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} & \\ y_u & y_v \end{array}\right| = |2-0| = 2$$

• area scaling factor

G is doubled in size by $\vec{T}$

4. **Convert and use theorem:**

$$f(x,y) = \frac{2x-y}{2} \to f(\vec{T}(u,v)) = \frac{2(u+v)-2v}{2} = u$$

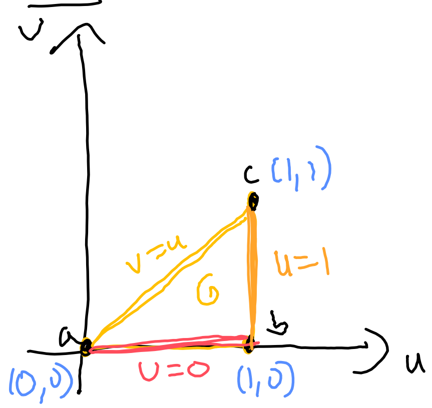
$$\int_0^4 \int_{y/2}^{y/2+1} \frac{2x-y}{2}\,dx\,dy = \int_0^1 \int_0^2 u \cdot 2\,dv\,du$$

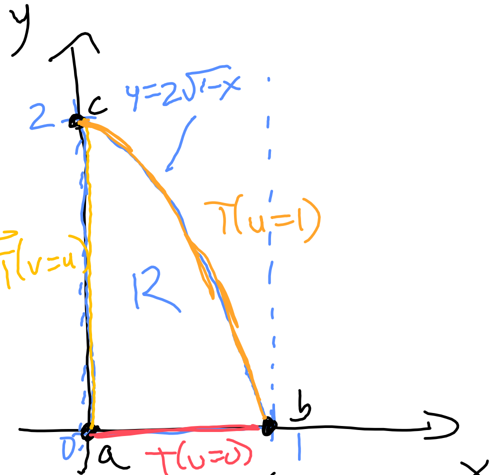
Example 97. Evaluate $\int_0^1 \int_0^{2\sqrt{1-x}} \sqrt{x^2+y^2}\ dy\ dx$ using the transformation $\mathbf{T}(u,v) = \langle u^2 - v^2, 2uv\rangle$ by showing that if G is the triangle with vertices $(0,0),(1,0),(1,1)$ in the uv-plane then $\mathbf{T}(G) = R$.

Filled in post-lecture

Goal: Show $\vec{T}(G) = R$

1) Sketch





$0 \le y \le 2\sqrt{1-x}$

$0 \le x \le 1$

- $T(0,0) = \langle 0-0, 2\cdot 0\cdot 0\rangle = \langle 0,0\rangle$
  $T(1,0) = \langle 1-0, 2\cdot 1\cdot 0\rangle = \langle 1,0\rangle$
  $T(1,1) = \langle 1-1, 2\cdot 1\cdot 1\rangle = \langle 0,2\rangle$

G is bounded by $v=u$ $(0\le u\le 1)$, $v=0$ $(0\le u\le 1)$, $u=1$ $(0\le v\le 1)$

On $v=u$:
$\vec{T}(u,u) = \langle u^2-u^2, 2\cdot u\cdot u\rangle = \langle 0, 2u^2\rangle$ for $0\le u\le 1$
$\Rightarrow x=0,\quad 0\le y\le 2(1)^2 = 2$
So $\vec{T}$ maps the $u=v$ side of G to the $x=0$ side of R

On $v=0$:
$\vec{T}(u,0) = \langle u^2-0, 2\cdot u\cdot 0\rangle = \langle u^2, 0\rangle$, for $0\le u\le 1$
$\Rightarrow 0\le x\le 1,\ y=0$
So $\vec{T}$ maps the $v=0$ side of G to the $y=0$ side of R

On $u=1$: $\vec{T}(1,v) = \langle 1-v^2, 2\cdot 1\cdot v\rangle = \langle 1-v^2, 2v\rangle$ for $0\le v\le 1$
if $y=2v$ & $x=1-v^2$, then $x = 1-\frac{y^2}{4}$
so $y = 2\sqrt{1-x}$

We still need the Jacobian:

$$|\det D\vec{T}(u,v)| = \left|\det\begin{pmatrix} 2u & -2v \\ 2v & 2u \end{pmatrix}\right| = |4u^2+4v^2| = 4(u^2+v^2)$$

And to convert the integrand:

$$\begin{aligned}\sqrt{x^2+y^2} &= \sqrt{(u^2-v^2)^2+(2uv)^2} \\ &= \sqrt{u^4-2u^2v^2+v^4+4u^2v^2} \\ &= \sqrt{u^4+2u^2v^2+v^4} \\ &= \sqrt{(u^2+v^2)^2} \\ &= u^2+v^2\end{aligned}$$

• This is why we made this choice: it makes the integrand a perfect square.

So,

$$\begin{aligned}\int_0^1\int_0^{2\sqrt{1-x}}\sqrt{x^2+y^2}\,dy\,dx &= \int_0^1\int_0^u (u^2+v^2)\cdot 4(u^2+v^2)\,dv\,du \\ &= 4\int_0^1\int_0^u u^4+2u^2v^2+v^4\,dv\,du \\ &= 4\int_0^1 u^4v+\frac{2}{3}u^2v^3+\frac{1}{5}v^5\Big|_0^u\,du \\ &= 4\int_0^1 u^5+\frac{2}{3}u^5+\frac{1}{5}u^5\,du \\ &= 4\cdot\frac{28}{15}\int_0^1 u^5\,du \\ &= \frac{4}{6}\cdot\frac{28}{15}\cdot u^6\Big|_0^1 \\ &= \boxed{\frac{56}{45}}\end{aligned}$$

Note: In polar coords:

R is $0 \le \theta \le \pi/2$
$0 \le r \le$ _____

but converting $y = 2\sqrt{1-x}$ is hard.

$$y^2 = 4-4x$$
$$r^2\sin^2\theta = 4-4r\cos\theta$$
$$r^2\sin^2\theta + 4r\cos\theta - 4 = 0$$
$$r = \frac{-4\cos\theta \pm \sqrt{16\cos^2\theta+16\sin^2\theta}}{2\sin^2\theta}$$

(want $r \ge 0$)
$$r = \frac{-4\cos\theta+4}{2\sin^2\theta} = -2\cot\theta\csc\theta + 2\csc^2\theta$$

So we have

$$\int_0^{\pi/2}\int_0^{-2\cot\theta\csc\theta+2\csc^2\theta} r^2\,dr\,d\theta$$
$$= \int_0^{\pi/2}\frac{1}{3}\left(-2\cot\theta\csc\theta+2\csc^2\theta\right)^3 d\theta$$

↑ not nice at all

Example 98. a) (Itempool) Find the Jacobian of the transformation

$$x = u + (1/2)v, \; y = v.$$

$$\det\begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix} = 1 - 0 = 1$$

b) (Itempool) Which transformation seems most suitable for the integral

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3(2x-y)e^{(2x-y)^2}\,dx\,dy?$$

i) $u = x, v = y$

ii) $u = \sqrt{x^2+y^2}, v = \arctan(y/x)$

iii) $u = 2x - y,\; v = y^3$

iv) $u = 2x - y,\; v = y$

$u + v = 2x \cancel{-y+y}$

$x = \frac{1}{2}u + \frac{1}{2}v$

$y = v$

$\det(DT(u,v)) = 1/2$

$y = 0 \quad v = 0$

$y = 2 \rightarrow v = 2$

$x = y/2 \quad \frac{1}{2}(u+v) = \frac{1}{2}v \rightarrow u = 0$

$x = (y+4)/2 \quad \frac{1}{2}(u+v) = \frac{1}{2}(v+4) \rightarrow u =$

c) Change variables in the integral above to one which is easier ~~to compute using~~ your work in a) and b).

$\boxed{\text{or } \vec{T}(u,v) = \langle (2x-y)^2, y \rangle}$

$$= \int_0^4 \int_0^2 v^3 u e^{u^2} \cdot \frac{1}{2}\,dv\,du$$

u-sub

-For next semester, find a way to add inverse substitutions here

$u = 2x - y$
$v = y$ $\rightarrow$ $v^3 u e^u \dfrac{1}{\left|\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix}\right|}$

$x = y/2 \rightarrow u = 0$

$x = \dfrac{y+4}{2} \rightarrow u = 4$

$y = 0 \rightarrow v = 0$

$y = 2 \rightarrow v = 2$

03/30/2023 Lecture 19

Last time: Midterm 3

Today: 16.1 Line integrals of scalar functions
16.2 Vector fields and line integrals

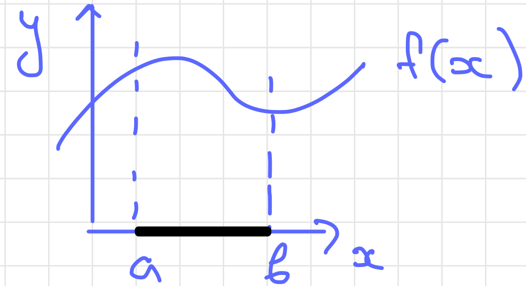
Due date:

Next week: Quiz: 16.1

16. Integrals and Vector fields

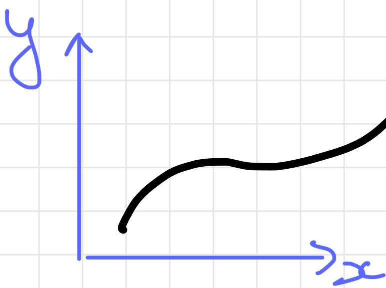
16.1 Line integrals of scalar functions

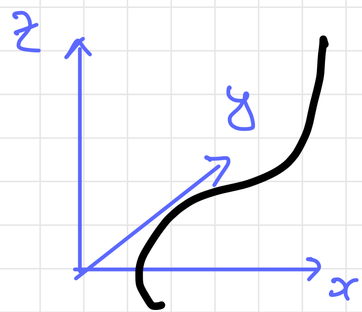
Reminder $\int_a^b f(x)\,dx$



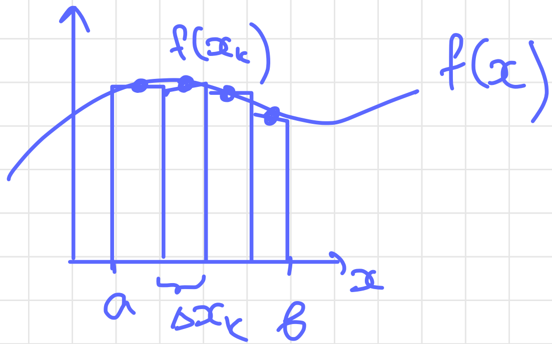
Application $m=\int_a^b \rho(x)\,dx$ - mass of a rod

$\Rightarrow \boxed{Q}$

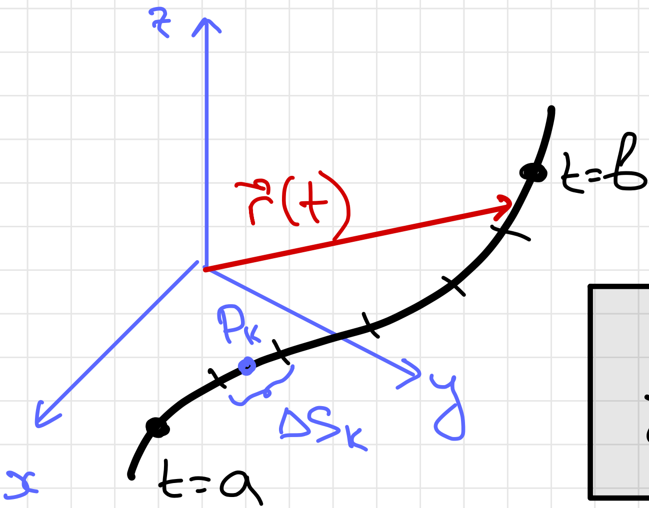




$\Rightarrow$ mass of a wire along a curve



$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x_k$$



C - a curve in space
w/ parameterization
$\vec{r}(t) = (x(t), y(t), z(t))$, $a \le t \le b$
$P_k(x_k, y_k, z_k)$

$$\boxed{\int_C f(x,y,z)\,ds = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k, z_k)\,\Delta S_k}$$

provided the limit exists

if f is cont, C is smooth $\Rightarrow$ the lim exists

$\dfrac{d}{dt}$

Reminder $\quad s(t) = \int_a^t |\vec{r}\,'(\tau)|\, d\tau$

how to evaluate $\int_C f(x,y,z)\, ds$?

