Day 1 - Lecture

MATH 2551 C/HP - Dr.H

Daily Announcements & Reminders:

- Introduce yourself to neighbors
- Make sure you can access Webwork & Ed Discussion
- Quiz 0 (practice only) tomorrow in studio
  - linear & single-var. calc topics
- Open Ed Discussion and answer warm-up poll

Goals for Today:

Sections 12.1, 12.3, 12.4

- Set classroom norms
- Describe the big-picture goals of the class
- Review $\mathbb{R}^3$ and the dot product
- Introduce the cross product and its properties

Class Values/Norms:

- Mistakes are a learning opportunity
- Mathematics is collaborative
- Make sure everyone is included
- Criticize ideas, not people
- Be respectful of everyone
- Stay organized
- Ask questions

Introduction to the Course

Big Idea: Extend differential & integral calculus.

What are some key ideas from these two courses?

Differential Calculus		Integral Calculus
Limits & continuity	Visualizing graphs	Integration Techniques
Derivatives		Polar / Parametric
Optimization	Fundamental Theorem of Calculus	Area under curves

Before: we studied **single-variable functions** $f : \mathbb{R} \to \mathbb{R}$ like $f(x) = 2x^2 - 6$.

$$f(x) = e^{\cos(3x)} + \ln(\sin(x))$$

Now: we will study **multi-variable functions** $f : \mathbb{R}^n \to \mathbb{R}^m$: each of these functions is a rule that assigns one output vector with m entries to each input vector with n entries.

$$\mathbb{R} \to \mathbb{R}^3 : \quad \vec{F}(t) = \begin{bmatrix} t+1 \\ \cos(t) \\ t^2 + e^t \end{bmatrix}$$

$$\mathbb{R}^3 \to \mathbb{R} : \quad f(x,y,z) = x^2 y + \cos(z)$$

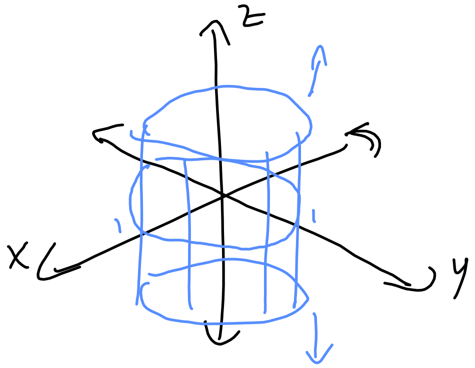
$$\mathbb{R}^2 \to \mathbb{R}^3 : \quad \vec{r}(s,t) = \begin{bmatrix} s+t \\ s^2 t \\ s+3t+\ln(t) \end{bmatrix}$$

linear algebra bridges the gap

Example 1. What shape is the set of solutions $(x, y, z) \in \mathbb{R}^3$ to the equation

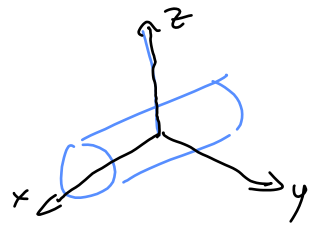
$$x^2 + y^2 = 1?$$

• circle (in x-y-plane) $\longrightarrow$ • cylinder (z is free)

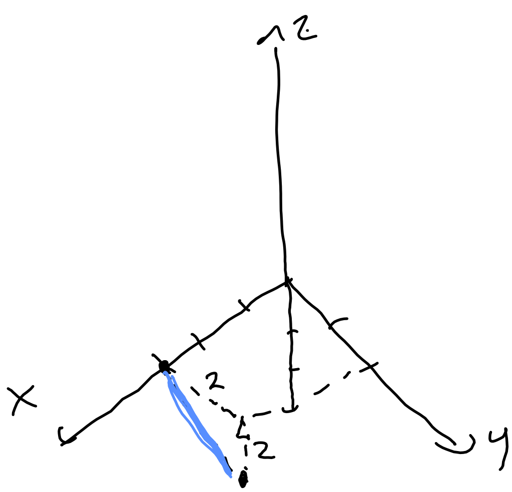


To get cylinder along x-axis

$y^2 + z^2 = 1$



Example 2. What is the distance from $(3, 2, -2)$ to the xy-plane? What is the distance to the x-axis?



dist from $(3,2,-2)$ to $z=0$: $|-2| = 2$

dist to x-axis is $\|(3,2,-2) - (3,0,0)\|$

$= \sqrt{0^2 + 2^2 + (-2)^2}$

$= \sqrt{8} = 2\sqrt{2}$

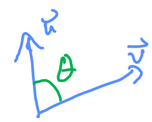
Section 12.3/4: Dot & Cross Products

Definition 3. The **dot product** of two vectors $\mathbf{u} = \langle u_1, u_2, \dots, u_n\rangle$ and $\mathbf{v} = \langle v_1, v_2, \dots, v_n\rangle$ is

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + \dots + u_nv_n = \vec{u}^T\vec{u} = [u_1 \ \dots \ u_n]\begin{bmatrix} u_1 \\ \vdots \\ v_n \end{bmatrix}$$

This product tells us about angle between vectors.