1) find a (smooth) parameterization of C

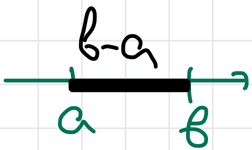
2) find $ds = |\vec{r}\,'(t)|\, dt$

3) then $\int_C f(x,y,z)\, ds = \int_a^b f(\vec{r}(t)) \cdot |\vec{r}\,'(t)|\, dt$

4) integrate

Ex $\int_C 1 \cdot ds = \int_a^b |\vec{r}\,'(t)|\,dt$ = arc length of C

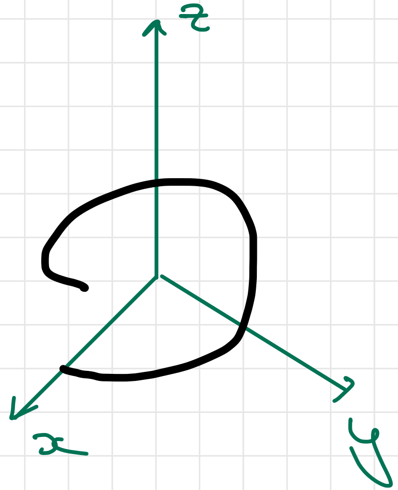
$$\int_a^b 1 \cdot dx = b-a$$



Ex Evaluate $\int_C (3x^2+3y^2+2z)\,ds$

C is parameterized by

① $\vec{r}(t) = (\cos t, \sin t, t)$,
$0 \le t \le 2\pi$ helix



(2) $\vec{r}\,'(t) = (-\sin t, \cos t, 1)$

$|\vec{r}\,'(t)| = \sqrt{(-\sin t)^2 + \cos^2 t + 1^2} = \sqrt{1+1} = \sqrt{2}$

(3) $f(\vec{r}(t)) = 3\cdot\cos^2 t + 3\cdot\sin^2 t + 2\cdot t = 3 + 2t$

$$\int_C (3x^2 + 3y^2 + 2z)\,ds = \int_0^{2\pi} (3+2t)\cdot\sqrt{2}\,dt$$

$$= \left(3\sqrt{2}\,t + 2\sqrt{2}\,\frac{t^2}{2}\right)\Big|_0^{2\pi}$$

$$= 3\sqrt{2}(2\pi - 0) + \sqrt{2}(4\pi^2 - 0) = 6\sqrt{2}\pi + 4\sqrt{2}\pi^2$$

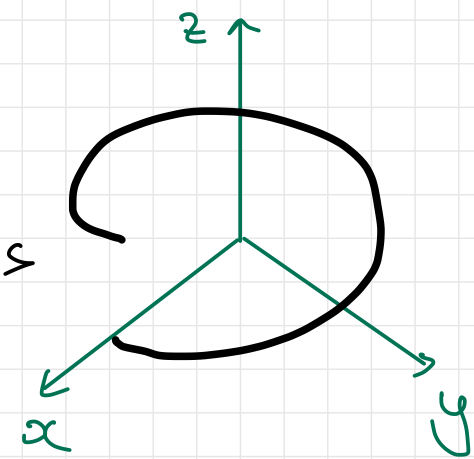
Ex Evaluate $\int_C (3x^2 + 3y^2 + 2z)\,ds$

$C: \vec{r}_1(t) = (\cos(2t), \sin(2t), 2t)$, $0 \le t \le \pi$

$$\int_C (3x^2+3y^2+2z)\,ds = 6\sqrt{2}\pi + 4\sqrt{2}\pi^2$$

b/c $(\cos t, \sin t, t)$ AND $(\cos(2t), \sin(2t), 2t)$

$0 \le t \le 2\pi$ $\qquad$ $0 \le t \le \pi$



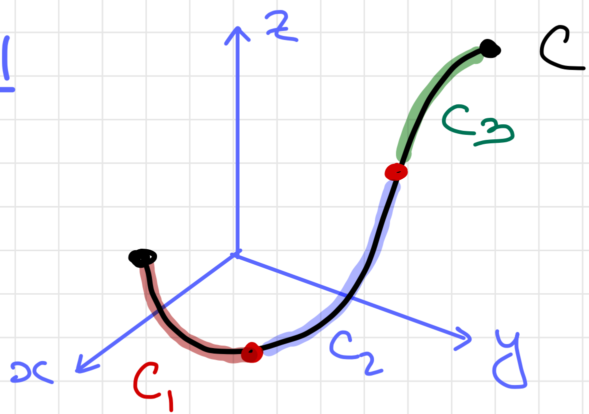
Different parameterizations of THE SAME CURVE

Reminder

same curve : same domain + same range

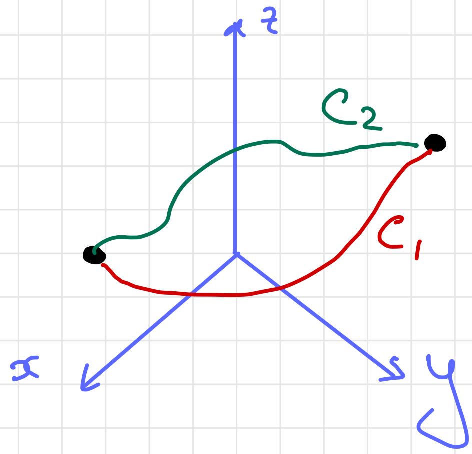
(x,y,z) $\qquad$ $f(x,y,z)$

Situation 1



$$\int_C f(x,y,z)\,ds = \int_{C_1} f(x,y,z)\,ds + \int_{C_2} f(x,y,z)\,ds + \int_{C_3} f(x,y,z)\,ds$$

Situation 2



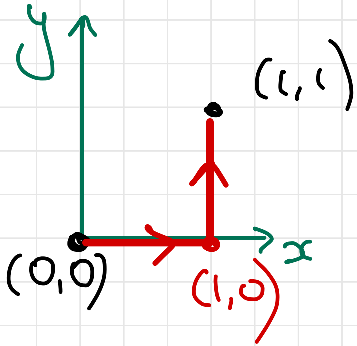
$$\int_{C_1} f(x,y,z)\,ds \neq \int_{C_2} f(x,y,z)\,ds$$

Ex Evaluate $\int_C (2x+y)\,ds$

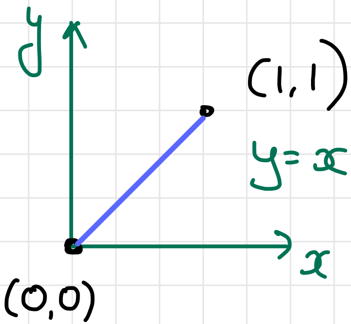
where C is a path connecting $(0,0)$ and $(1,1)$ when

a) C is the shortest path b/w $(0,0)$ and $(1,1)$

b)



a)



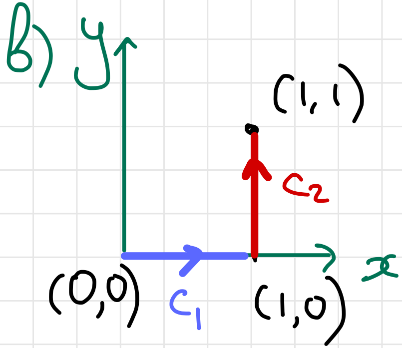
$$\vec{r}(t) = (t, t) = (1,1)t + (0,0)$$

$$0 \leq t \leq 1$$

$$|\vec{r}\,'(t)| = |(1,1)| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\int_c (2x+y)\,ds = \int_0^1 (2t+t)\cdot\sqrt{2}\,dt = 3\sqrt{2}\,\frac{t^2}{2}\Big|_0^1 = \frac{3\sqrt{2}}{2}$$

b)



$$\int_C (2x+y)\,ds = \int_{C_1} (2x+y)\,ds + \int_{C_2} (2x+y)\,ds$$

$$C_1 : \vec{r}_1(t) = (t, 0) = (1,0)t + (0,0)$$

$$0 \le t \le 1$$

$$|\vec{r}\,'(t)| = |(1,0)| = 1$$

$$\int_{C_1} (2x+y)\,ds = \int_0^1 2t \cdot 1\,dt = t^2 \Big|_0^1 = 1$$

$$C_2 : \vec{r}_2(t) = (1, t) = (1,0) + (0,1)t$$

$$|\vec{r}_2\,'(t)| = |(0,1)| = 1$$

$$\int_{C_2} (2x+y)\,ds = \int_0^1 (2\cdot 1 + t)\cdot 1\,dt = \left(2t + \frac{t^2}{2}\right)\Big|_0^1 = 2 + \frac{1}{2} = \frac{5}{2}$$

$$\int_C (2x+y)\,ds = 1 + \frac{5}{2} = \frac{7}{2}$$

Applications

Let $\delta(x,y,z)$ – density $\left(\frac{kg}{meter}, \frac{g}{cm}, \ldots\right)$

1) Mass $\quad M = \int_C \delta \, ds$

2) First moments $\quad M_{yz} = \int_C x \, \delta \, ds$

$M_{xz} = \int_C y \, \delta \, ds$

$M_{xy} = \int_C z \, \delta \, ds$

3) Coord. of the center of mass $\quad \bar{x} = \frac{M_{yz}}{M}$

$\bar{y} = \frac{M_{xz}}{M}$

$\bar{z} = \frac{M_{xy}}{M}$

4) Moments of inertia $\quad I_L = \int_C r^2 \delta \, ds$

$r(x,y,z)$ – dist from (x,y,z) to the line L

Reminder

same curve: same domain + same range

(x, y, z) $f(x, y, z)$

same initial point + same terminal point

$$\int_a^b f(x)\,dx$$

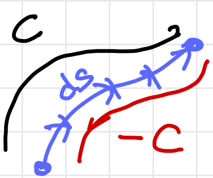
$\Downarrow$

orientation of curve

(direction along the curve)

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

(dx – a step in $+x$ direction)



$$\int_C f(x,y,z)\,ds = -\int_{-C} f(x,y,z)\,ds$$

16.2 Vector fields and line integrals

work done by a force in moving an object along a path

Scalar field: scalar function – temperature

- $f(x,y,z)$

Vector field: at each point vector – wind
magnitude + direction

- gravity/electric field
- tangent vector along a curve
- normal vectors to a surface
- $\nabla f(x,y,z) = (f_x(x,y,z), f_y(x,y,z), f_z(x,y,z))$

04/04/2023 Lecture 20

Last time : 16.1 Line integrals of scalar functions

Today : 16.2 Vector fields and line integrals
16.3 Conservative fields

Due date :

Tomorrow : Quiz : 16.1

16.2 Vector fields and line integrals

work done by a force in moving an object along a path

Scalar field: scalar function - temperature

- $f(x,y,z)$

Vector fields: at each point vector - wind

magnitude + direction

- gravity/electric field
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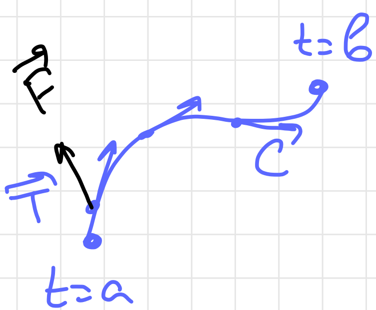
- normal vectors to a surface
- $\nabla f(x,y,z) = (f_x(x,y,z), f_y(x,y,z), f_z(x,y,z))$

vector field

$$\vec{F}(x,y,z) = P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + R(x,y,z)\,\vec{k}$$

$\vec{F}$ cont $\Leftrightarrow$ P, Q, R cont

$\vec{F}$ differentiable $\Leftrightarrow$ P, Q, R dif



$$\vec{r}(t) = (x(t), y(t), z(t))$$

(unit) tangent vector $\vec{T} = \dfrac{d\vec{r}}{ds} = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$

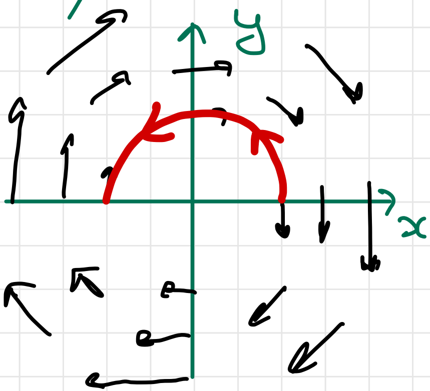
line integral of $\vec{F}$ along C

$$\int_C \underbrace{\vec{F}\cdot\vec{T}}\, ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} \cdot |\vec{r}\,'(t)|\, dt = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

scalar tangential component of $\vec{F}$ along C

Ex Evaluate $\int_C \vec{F} \cdot \vec{r}\,'(t)\,dt$

a) where $\vec{r}(t) = (\cos t, \sin t)$, $\quad 0 \le t \le \pi$



$\vec{r}\,'(t) = (-\sin t, \cos t)$

$\underbrace{\vec{F} = (y, -x)}_{\text{given}} \Rightarrow \vec{F}(\vec{r}(t)) = (\sin t, -\cos t)$