$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\cdot\|\vec{v}\|\cos(\theta)$$



$u\cdot\vec{v} > 0 \iff \theta$ is acute

$u\cdot\vec{v} < 0 \iff \theta$ is obtuse

$$\|\vec{u}\| = \sqrt{\vec{u}\cdot\vec{u}} = \sqrt{u_1^2 + u_2^2 + \dots + u_n^2}$$

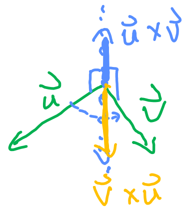
In particular, two vectors are **orthogonal** if and only if their dot product is 0.

Example 4. Are $\mathbf{u} = \langle 1, 1, 4\rangle$ and $\mathbf{v} = \langle -3, -1, 1\rangle$ orthogonal?

$$\vec{u}\cdot\vec{v} = (1)(-3) + (1)(-1) + 4(1) = -3 - 1 + 4 = 0$$

so $\vec{u}$ & $\vec{v}$ are orthogonal!

Goal: Given two vectors, produce a vector orthogonal to both of them in a "nice" way.



$\vec i \times \vec j = \vec k$
$\vec j \times \vec k = \vec i$
$\vec k \times \vec i = \vec j$

1. Right Handed

=> "antisymmetric"
$\vec u \times \vec v = -(\vec v \times \vec u)$

2. Algebraically Product

-> $\vec u \times (\vec v \times \vec w) \cancel{=} (\vec u \times \vec v) \times \vec w$

-> $c(\vec u \times \vec w) = (c\vec u) \times \vec w = \vec u \times (c\vec w)$

-> $\vec u \times (\vec v + \vec w) = (\vec u \times \vec v) + (\vec v \times \vec w)$

not $\vec 0$

$\vec u \times (\vec u \times \vec v) \neq (\vec u \times \vec u) \times \vec v \leftarrow \vec 0$

$\vec i = \langle 1,0,0 \rangle$
$\vec j = \langle 0,1,0 \rangle$
$\vec k = \langle 0,0,1 \rangle$

Definition 5. The **cross product** of two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ in $\mathbb{R}^3$ is

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \vec i & \vec j & \vec k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

- output is vector

- $\|\vec u \times \vec v\| = \|\vec u\| \cdot \|\vec v\| |\sin(\theta)|$

Example 6. Find $\langle 1,2,0\rangle \times \langle 3,-1,0\rangle$.

$$\langle 1,2,0\rangle \times \langle 3,-1,0\rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ 3 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ -1 & 0 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} \vec{k}$$

$$= (0-0)\vec{i} - (0-0)\vec{j} + (-1-6)\vec{k}$$

$$= -7\vec{k}$$

$$= \langle 0,0,-7\rangle$$

Day 2 Lecture

Daily Announcements & Reminders:

- Solutions to Quiz 0 posted
- HW 12.2/12.3 due tonight at 10pm
- Next week attendance starts in studio
- Do warmup on Ed Discussion →

Goals for Today:

Sections 12.5-12.6

- Apply the cross product to solve problems
- Learn the equations that describe lines, planes, and quadric surfaces in $\mathbb{R}^3$
- Solve problems involving the equations of lines and planes
- Sketch quadric surfaces in $\mathbb{R}^3$

Example 8. Find a set of parametric equations for the line through the point $(1, 10, 100)$ which is parallel to the line with vector equation

$$\mathbf{r}(t) = \langle 1, 4, -3 \rangle t + \langle 0, -1, 1 \rangle$$

$$\boxed{\begin{aligned} x(t) &= 1 + t \\ y(t) &= 10 + 4t \\ z(t) &= 100 - 3t \end{aligned}}$$