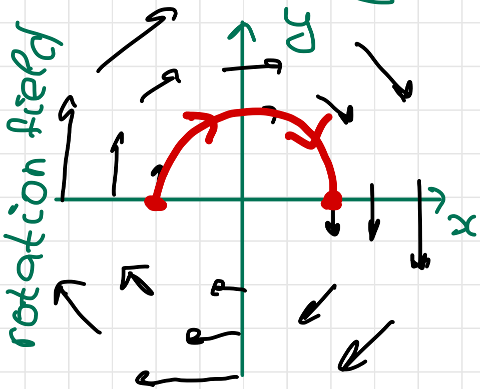
$$\int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$= \int_0^{\pi} (\sin t, -\cos t) \cdot (-\sin t, \cos t)\,dt$$

$$= \int_0^{\pi} (-\sin^2 t - \cos^2 t)\,dt = -t\Big|_0^{\pi} = -(\pi - 0) = -\pi$$

b) where $\vec{r}(t) = (\cos(t+\pi), \sin t)$ $\quad 0 \le t \le \pi$

$\vec{F} = (y, -x)$ $\quad$ (reversing the direction)



$\vec{r}(0) = (\cos(0+\pi), \sin 0) = (-1, 0)$

$\vec{r}(\pi) = (\cos(\pi+\pi), \sin \pi) = (1, 0)$

$\vec{r}'(t) = (-\sin(t+\pi), \cos t)$

$$\int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$

$$= \int_0^{\pi} (\sin t, -\cos(t+\pi)) \cdot (-\sin(t+\pi), \cos t)\, dt$$

$$= \int_0^{\pi} (\sin t, \cos t) \cdot (\sin t, \cos t)\, dt = \int_0^{\pi} (\sin^2 t + \cos^2 t)\, dt$$

$$= t \Big|_0^{\pi} = \pi$$

Work done by a force over a curve

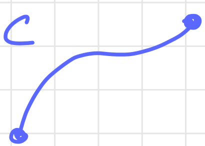
$$\boxed{W = \int_C \vec{F} \cdot \vec{T}\, ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt}$$

Flow and Circulation

if $\vec{F}(x,y,z)$ is the velocity field of a fluid flowing in the region

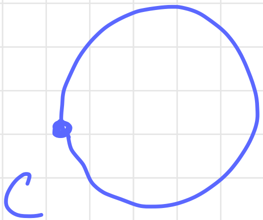
then

C



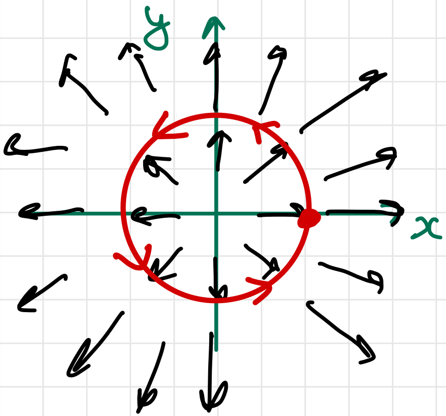
$$\Rightarrow \int_C \vec{F} \cdot \vec{T}\, ds = \text{flow integral}$$

i.e. flow along the curve from $\vec{r}(a)$ to $\vec{r}(b)$



$$\Rightarrow \oint_C \vec{F}\cdot\vec{T}\,ds = \text{circulation around the curve}$$

Ex radial field $\vec{F} = x\vec{i} + y\vec{j}$



$\vec{r}(t) = (\cos t, \sin t)$, $\quad 0 \le t \le 2\pi$

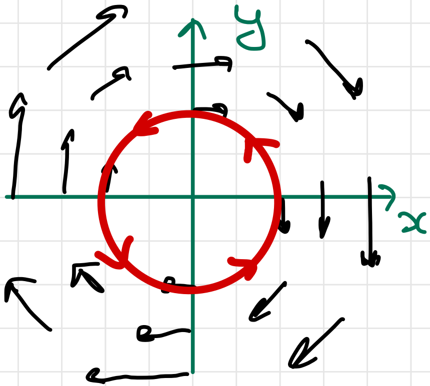
$\vec{r}(0) = (\cos 0, \sin 0) = (1, 0)$

$$\int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$= \int_0^{2\pi} (\cos t, \sin t) \cdot (-\sin t, \cos t)\,dt$$

$$= \int_0^{2\pi} (-\cos t \sin t + \sin t \cos t)\,dt = 0$$

Ex rotation field $\vec{F} = -y\vec{i} + x\vec{j}$



$\vec{r}(t) = (\cos t, \sin t)$, $0 \le t \le 2\pi$

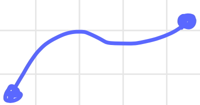
$$\int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$= -t \Big|_0^{2\pi} = -2\pi + 0 = -2\pi$$

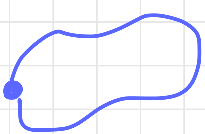
(we evaluated it earlier)

Flux across a simple closed plane curve

xy-plane, C-curve:

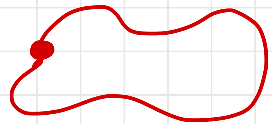
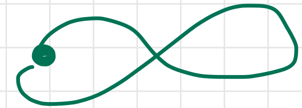
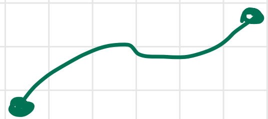
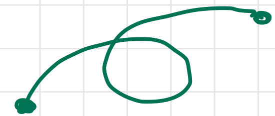


does not cross itself – SIMPLE

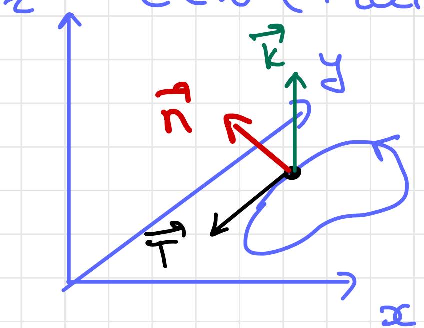


initial point = terminal point – CLOSED (loop)

Ex

	Simple	Not Simple
Closed		
Not Closed		

CCW (+dir)

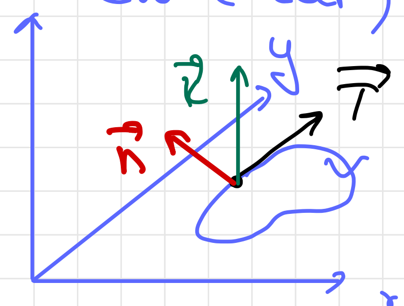


$\vec{n} = \vec{T} \times \vec{k}$

OUTWARD-pointing normal vector

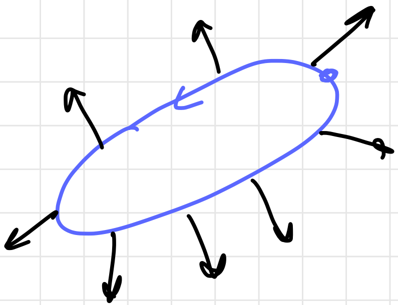
$$\Rightarrow \vec{n} = \pm \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{x'}{|\vec{r}'|} & \frac{y'}{|\vec{r}'|} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \pm \frac{(y'(t), -x'(t))}{|\vec{r}'(t)|}$$

CW (-dir)



$\vec{n} = \vec{k} \times \vec{T} = -\vec{T} \times \vec{k}$

Flux of $\vec{F}$ across $C = \int_C \underbrace{\vec{F} \cdot \vec{n}}_{\text{scalar normal component of } \vec{F}} ds$



„fluid entering or leaving C"
(−) (+)

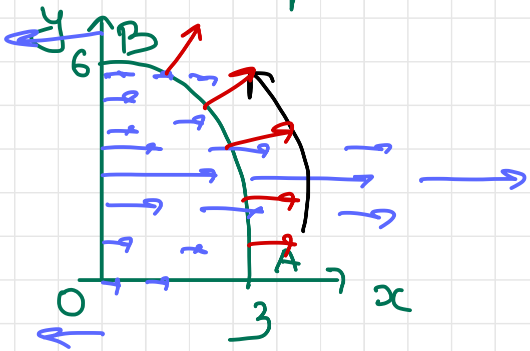
$$\int_C \vec{F} \cdot \vec{n} \, ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \left(\pm \frac{(y'(t), -x'(t))}{|\vec{r}'(t)|} \right) |\vec{r}'(t)| \, dt$$

$\vec{F} = P(x,y)\vec{i} + Q(x,y)\vec{j}$, $y'(t) = \frac{dy}{dt}$, $x'(t) = \frac{dx}{dt}$

$$\Rightarrow \int_C \vec{F} \cdot \vec{n} \, ds = \oint P \, dy - Q \, dx$$

Ex Evaluate flux of $\vec{F}(x,y) = (\underbrace{3+2y-\frac{y^2}{3}}_{P}, \underbrace{0}_{Q})$ across

ⓐ the quarter of ellipse



$\vec{r}(t) = (3\cos t, 6\sin t)$

$0 \le t \le \frac{\pi}{2}$

$\vec{r}'(t) = (\underbrace{-3\sin t}_{x'(t)}, \underbrace{6\cos t}_{y'(t)})$

$$\text{flux}_{AB} = \int_C \vec{F}(\vec{r}(t)) \cdot \vec{n}\, ds = \int P\,dy - Q\,dx$$

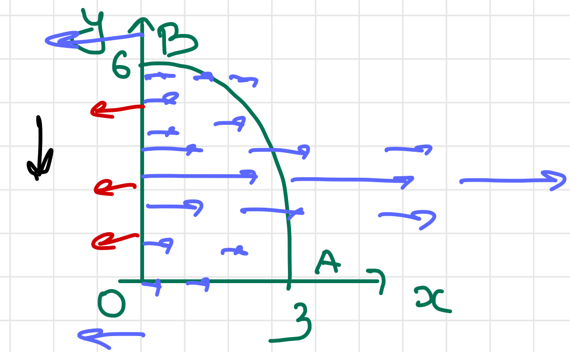
$$= \int_0^{\pi/2} \left(P y'(t)\,dt - Q x'(t)\,dt \right)$$

$$= \int_0^{\pi/2} \left[\left(3 + 2\cdot 6\sin t - \frac{(6\sin t)^2}{3}\right)\cdot(6\cos t)\,dt - 0\cdot(-3\sin t)\,dt\right]$$

$$= \left(18\sin t + 72\cdot\frac{\sin^2 t}{2} - 72\,\frac{\sin^3 t}{3}\right)\Big|_0^{\pi/2}$$

$$= 18 + 36 - 24 - 0 = 30 > 0 \Rightarrow$$ outward direction „leaving"

(6) $x = 0$, $0 \le y \le 6$ $\qquad \vec{F}(x,y) = \left(3 + 2y - \frac{y^2}{3}, 0\right)$



$$\vec{r}(t) = (0,6) + (0,-6)\,t \;,\quad 0 \le t \le 1$$
$$= (0,\ 6-6t)$$

$$\vec{r}\,'(t) = (\underbrace{0}_{x'},\ \underbrace{-6}_{y'})$$

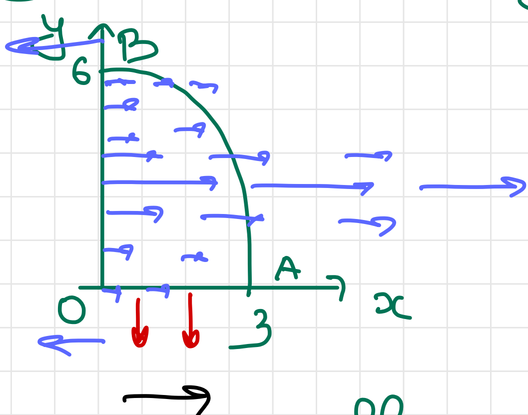
$$\text{flux}_{BO} = \int P\,\overbrace{dy}^{y'dt} - Q\,\overbrace{dx}^{x'dt}$$

$$= \int_0^1 \left\{ \left(3 + 2(6-6t) - \frac{(6-6t)^2}{3}\right)\cdot(-6)\,dt - 0\cdot 0\,dt \right\}$$

$$= -\int_0^1 \left(15 - 12t - 12(1-t)^2\right) 6\,dt = \left(15 - 6t + 12\frac{(1-t)^3}{3}\right)\Big|_0^1 \cdot (-6)$$

$$= -30 \qquad \text{„entering"}$$

c



$y = 0, \quad 0 \le x \le 3, \quad \vec{F}(x,y) = (\underbrace{3 + 2y - \frac{y^2}{3}}_{P}, \underbrace{0}_{Q})$