parametric equations

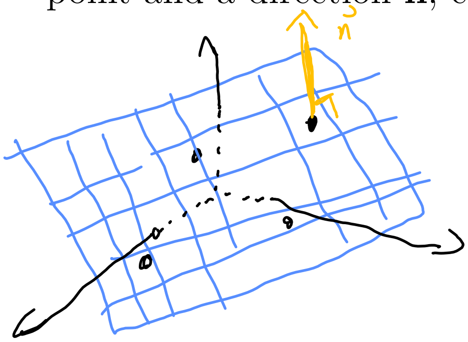
symmetric equations

$$x - 1 = \frac{y - 10}{4} = \frac{z - 100}{-3}$$

Section 12.5 Planes

Planes in $\mathbb{R}^3$

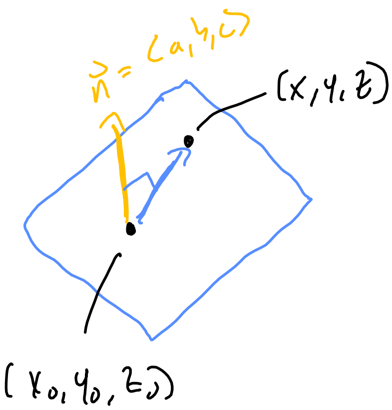
Conceptually: A plane is determined by either three points in $\mathbb{R}^3$ or by a single point and a direction $\mathbf{n}$, called the *normal vector*.



- 3 points
- 2 directions in plane & 1 point
- 1 direction $\perp$ plane & 1 point

Algebraically: A plane in $\mathbb{R}^3$ has a *linear* equation (back to Linear Algebra! imposing a single restriction on a 3D space leaves a 2D linear space, i.e. a plane)

$$ax + by + cz = d$$



$$\langle x-x_0, y-y_0, z-z_0\rangle \cdot \langle a,b,c\rangle \overset{\perp}{=} 0$$

$$\boxed{a(x-x_0)+b(y-y_0)+c(z-z_0)=0}$$

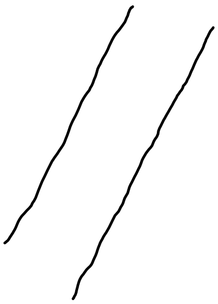
Std eqn of plane w/ $\vec n = \langle a,b,c\rangle$ through point (x_0,y_0,z_0)

Plane which is orthogonal to the line $\underbrace{\langle 1,4,-3\rangle}_{\text{parallel to } \vec n} t + \langle 1,10,100\rangle$ through the point $(1,1,1)$

is $\boxed{1(x-1)+4(y-1)+(-3)(z-1)=0}$

In $\mathbb{R}^3$, a pair of lines can be related in three ways:

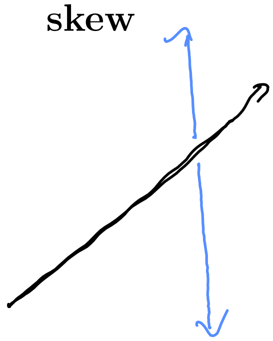
parallel



same direction
no intersection

parallel direction vectors

skew



not same direction
but no intersection

Q: How do we know if they are skew?
set up system!

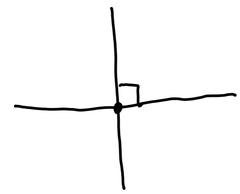
$x: \quad t = 2s$

$y: \quad t = -s$

$z: \quad t+1 = -s$

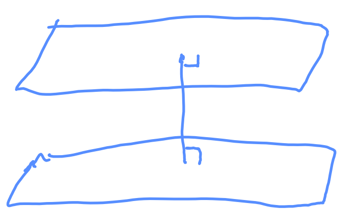
intersecting

↳ orthogonal

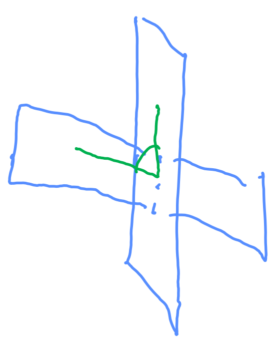


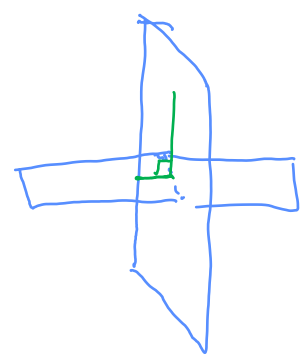
On the other hand, a pair of planes can be related in just two ways:

parallel



intersecting





orthogonal

$\Longleftrightarrow$

$\vec{n}_1 \perp \vec{n}_2$

$\Longleftrightarrow$

$\vec{n}_1 \cdot \vec{n}_2 = 0$

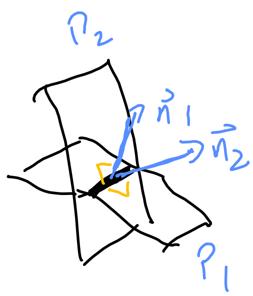
Example 9. Consider the planes $\overset{P_1}{y - z = -2}$ and $\overset{P_2}{x - y = 0}$. Show that the planes intersect and find an equation for the line passing through the point $P = (-8, 0, 2)$ which is parallel to the line of intersection of the planes.