$\text{flux}_{OA} = 0$

$\vec{F} \perp \vec{n}$

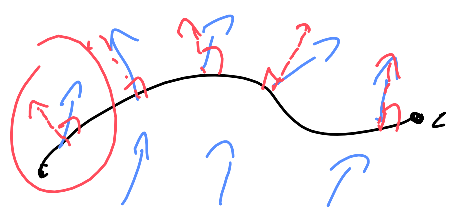
$\text{flux}_{OAB} = 30 - 30 + 0 = 0$

MATH 2551 Lecture 21 Notes – 4/6 – 16.3

Reminders & Announcements

- HW 16.1 due tonight
- Exam 3 graded next W or F
- Back to guided notes next week

Flux:



- to get normal component of $\vec{F}$ to C

$$\vec{F} \cdot \vec{n}$$

$\vec{n}$: unit normal to C

$\vec{r}(t) = \langle x(t), y(t) \rangle$

$\vec{N}(t) = \langle y'(t), -x'(t) \rangle$

(Exam 1, $\perp$ to $\vec{r}\,'(t)$)

$$\text{flux} = \int_C \vec{F} \cdot \vec{n}\, ds$$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{\langle y'(t), -x'(t) \rangle}{\cancel{\sqrt{x'(t)^2 + y'(t)^2}}} \cancel{|\vec{r}\,'(t)|}\, dt$$

- $\vec{F}(x,y,z) = P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + R(x,y,z)\,\vec{k}$

$$= \int_C P\, dy - Q\, dx$$

Conservative Vector Fields

Def: $\vec{F}$ is conservative on an open region D if

$\int_C \vec{F} \cdot d\vec{r}$ is the same for all paths C in D that have the same endpoints.

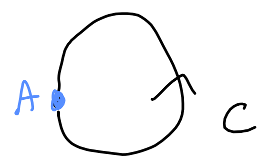
$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r} = \int_{C_3} \vec{F}\cdot d\vec{r} \ldots$

$\int_{C_4} \vec{F}\cdot d\vec{r}$ need not be the same as

- gravitational fields are cons
- state functio.

- If $\vec{F}$ is cons, can use simplest from A to B to compute a line integral.

ex: If C is a closed path and $\vec{F}$ is conservative around C, what is $\int_C \vec{F}\cdot d\vec{r}$?



$\int_C \vec{F}\cdot d\vec{r} = 0$ b/c simplest path from A to A is not moving.

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\vec{T}\,ds = \int_C P\,dx + Q\,dy + R\,dz$$

(work/circulation/flow/tangential integral)

ex: Let $\vec{F} = \langle x, y\rangle$. Is $\vec{F}$ conservative? (on its domain)

– Better tool needed.

ex: Is $\vec{F}$ a gradient vector field? E.g. is there f such that $\nabla f = \vec{F}$,

f is called a potential function for $\vec{F}$.

Need: $f_x = x$
$f_y = y$

1) Integrate $f_x = x$ wrt x

$$\int f_x\,dx = \int x\,dx$$

$f(x,y) = \frac{1}{2}x^2 + g(y)$

2) Take $\frac{\partial}{\partial y}$ & compare to $\vec{F}$

$$\frac{\partial}{\partial y}(f) = \frac{\partial}{\partial y}\left(\frac{1}{2}x^2 + g(y)\right) = y$$

$$0 + g'(y) = y$$

3) Integrate $g'(y) = y$ wrt y

$$g(y) = \frac{1}{2}y^2 + C$$

So our potential is $f(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + C$

ex) $\vec{F} = \langle x+y, x^2+y^2 \rangle$ is NOT conservative.

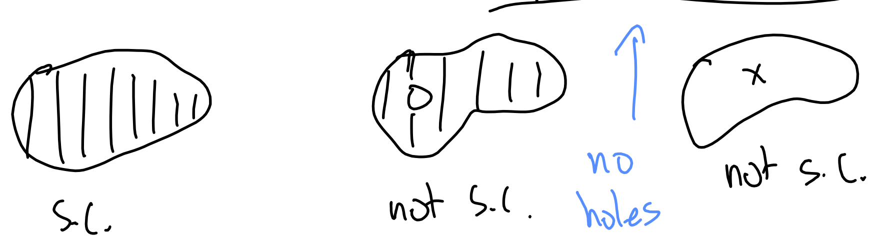
Clairaut: $f_{xy} = f_{yx}$

Test for gradient field: If $\vec{F} = \langle f_x, f_y \rangle = \langle P, Q \rangle$ then $P_y = f_{xy} = f_{yx} = Q_x$

- $P_y = 1$ but $Q_x = 2x$
- This is the $\boxed{\text{mixed-partials test}}$
- If $\vec{F} = \langle P, Q, R \rangle$ need

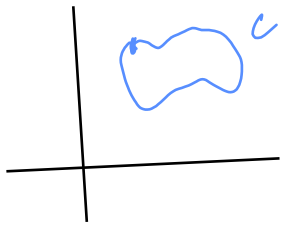
$$P_y = Q_x \quad (f_{xy} = f_{yx})$$
$$P_z = R_x \quad (f_{xz} = f_{zx})$$
$$Q_z = R_y \quad (f_{yz} = f_{zy})$$

- Why do we care? $\vec{F}$ is conservative $\Leftrightarrow$ $\vec{F}$ is a gradient of f on simply-connected domain



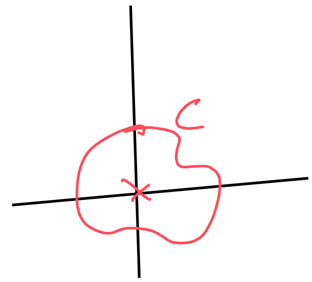
- ex: $f = \dfrac{1}{x^2+y^2} \Rightarrow \nabla f = \left\langle \dfrac{-2x}{(x^2+y^2)^2}, \dfrac{-2y}{(x^2+y^2)^2} \right\rangle = \vec{F}$

- Is conservative away from $(0,0)$:



$\int_C \vec{F} \cdot d\vec{r} = 0$

- Is not conservative around $(0,0)$

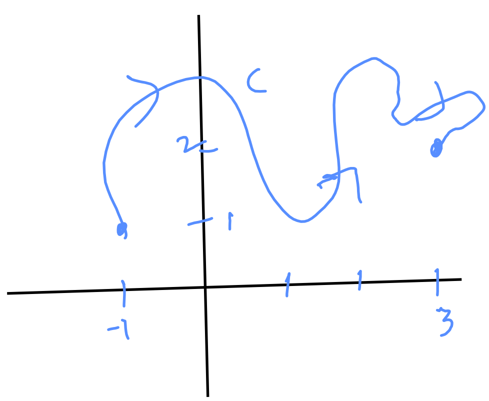


$\int_C \vec{F} \cdot d\vec{r}$ need not be 0

Fundamental Theorem of Line Integrals

If C is a path from A to B, $\int_C \nabla f \cdot d\vec{r} = f(B) - f(A)$

ex: Compute $\int_C \langle x, y \rangle \cdot d\vec{r}$ for C shown below.



- We showed $\langle x, y \rangle = \nabla \left(\frac{1}{2}x^2 + \frac{1}{2}y^2 \right)$ ← f

So FToLI says

$$\int_C \langle x, y \rangle \cdot d\vec{r} = f(3,2) - f(-1,1)$$
$$= \tfrac{1}{2}(9+4) - \tfrac{1}{2}(1+1)$$
$$= \tfrac{11}{2}$$

$\boxed{\text{Summarize}}$

- $\vec{F}$ is conservative if line integrals depend only on endpoints of path
- $\vec{F}$ is conservative $\Leftrightarrow \nabla f = \vec{F}$ on simply-connected set
- Use FToLI to determine the value of such a line integrals
- Use mixed-partials test to check for conservativeness

ex: Determine if $\vec{F}(x,y) = \langle x^2+y^2, e^x+e^y \rangle$ is conservative.

Check: $P_y \overset{?}{=} Q_x$

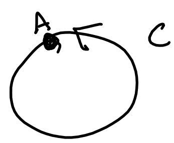
$2y \overset{?}{=} e^x$ No.

So this $\vec{F}$ is not conservative.

ex: Last time, you found $\int_C \langle -y, x \rangle \cdot d\vec{r}$ where C the unit circle oriented CCW to be 2π.

Is $\vec{F} = \langle -y, x \rangle$ conservative on a domain containing C.

No, b/c C is closed and $\int_C \vec{F} \cdot d\vec{r} \neq 0$.

• If C is closed curve: 

& $\vec{F} = \nabla f$:

$$\int_C \vec{F} \cdot d\vec{r} \underset{\text{FTOLI}}{=} f(A) - f(A) = 0$$

Daily Announcements & Reminders:

- 16.2 HW due tonight
- Quiz 9 on 16.2, 16.3 tomorrow
- Worksheet content was misaligned, hopefully fixed now

Goals for Today: Section 16.4

- Define the divergence and curl of a vector field
- Interpret divergence and curl geometrically
- Apply Green's Theorem to compute line integrals over the boundary of a simply-connected region

Useful notation: $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$

So if $f(x, y, z)$ is a function of three variables, $\nabla f = \left\langle \frac{\partial}{\partial x}(f), \frac{\partial}{\partial y}(f), \frac{\partial}{\partial z}(f) \right\rangle$

If $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ is a vector field:

- Divergence $\nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

  - works for any # of components

- Curl $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, -(R_x - P_z), Q_x - P_y \rangle$

How do we measure the change of a vector field?

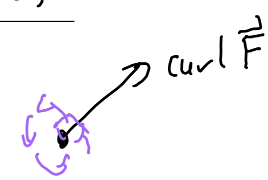
1. **Divergence** (in any $\mathbb{R}^n$)
   - Tells us flux density
   - Measures compression (−) / expansion (+)
   - Is a scalar
   - Is the instantaneous rate of change of strength of field in the direction of flow
   - div $\mathbf{F}$ = $\nabla \cdot \vec{F} = P_x + Q_y + R_z$

e.g. $\vec{F} = \langle 2x+y, y^2 \rangle \Rightarrow \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(2x+y) + \frac{\partial}{\partial y}(y^2)$

$= 2 + 2y$

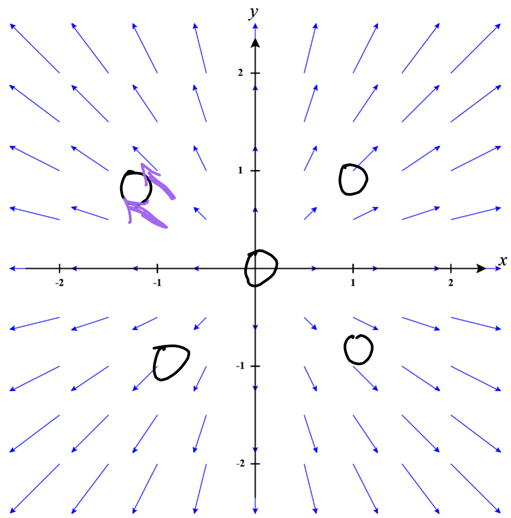
- If div $\vec{F} = 0$ everywhere, flow is incompressible

2. **Curl** (in $\mathbb{R}^3$)
   - Tells us circulation density
   - Measures rate and direction of local circulation
   - Is a vector
   - Direction gives RHR axis of rotation
   - Magnitude gives rate of rotation
   - curl $\mathbf{F}$ = $\nabla \times \vec{F}$
   - If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$: we use $\nabla \times \mathbf{F} = \nabla \times \langle P, Q, 0 \rangle = \langle 0, 0, Q_x - P_y \rangle$



- $\vec{F}$ is conservative if curl $\vec{F} = \vec{0}$ everywhere
- scalar curl : curl $\vec{F} \cdot \vec{k} = Q_x - P_y$

Example 106. Let $\mathbf{F}(x, y) = \langle x, y\rangle$. Based on the visualization of this vector field below, what can we say about the sign (+,-,0) of the divergence and curl of this vector field? Verify by computing the divergence and curl.



• B/c more flow out of each small circle than in,

$\operatorname{div} \vec{F} > 0$ at all points.

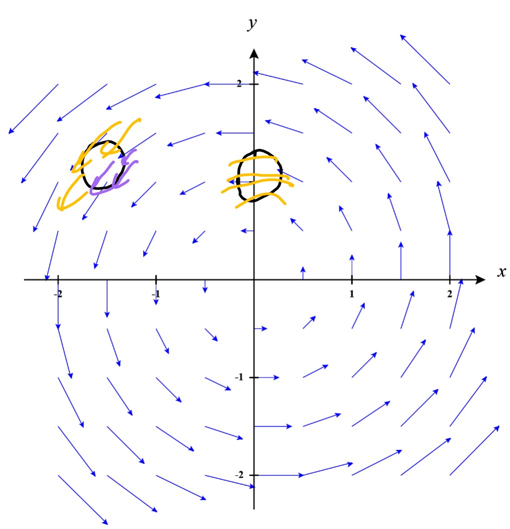
• B/c no circles rotate, $\operatorname{curl} \vec{F} = \vec{0}$.

• $\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\rangle \cdot \langle x, y\rangle$

$= 1 + 1 = 2$

• $\operatorname{curl} \vec{F} = \langle 0, 0, Q_x - P_y\rangle = \langle 0, 0, 0-0\rangle$

Example 107. (Itempool) Let $\mathbf{F}(x, y) = \langle -y, x\rangle$. Based on the visualization of this vector field below, what can we say about the sign (+,-,0) of the divergence and curl of this vector field? Verify by computing the divergence and curl.



$\operatorname{div} F = 0 \; : \; \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) = 0$

$\operatorname{curl} F \cdot k = Q_x - P_y = 1 - (-1) = 2$

Question: How is this useful?

Answer: We can relate rates of change of a vector field inside a region to the behavior of the vector field on the boundary of the region.

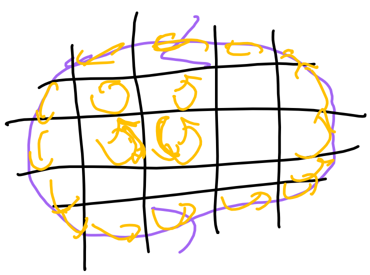
Theorem 108 (Green's Theorem). *Suppose C is a piecewise smooth, simple, closed curve enclosing on its left a region R in the plane. If $\mathbf{F} = \langle P, Q\rangle$ has continuous partial derivatives around R, then*

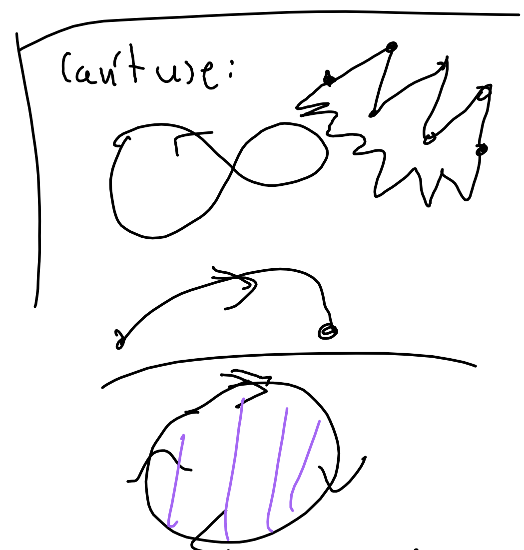
a) Circulation form:

$$\int_C \mathbf{F}\cdot\mathbf{T}\, ds = \int_C P\, dx + Q\, dy = \iint_R (\nabla\times\mathbf{F})\cdot\mathbf{k}\, dA = \iint_R Q_x - P_y\, dA$$

b) Flux form:

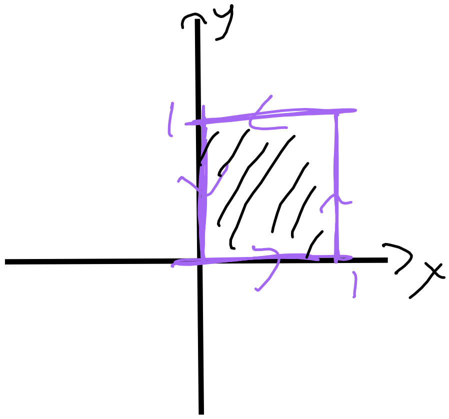
$$\int_C \mathbf{F}\cdot\mathbf{n}\, ds = \int_C P\, dy - Q\, dx = \iint_R (\nabla\cdot\mathbf{F})\, dA = \iint_R P_x + Q_y\, dA$$





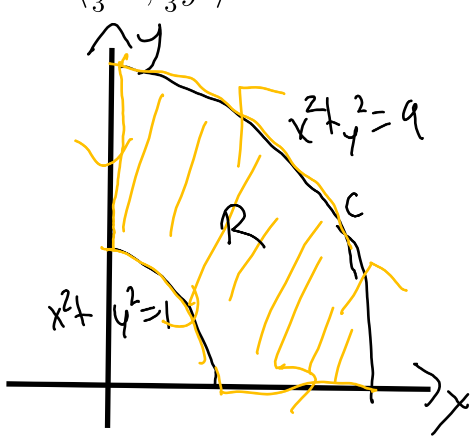
Reverse orientation & reverse sign to use theorem

Example 109. Evaluate the line integral $\int_C \mathbf{F} \cdot \mathbf{T}\; ds$ for the vector field $\mathbf{F} = \langle -y^2, xy \rangle$ where C is the boundary of the square bounded by $x = 0, x = 1, y = 0$, and $y = 1$.



$$\text{circulation} = \int_C \vec{F}\cdot\vec{T}\, ds$$
$$= \iint_R \text{curl}\,\vec{F}\cdot\vec{k}\; dA$$
$$= \int_0^1 \int_0^1 Q_x - P_y \; dx\, dy$$
$$= \int_0^1 \int_0^1 y + 2y \; dx\, dy$$
$$= \int_0^1 3xy \Big|_0^1 dy$$
$$= \int_0^1 3y\, dy$$
$$= \frac{3}{2}y^2\Big|_0^1 = \boxed{\frac{3}{2}}$$

Example 110. Compute the flux out of the region R which is the portion of the annulus between the circles of radius 1 and 3 in the first ~~octant~~ Quadrant for the vector field $\mathbf{F} = \langle \frac{1}{3}x^3, \frac{1}{3}y^3 \rangle$.

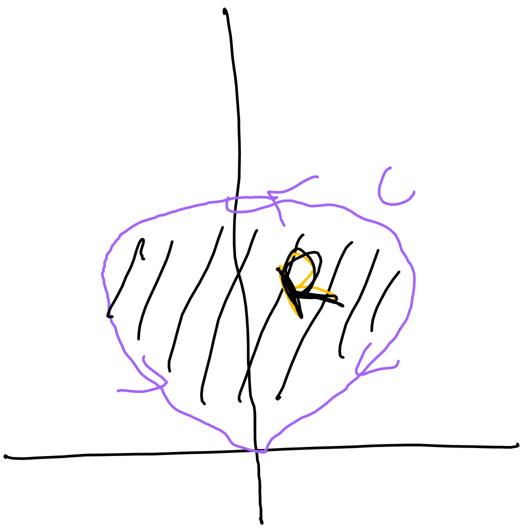


• If asked "flux into ..."

swap sign.

$$\text{flux} = \int_C \vec{F}\cdot\vec{n}\, ds$$
$$= \iint_R \nabla\cdot\vec{F}\; dA$$
$$= \iint_R \frac{\partial}{\partial x}\left(\frac{1}{3}x^3\right) + \frac{\partial}{\partial y}\left(\frac{1}{3}y^3\right) dA$$
$$= \iint_R x^2 + y^2 \; dA$$
$$= \int_0^{\pi/2}\int_1^3 r^2 \cdot r\, dr\, d\theta$$
$$= \int_0^{\pi/2} \frac{1}{4}r^4\Big|_1^3 d\theta$$
$$= \frac{\pi}{2}\cdot\left(\frac{81}{4} - \frac{1}{4}\right) = \boxed{10\pi}$$

Example 111. Let R be the region bounded by the curve $\mathbf{r}(t) = \langle \sin(2t), \sin(t) \rangle$ for $0 \leq t \leq \pi$. Find the area of R, using Green's Theorem applied to the vector field $\mathbf{F} = \frac{1}{2}\langle x, y \rangle$.



$$\text{Area}(R) = \iint_R 1 \, dA = \iint_R \operatorname{div} \vec{F} \, dA$$

$$\operatorname{div} \vec{F} = \frac{1}{2} + \frac{1}{2} = 1$$

$$\overset{\text{Green's Thm}}{=} \int_C \vec{F} \cdot \vec{n} \, ds$$

- $\vec{r}\,'(t) = \langle 2\cos(2t), \cos(t) \rangle$
- $\vec{F}(\vec{r}(t)) = \frac{1}{2}\langle \sin(2t), \sin(t) \rangle$

$\vec{n} = \langle y'(t), -x'(t) \rangle$

$$= \int_0^{\pi} \frac{1}{2}\langle \sin(2t), \sin(t) \rangle \cdot \langle \cos(t), -2\cos(2t) \rangle \, dt$$

$$= \int_0^{\pi} \frac{1}{2} \sin(2t)\cos(t) - \sin(t)\cos(2t) \, dt$$

$$= \int_0^{\pi} \sin(t)\cos^2(t) - \sin(t)(2\cos^2(t) - 1) \, dt$$

$u = \cos(t)$ $\quad du = -\sin(t)\, dt$

$$= \int_1^{-1} (-u^2 + 2u^2 - 1) \, du$$

$$= \frac{1}{3}u^3 - u \Big|_1^{-1} = \left(-\frac{1}{3} + 1\right) - \left(\frac{1}{3} - 1\right)$$

$$= \boxed{\frac{4}{3}}$$

Note: This is the idea behind the operation of the measuring instrument known as a planimeter.

Daily Announcements & Reminders:

- 16.3 HW due tonight
- Finish new material next R
- Exam 3 grades out tomorrow
- $P\,dx + Q\,dy + R\,dz$ is exact $\Longleftrightarrow$ $\vec{F} = \langle P, Q, R \rangle$ is conservative

Note: $\int_C P\,dx + Q\,dy + R\,dz = \int_C \vec{F} \cdot \vec{T}\, ds$ if $\vec{F} = \langle P, Q, R \rangle$

Goals for Today: Sections 16.5/16.6

- Describe surfaces in $\mathbb{R}^3$ parametrically
- Define and compute surface integrals
- Use surface integrals to compute meaningful quantities: surface areas, masses, flux, etc.

Different ways to think about curves and surfaces:

	Curves	Surfaces
Explicit:	$y = f(x)$ e.g. $y = \sin(x)$	$z = f(x, y)$ e.g. $z = \sqrt{x^2 + y^2}$ 
Implicit: (level curve/surface)	$F(x, y) = 0$ e.g. $\sin(x) - y = 0$ $x^2 + y^2 = 1$	$F(x, y, z) = 0$ e.g. $\dfrac{\sqrt{x^2 + y^2}}{z} - 1 = 0$ $x^2 + y^2 + z^2 - 1 = 0$
Parametric Form:	$\mathbf{r}(t) = \langle x(t), y(t) \rangle$ e.g. $\mathbf{r}(t) = \langle t, \sin(t) \rangle, t \in \mathbb{R}$	$\vec{r}(u, v) = \langle x(u,v), y(u,v), z(u,v) \rangle$ e.g. $\vec{r}(u, v) = \langle u, v, \sqrt{u^2 + v^2} \rangle$ $u, v \in \mathbb{R}$

Example 112. Give parameteric representations for the surfaces below.

a) $x = y^2 + \frac{1}{2}z^2 - 2$

$\vec{r}(u,v) = \langle u^2 + \frac{1}{2}v^2 - 2, u, v \rangle \quad u,v \in \mathbb{R}$

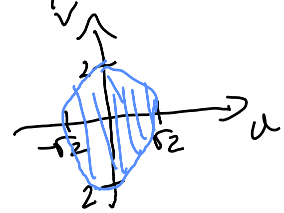
$\vec{r}(s,t) = \langle 4s^2 + 2t^2 - 2, 2s, 2t \rangle \quad s,t \in \mathbb{R}$

b) The portion of the surface $x = y^2 + \frac{1}{2}z^2 - 2$ which lies behind the yz-plane.

need: $x < 0$

$\vec{r}(u,v) = \langle u^2 + \frac{1}{2}v^2 - 2, u, v \rangle,$ $\quad u^2 + \frac{1}{2}v^2 - 2 < 0$

$\frac{u^2}{2} + \frac{v^2}{4} < 1$



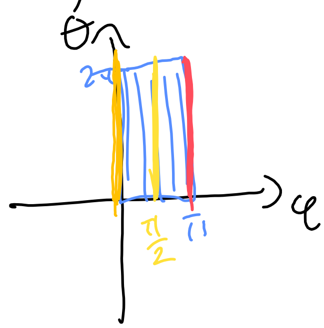
c) $x^2 + y^2 + z^2 = 9$

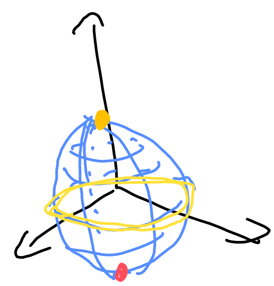
$\rho = 3$

$\vec{r}(\varphi,\theta) = \langle 3\sin\varphi\cos\theta, 3\sin\varphi\sin\theta, 3\cos\varphi \rangle$

$0 \le \varphi \le \pi$

$0 \le \theta \le 2\pi$





d) $x^2 + y^2 = 25$ $\quad r = 5$

$\vec{r}(\theta,z) = \langle 5\cos\theta, 5\sin\theta, z \rangle$

$0 \le \theta \le 2\pi$

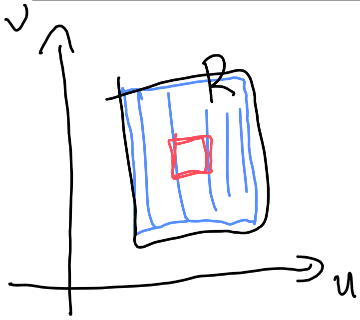
$z \in \mathbb{R}$

What can we do with this?

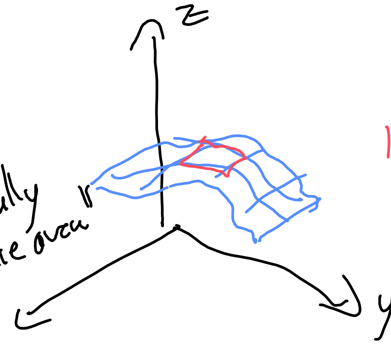
$\vec{r}_u(u,v) = \langle x_u(u,v), y_u(u,v), z_u(u,v) \rangle$

If our parameterization is **smooth** ($\mathbf{r}_u, \mathbf{r}_v$ not parallel in the domain), then:

- $\mathbf{r}_u \times \mathbf{r}_v$ is a normal vector to surface
- A rectangle of size $\Delta u \times \Delta v$ in the uv-domain is mapped to a ~~rectangle~~ parallelogram of size $|\vec{r}_u \times \vec{r}_v|\, \Delta u \Delta v$ on the surface in $\mathbb{R}^3$.



$d\sigma$ = "infinitesimally small surface area"



Area $= |\vec{r}_u|\Delta u\, |\vec{r}_v|\Delta v \sin\theta$

$= |\vec{r}_u \times \vec{r}_v|\, \Delta u \Delta v$

- Thus, $Area(S) = \iint_S 1 \, d\sigma = \iint_R |\vec{r}_u \times \vec{r}_v| \, dA$

$|\vec{r}_u \times \vec{r}_v| \, dA = d\sigma$

Example 113. (Itempool) Find the area of the portion of the cylinder $x^2 + y^2 = 25$ between $z = 0$ and $z = 1$.

$\vec{r}(\theta, z) = \langle 5\cos\theta, 5\sin\theta, z \rangle$

$0 \le \theta \le 2\pi$

$0 \le z \le 1$

$\vec{r}_\theta = \langle -5\sin\theta, 5\cos\theta, 0 \rangle$

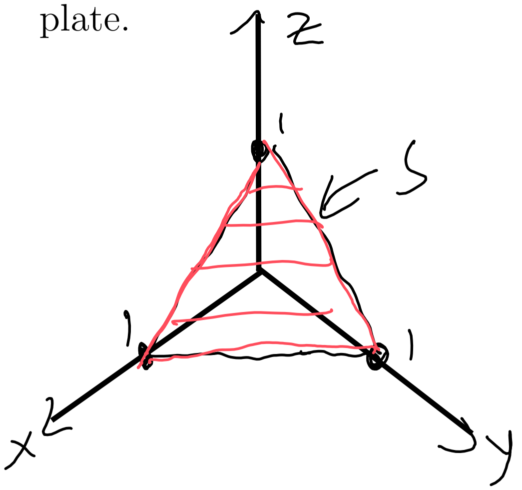
$\vec{r}_z = \langle 0, 0, 1 \rangle$

$\vec{r}_\theta \times \vec{r}_z = \langle 5\cos\theta, 5\sin\theta, 0 \rangle$

$|\vec{r}_\theta \times \vec{r}_z| = 5$

$Area(S) = \int_0^{2\pi} \int_0^1 5 \, dz \, d\theta = 10\pi$

Example 114. Suppose the density of a thin plate S in the shape of the portion of the plane $x + y + z = 1$ in the first octant is $\delta(x, y, z) = 6xy$. Find the mass of the plate.



$$\text{mass} = \iint_S \delta(x,y,z)\, d\sigma$$

$$= \iint_S 6xy\, d\sigma$$

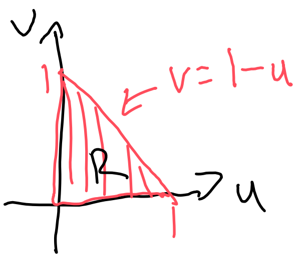
1) Parameterize S:

$z = 1 - x - y$

$\vec{r}(u,v) = \langle u, v, 1-u-v \rangle$

$0 \le u \le 1$

$0 \le v \le 1-u$



2) Find $|\vec{r}_u \times \vec{r}_v|$:

$\vec{r}_u = \langle 1, 0, -1 \rangle$

$\vec{r}_v = \langle 0, 1, -1 \rangle$

$\vec{r}_u \times \vec{r}_v = \langle 1, 1, 1 \rangle$

$|\vec{r}_u \times \vec{r}_v| = \sqrt{3}$

3) Plug in: $\iint_S f(x,y,z)\, d\sigma$

$$= \iint_R f(\vec{r}(u,v))\, |\vec{r}_u \times \vec{r}_v|\, dA$$

$$\text{mass} = \int_0^1 \int_0^{1-u} 6uv \cdot \sqrt{3}\, dv\, du = \frac{\sqrt{3}}{4}$$

Goal: If $\mathbf{F}$ is a vector field in $\mathbb{R}^3$, find the total flux of $\mathbf{F}$ through a surface S.

Note: If the flux is positive, that means the net movement of the field through S is in the direction of the normal vector

If $\mathbf{r}(u, v)$ is a smooth parameterization of S with domain R, we have

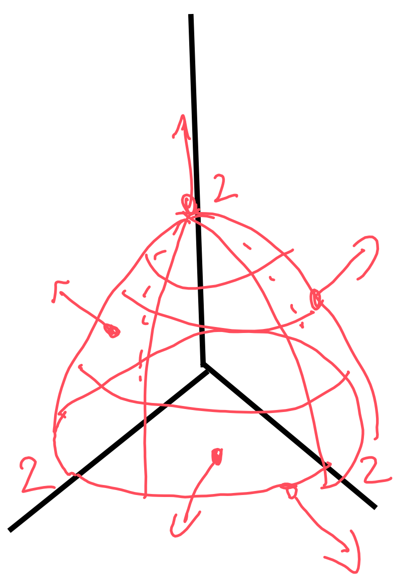
$$\text{flux of } \mathbf{F} \text{ through } S = \iint_S (\mathbf{F} \cdot \mathbf{n})\, d\sigma = \iint_R \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA.$$

unit normal to S

F·n => gives portion of $\vec{F}$ ⊥ S

Example 115. Find the flux of $\mathbf{F} = \langle x, y, z\rangle$ through the upper hemisphere of $x^2 + y^2 + z^2 = 4$, oriented away from the origin.

no bottom cap



1) Parameterize hemisphere

2) Compute $\vec{r}_u \times \vec{r}_v$, check orientation

3) Apply formula

Daily Announcements & Reminders:

- HW: 16.4/5 due tonight
  - look at how many "practice problems" you need for 200 pts
- Quiz 10 tomorrow: Green's Thm & surface parameterizations
- CIOS is open - please take 10-15 minutes to fill it out
  - Guided notes, HW improvements came from feedback last sem.
- Final Exam: Th, May 4, 8-10:50, usual room
  - 50% new material, 50% units 1-3
  - Roughly double previous exams

Goals for Today: Section 16.6/16.7

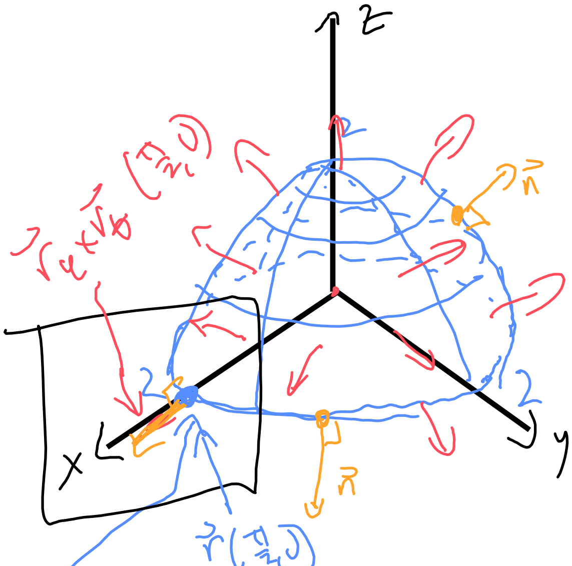
- Compute flux surface integrals
- Interpret the physical significance of flux surface integrals
- Introduce and apply Stokes' Theorem for surface integrals

Recall: If $\mathbf{r}(u, v)$ is a smooth parameterization of S with domain R, we have

$$\text{flux of } \mathbf{F} \text{ through } S = \iint_S (\mathbf{F} \cdot \mathbf{n})\, d\sigma = \iint_R \mathbf{F}(\mathbf{r}(u, v)) \cdot \underbrace{(\mathbf{r}_u \times \mathbf{r}_v)}\, dA.$$

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{\cancel{|\vec{r}_u \times \vec{r}_v|}} \qquad d\sigma = \cancel{|\vec{r}_u \times \vec{r}_v|}\, dA$$

Example 116. Find the flux of $\mathbf{F} = \langle x, y, z \rangle$ through the upper hemisphere of $x^2 + y^2 + z^2 = 4$, oriented away from the origin.



$\vec{F} = \langle x, y, z \rangle$

1) Parameterize S: use spherical coords b/c hemisphere

$\rho = 2$

$\vec{r}(\varphi, \theta) = \langle 2\sin\varphi\cos\theta, 2\sin\varphi\sin\theta, 2\cos\varphi \rangle$

$0 \leq \varphi \leq \pi/2$

$0 \leq \theta \leq 2\pi$

2) Compute $\vec{r}_\varphi \times \vec{r}_\theta$:

$\vec{r}_\varphi = \langle 2\cos\varphi\cos\theta, 2\cos\varphi\sin\theta, -2\sin\varphi \rangle$

$\vec{r}_\theta = \langle -2\sin\varphi\sin\theta, 2\sin\varphi\cos\theta, 0 \rangle$

$\vec{r}_\varphi \times \vec{r}_\theta = \langle 0 + 4\sin^2\varphi\cos\theta, -(0 - 4\sin^2\varphi\sin\theta), 4\sin\varphi\cos\varphi \rangle$

check orientation: • $\varphi = \frac{\pi}{2}, \theta = 0$: $\vec{r}_\varphi \times \vec{r}_\theta(\frac{\pi}{2}, 0) = \langle 4, 0, 0 \rangle$ ✓ (normal vector)

• $4\sin\varphi\cos\varphi = 2\sin 2\varphi > 0$ if $0 \leq \varphi \leq \pi/2$

3) Plug in:

$$\iint_S \vec{F} \cdot \vec{n}\, d\sigma = \int_0^{2\pi}\int_0^{\pi/2} F(\vec{r}(\varphi,\theta)) \cdot (\vec{r}_\varphi \times \vec{r}_\theta)\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \langle 2\sin\varphi\cos\theta, 2\sin\varphi\sin\theta, 2\cos\varphi \rangle \cdot \langle 4\sin^2\varphi\cos\theta, 4\sin^2\varphi\sin\theta, 4\sin\varphi\cos\varphi \rangle\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} 8\sin^3\varphi\cos^2\theta + 8\sin^3\varphi\sin^2\theta + 8\sin\varphi\cos^2\varphi\, d\varphi\, d\theta$$

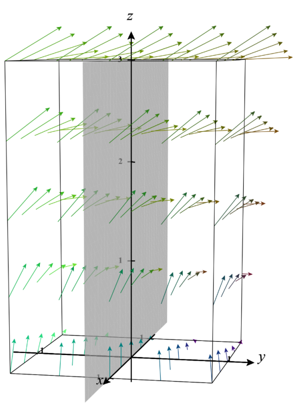
$8\sin^3\varphi(1) + 8\sin\varphi\cos^2\varphi$

$8\sin\varphi(\underbrace{\sin^2\varphi + \cos^2\varphi}_{1})$

$$= \int_0^{2\pi}\int_0^{\pi/2} 8\sin\varphi\, d\varphi\, d\theta = 16\pi$$

Example 117. (Itempool) Suppose S is a smooth surface in $\mathbb{R}^3$ and $\mathbf{F}$ is a vector field in $\mathbb{R}^3$. **True or False:** If $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma > 0$, then the angle between $\mathbf{F}$ and $\mathbf{n}$ is acute at all points on S.

Example 118. (Itempool) Based on the plot of the vector field $\mathbf{F}$ and the surface S below, oriented in the positive y-direction, is the flux integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$ positive, negative, or zero?



Recall: If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field, we defined its:

1. *divergence:* $\nabla \cdot \mathbf{F} = P_x + Q_y + R_z$
2. *curl:* $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$

Example 119. (Itempool) Suppose $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field in $\mathbb{R}^3$ with continuous partial derivatives. Compute the divergence of the curl of $\mathbf{F}$, i.e. $\nabla \cdot (\nabla \times \mathbf{F})$.

$$\nabla \cdot (\langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle)$$
$$= (\cancel{R_{yx}} - \cancel{Q_{zx}}) + (\cancel{P_{zy}} - \cancel{R_{xy}}) + (\cancel{Q_{xz}} - \cancel{P_{yz}})$$
$$= 0$$

- $\nabla \times (\nabla f) = \vec{0}$
- $\nabla \times (\underbrace{\nabla \cdot \vec{F}}_{\text{not a vector field}}) = \ ?$ DNE

Theorem 120 (Stokes' Theorem)**.** *Let S be a smooth oriented surface and C be its compatibly oriented boundary. Let* **F** *be a vector field with continuous partial derivatives. Then*

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \int_C \mathbf{F} \cdot \mathbf{T}\, ds.$$

"flux of the curl of $\vec{F}$ through S" = "circulation of $\vec{F}$ around boundary of S"

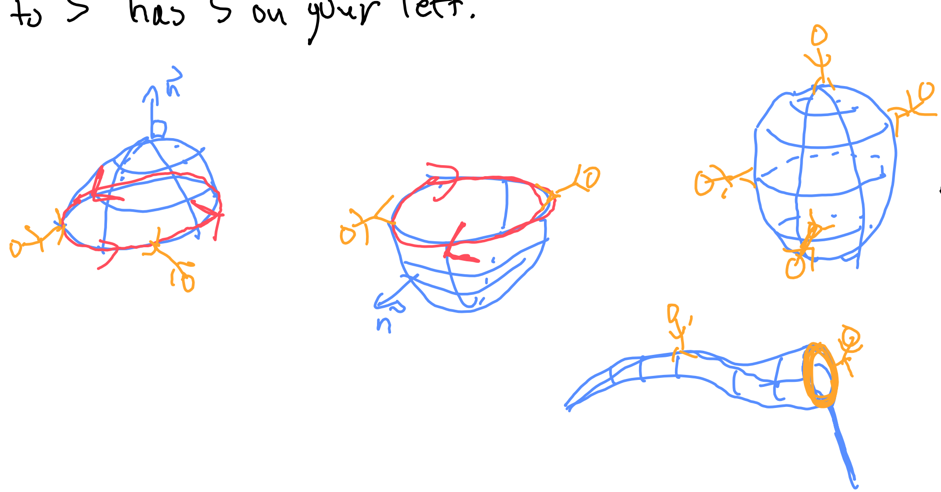
- If S is a region R in the xy-plane, then we get:

$$\iint_R (\nabla \times \vec{F}) \cdot \hat{k}\, dA = \int_C \vec{F} \cdot \vec{T}\, ds$$ ← circulation form of Green's Thm

- An **oriented surface** is one where the normal vectors are consistent across all of S

- S and C are oriented compatibly if:

walking along C in its orientation with your head in the direction of $\vec{n}$ to S has S on your left.



- has no boundary, it is closed

Example 121 (DD)**.** Let $\mathbf{F} = \langle -y, x+(z-1)x^{x\sin(x)}, x^2+y^2 \rangle$. Find $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma$ over the surface S which is the part of the sphere $x^2 + y^2 + z^2 = 2$ above $z = 1$, oriented away from the origin.

Start here on
Thursday

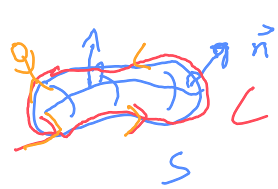
Daily Announcements & Reminders:

- 16.6 HW due tonight
- 16.7, 16.8, "Practice Problems" due T
- Final Exam detailed info tomorrow or Monday
- CIOS completion at 33%
  - 85% for bonus (by May 5)
- See Canvas for office hour info

Goals for Today: Section 16.7/16.8

- Apply Stokes' Theorem to flux integral problems.
- Use Stokes' Theorem to simplify flux integrals
- Introduce and apply the Divergence Theorem to flux integral problems

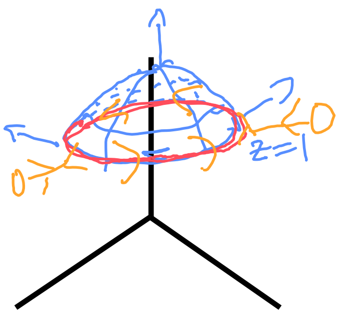
Theorem 122 (Stokes' Theorem)**.** *Let S be a smooth oriented surface and C be its compatibly oriented boundary. Let* $\mathbf{F}$ *be a vector field with continuous partial derivatives. Then*



$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \int_C \mathbf{F} \cdot \mathbf{T}\, ds.$$

- intuition is same as for Green's Thm

Example 123 (DD)**.** Let $\mathbf{F} = \langle -y, x+(z-1)x^{x\sin(x)}, x^2+y^2\rangle$. Find $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,d\sigma$ over the surface S which is the part of the sphere $x^2+y^2+z^2 = 2$ above $z = 1$, oriented away from the origin.



1) Option 1: Parameterize S, compute curl F, plug in.

$$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x+(z-1)x^{x\sin(x)} & x^2+y^2 \end{vmatrix}$$

$$= \underbrace{\left\langle 2y - x^{x\sin(x)}, -(2x-0), \right.}$$

- Very hard to integrate

2) Use Stokes' Thm: $\iint_S (\nabla\times\vec{F})\cdot\vec{n}\,d\sigma = \int_C \vec{F}\cdot\vec{T}\,ds$

- Orient C counterclockwise
- Parameterize C: $\vec{r}(t) = \langle \cos(t), \sin(t), 1\rangle$ $\quad \vec{r}\,'(t) = \langle -\sin(t), \cos(t), 0\rangle$

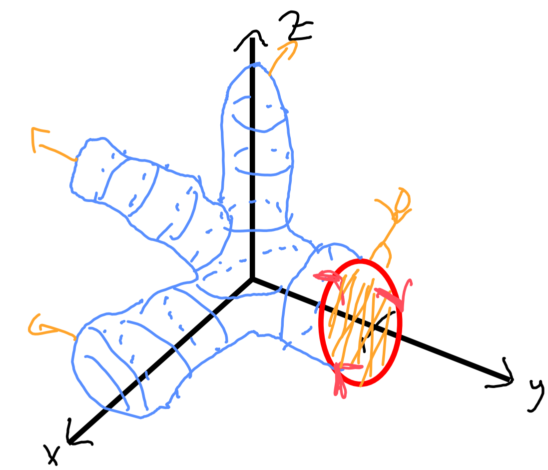
$x^2+y^2+z^2=2$ and $z=1$ => $x^2+y^2=1$

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$
$$= \int_0^{2\pi} \langle -\sin(t), \cos(t)+0, 1\rangle\cdot\langle -\sin(t), \cos(t), 0\rangle\,dt$$
$$= \int_0^{2\pi} \sin^2(t)+\cos^2(t)\,dt$$
$$= 2\pi$$

Question: What can we say if two different surfaces S_1 and S_2 have the same oriented boundary C?

$$\iint_{S_1} (\nabla \times \vec{F}) \cdot \vec{n}_1 \, d\sigma = \int_C \vec{F} \cdot \vec{T} \, ds = \iint_{S_2} (\nabla \times \vec{F}) \cdot \vec{n}_2 \, d\sigma$$

Example 124. Suppose $\operatorname{curl} \mathbf{F} = \langle \overset{P}{y^{y^y} \sin(z^2)}, \overset{Q}{(y-1)e^{x^{x^x}} + 2}, \overset{R}{ze^{x^{x^y}}} \rangle$. Compute the net flux of the curl of $\mathbf{F}$ over the surface pictured below, which is oriented outward and whose boundary curve is a unit circle centered on the y-axis in the plane $y = 1$.



1) Do it. Parameterizing is ? X

2) Stokes' Thm:
- boundary nice ✓
- need $\vec{F}$? • possible
- $\vec{F}$ is $\nabla \times \vec{G} \iff \nabla \cdot \vec{F} = 0$

3) Replace S with disk.

$$\iint_S \operatorname{curl} \vec{F} \cdot \vec{n} \, d\sigma = \iint_{S_2} \operatorname{curl} \vec{F} \cdot \vec{n} \, d\sigma$$

$\langle P, Q, R \rangle \cdot \langle 0, -1, 0 \rangle$

$$= \iint_{S_2} \operatorname{curl} \vec{F} \cdot (-\vec{j}) \, d\sigma$$

$$= \iint_{S_2} (P) \cdot 0 + 2 \cdot (-1) + R \cdot 0 \, d\sigma = \iint_{S_2} -2 \, d\sigma$$

$$= -2 \iint_{S_2} d\sigma$$

$$= -2(\pi)$$

Theorem 125 (Divergence Theorem)**.** *Let S be a closed surface oriented outward, D be the volume inside S, and $\mathbf{F}$ be a vector field with continuous partial derivatives. Then*

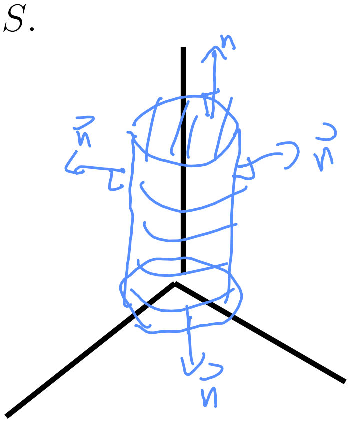
$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot \mathbf{F}\, dV.$$

"net flux of F through closed surface S" = "sum of local flux of F inside S"

Sphere is closed

• DO NOT APPLY TO NON-CLOSED SURFACES

Example 126. Let $\mathbf{F} = \langle y^{1234}e^{\sin(yz)}, y - x^{z^x}, z^2 - z\rangle$ and S be the surface consisting of the portion of cylinder of radius 1 centered on the z-axis between $z = 0$ and $z = 3$, together with top and bottom disks, oriented outward. Find the flux of $\mathbf{F}$ through S.



$$\iint_S \vec{F}\cdot\vec{n}\, d\sigma = \iiint_D \nabla\cdot\vec{F}\, dV$$

$$= \iiint_D 0 + 1 + (2z-1)\, dV$$

$$= \iiint_D 2z\, dV$$

$$= \int_0^{2\pi}\int_0^1\int_0^3 2zr\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^1 2r\, dr \cdot \int_0^3 z\, dz$$

$$= 2\pi \cdot (1-0) \cdot \left(\tfrac{9}{2} - 0\right)$$

$$= 9\pi$$

ex: S = as in Ex 123. $F = \langle 1, 1, 3z \rangle$

Find $\iint_S \vec{F}\cdot\vec{n}\, d\sigma$.

$$\iint_{S \cup S_2} \vec{F}\cdot\vec{n}\, d\sigma = \iiint_D \nabla\cdot\vec{F}\, dV = \boxed{\iiint_D 3\, dV}$$

$= 3 \cdot \text{vol}(D)$

$$\parallel$$

$$\iint_S \vec{F}\cdot\vec{n}\, d\sigma + \boxed{\iint_{S_2} \vec{F}\cdot\vec{n}\, d\sigma}$$

If these integrals are easier than original,

$x^2+y^2+z^2=2$

$\rho = \sqrt{2}$

$\rho\cos\varphi = 1$

$\cos\varphi = \frac{1}{\sqrt{2}}$

$\varphi = \pi/4$

$$\iint_S \vec{F}\cdot\vec{n}\, d\sigma = \iiint_D 3\, dV - \iint_{S_2} \vec{F}\cdot\vec{n}\, d\sigma$$

$$= \int_0^{2\pi}\int_0^{\pi/4}\int_{\sec\varphi}^{\sqrt{2}} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$- \iint_{S_2} \vec{F}\cdot(-\vec{k})\, d\sigma = \iint_{S_2} -3z\, d\sigma$$

$$= \int_0^{2\pi}\int_0^{\pi/4}\int_{\sec\varphi}^{\sqrt{2}} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta + \iint_{S_2} 3z\, d\sigma$$

Daily Announcements & Reminders:

- Remaining HW (16.7, 16.8, practice) due tonight
- CIOS completion at 62% (85% for grade incentive)
- Studio attendance computed today or tomorrow
- New metacognitive reflective bonus on Canvas
  - due 5/4 8 am, worth 3% bonus to final exam
- Final Exam 5/4 8am in usual lecture hall
- Studypalooza tomorrow in Clough 152 at 9 am

Goals for Today:

- Answer student questions about the course/unit.
- Review the core ideas of the course/unit.
- Practice problems from the course/unit.

Responses to questions:

Line Integral Methods

Methods for vector line integrals

- Stokes' Thm
- Green's Thm
- Parameterize / compute
- Check for conservative, find potential, FTOLI

1) Scalar or vector?

Scalar:
- mass
- arc length
- moments
- $\int_C f(x,y,z)\,ds$

Vector:
- flow
- work
- circulation
- flux
- $\int_C \vec{F}\cdot\vec{T}\,ds$, $\int_C \vec{F}\cdot\vec{n}\,ds$, ...

- Just do it
  - Parameterize C
  - Plug in $f(\vec{r}(t))$
    $ds = |\vec{r}\,'(t)|\,dt$

2) Check to use Green's Thm?

- C is simple & closed, CCW orientation, in xy-plane

If computing flow/work/circulation

$$\int_C \vec{F}\cdot\vec{T}\,ds = \iint_R \text{curl}\,\vec{F}\cdot\vec{k}\,dA = \iint_R Q_x - P_y\,dA$$

If computing flux:

$$\int_C \vec{F}\cdot\vec{n}\,ds = \iint_R \text{div}\,\vec{F}\,dA = \iint_R P_x + Q_y\,dA$$

3) Check conservative / use FTOLI.

- must be computing flow/work/circulation, not flux
- Mixed Partials Test: $P_y = Q_x$, $P_z = R_x$, $Q_z = R_y$
  OR
  $\nabla\times\vec{F} = \vec{0}$
- Find potential f: $\nabla f = \vec{F}$, $\int_C \nabla f\cdot\vec{T}\,ds = f(B) - f(A)$

4) Parameterize & compute:

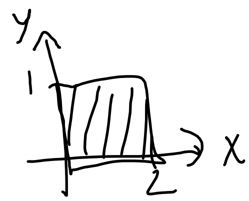
$\vec{r}(t)$ parameterizes C

circulation or work $\int_C \vec{F}\cdot\vec{T}\,ds = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$

flux $\int_C \vec{F}\cdot\vec{n}\,ds = \int_a^b \vec{F}(\vec{r}(t))\cdot\langle y'(t), -x'(t)\rangle\,dt$

Parameterizing surfaces

- Similar to finding bounds on triple integrals



1) If $z = f(x,y)$ is our surface:

$\vec{r}(u,v) = \langle u, v, f(u,v)\rangle$

e.g. S is the part of the paraboloid $z = x^2+y^2$ above $[0,2]\times[0,1]$

x-interval × y-interval

$\vec{r}(u,v) = \langle u, v, u^2+v^2\rangle$, $0 \le u \le 2$, $0 \le v \le 1$

$|\vec{r}_u \times \vec{r}_v| = \sqrt{f_x^2 + f_y^2 + 1}$

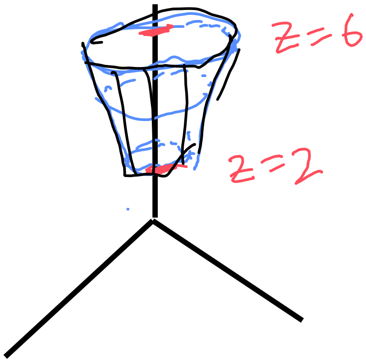
2) Otherwise, need ingenuity

- often thinking about other coords is useful

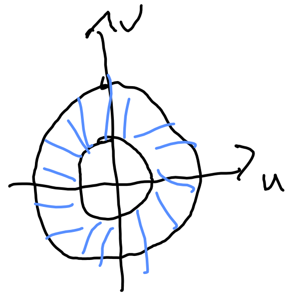
e.g. S is the surface formed by

19. Cone frustum The portion of the cone $z = 2\sqrt{x^2+y^2}$ between the planes $z = 2$ and $z = 6$

• no caps, not closed



1) $\vec{r}(u,v) = \langle u, v, 2\sqrt{u^2+v^2} \rangle$



$1 \le \sqrt{x^2+y^2} \le 3$

2) $\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 2r \rangle$

$1 \le r \le 3$

$0 \le \theta \le 2\pi$

3) $\vec{r}(\rho,\theta) = \langle \rho \frac{1}{\sqrt{5}}\cos\theta, \rho \cdot \frac{1}{\sqrt{5}}\sin\theta, \rho \cdot \frac{2}{\sqrt{5}} \rangle$

$\rho\cos\varphi = 2\rho\sin\varphi$

$\tan\varphi = \frac{1}{2}$

$\rho = 2\sec\varphi = \sqrt{5}$ → $\sqrt{5} \le \rho \le 3\sqrt{5}$

$\rho = 6\sec\varphi = 3\sqrt{5}$ $\quad 0 \le \theta \le 2\pi$

on cone:

$\sin\varphi = \frac{1}{\sqrt{5}}$

$\cos\varphi = \frac{2}{\sqrt{5}}$

Example 127. Evaluate the integral $\int_C y^2\,dx + x^2\,dy$ where C is the circle $x^2+y^2=4$.

As a flow integral: $\int_C P\,dx + Q\,dy \Rightarrow \int_C \vec{F}\cdot\vec{T}\,ds$

$\vec{F} = \langle P, Q\rangle$

In this case: $\vec{F} = \langle y^2, x^2\rangle$

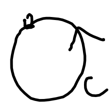
As a flux integral: $\int_C P\,dy - Q\,dx \Rightarrow \int_C \vec{F}\cdot\vec{n}\,ds$

$\vec{F} = \langle P, Q\rangle$

In this case: $\vec{F} = \langle x^2, -y^2\rangle$

- Could use polar coords to parameterize:

  $\vec{r}(t) = \langle 2\cos(t), 2\sin(t)\rangle \quad 0 \le t \le 2\pi$

- Use Green's Thm: 

- Conservative: need to use flow: $\vec{F} = \langle y^2, x^2\rangle$

  $\operatorname{curl}\vec{F}\cdot\vec{k} = \frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(y^2)$

  $= 2x - 2y \neq 0$

  $\Rightarrow \vec{F}$ not conservative

Green's Thm

$$\int_C y^2\,dx + x^2\,dy = \iint_R 2x - 2y\,dA$$

$$= \int_0^{2\pi}\int_0^2 2(r\cos\theta - r\sin\theta)\,r\,dr\,d\theta$$

$$\int_C \vec{F}\cdot\vec{T}\,ds = -\int_{-C} \vec{F}\cdot\vec{T}\,ds$$

Example 128. Find the outward flux of $\mathbf{F} = 2xy\mathbf{i}+2yz\mathbf{j}+2xz\mathbf{k}$ across the boundary of the cube cut from the first octant by the planes $x = 1, y = 1, z = 1$.

Example 129. Find the work done by $\mathbf{F} = \dfrac{x\mathbf{i} + y\mathbf{j}}{(x^2+y^2)^{3/2}}$ on an object moving along the plane curve $\mathbf{r}(t) = \langle e^t\cos(t), e^t\sin(t)\rangle$ from the point $(1, 0)$ to the point $(e^{2\pi}, 0)$.

suggests using FTOLI

Conservative?

$$P_y = \frac{\partial}{\partial y}\left(\frac{x}{(x^2+y^2)^{3/2}}\right) = -\frac{3}{2}\,\frac{x\cdot 2y}{(x^2+y^2)^{5/2}}$$

$$\parallel$$

$$Q_x = \frac{\partial}{\partial x}\left(\frac{y}{(x^2+y^2)^{3/2}}\right) = -\frac{3}{2}\,\frac{y\cdot 2x}{(x^2+y^2)^{5/2}}$$

So $\vec{F}$ is conservative away from $(0,0)$

Find potential f:

- $f = \int f_x\,dx = \int \frac{x}{(x^2+y^2)^{3/2}}\,dx = \int \frac{1}{2}\frac{du}{u^{3/2}} = -u^{-1/2}$

$$= -\frac{1}{\sqrt{x^2+y^2}} + g(y)$$

$u = x^2+y^2$

$$\frac{y}{(x^2+y^2)^{3/2}} = f_y = \frac{y}{(x^2+y^2)^{3/2}} + g'(y)$$

$$0 = g'(y)$$

$$C = g(y)$$

$$\Rightarrow f = -\frac{1}{\sqrt{x^2+y^2}}$$

So $\int_C \vec{F}\cdot\vec{T}\,ds = f(e^{2\pi},0) - f(1,0)$

$$= -\frac{1}{\sqrt{e^{4\pi}}} + \frac{1}{\sqrt{1}}$$

$$= \boxed{1 - \frac{1}{e^{2\pi}}}$$

Exercise: Compute directly to verify.

Post class

$$\vec{r}\,'(t) = e^t\langle \cos(t), \sin(t)\rangle + e^t\langle -\sin(t), \cos(t)\rangle$$

$$= e^t\langle \cos(t) - \sin(t), \cos(t) + \sin(t)\rangle$$

$$\vec{F}(\vec{r}(t)) = \frac{e^t\langle \cos(t), \sin(t)\rangle}{\left(e^{2t}(\cos^2(t) + \sin^2(t))\right)^{3/2}} = \frac{\langle \cos(t), \sin(t)\rangle}{e^{2t}}$$

$$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = \frac{1}{e^t}\left(\cos^2(t) - \cancel{\sin(t)\cos(t)} + \cancel{\sin(t)\cos(t)} + \sin^2(t)\right)$$

$$= \frac{1}{e^t}$$

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_0^{2\pi} \frac{1}{e^t}\,dt = -e^{-t}\Big|_0^{2\pi} = -e^{-2\pi} + 1 = \boxed{1 - \frac{1}{e^{2\pi}}}$$

Yay! It is the same.

Example 130. Find the flux of the field $\mathbf{F} = \langle 2xy + x, xy - y \rangle$ outward across the boundary of the square bounded by $x = 0, x = 1, y = 0, x = 1$.

Example 131. Find the flux of $\mathbf{F} = xz\mathbf{i} + yz\mathbf{j} + \mathbf{k}$ across the upper cap cut from the sphere $x^2 + y^2 + z^2 = 25$ by the plane $z = 3$, oriented away from the xy-plane.