1) Intersect? $\vec{n}_1 = \langle 0, 1, -1 \rangle$, $\vec{n}_2 = \langle 1, -1, 0 \rangle$ not parallel, so P_1 is not $\parallel$ to P_2 and so they intersect

OR give (x,y,z) solving both eqns!

$(0, 0, 2)$ is on both

2) Line equation



• direction for line of intersection is

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix} = \langle 0 - 1, -(0+1), (0-1) \rangle$$
$$= \langle -1, -1, -1 \rangle$$

• point is $(-8, 0, 2)$

$$\boxed{\ell(t) = \langle -1, -1, -1 \rangle t + \langle -8, 0, 2 \rangle \qquad t \in \mathbb{R}}$$

Section 12.6 Quadric Surfaces

Definition 10. A **quadric surface** in $\mathbb{R}^3$ is the set of points that solve a quadratic equation in x, y, and z.

You know several examples already:

- $x^2 + y^2 = 1$ (circular cylinder)
- $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$ (sphere, radius $= r$, center (x_0, y_0, z_0))

The most useful technique for recognizing and working with quadric surfaces is to examine their cross-sections.

Example 11. Use cross-sections to sketch and identify the quadric surface $x = z^2 + y^2$.

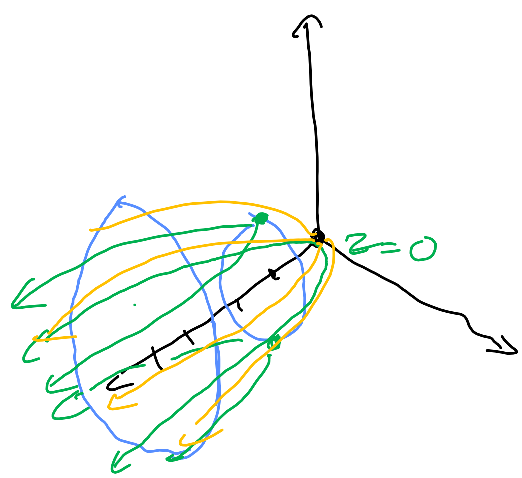
Fix values of x:

$x = 0$: $0 = z^2 + y^2$

$x = 1$: $1 = z^2 + y^2$

$x = 4$: $4 = z^2 + y^2$

$x = -1$: $-1 = z^2 + y^2$



Fix value of z:

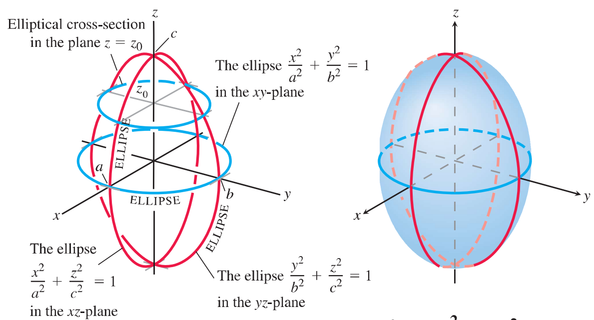
$z = 0$: $x = y^2$

$z = 1$: $x = 1 + y^2$

$z = -1$: $x = 1 + y^2$

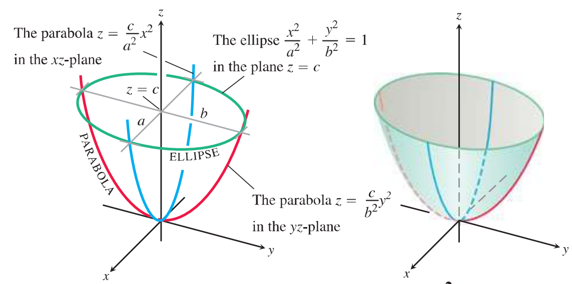
elliptical paraboloid

TABLE 12.1 Graphs of Quadric Surfaces



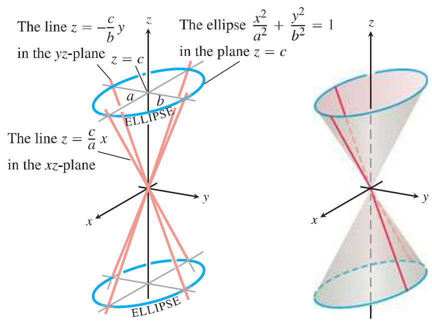
ELLIPSOID $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

has constant, no linear, all same sign



ELLIPTICAL PARABOLOID $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$

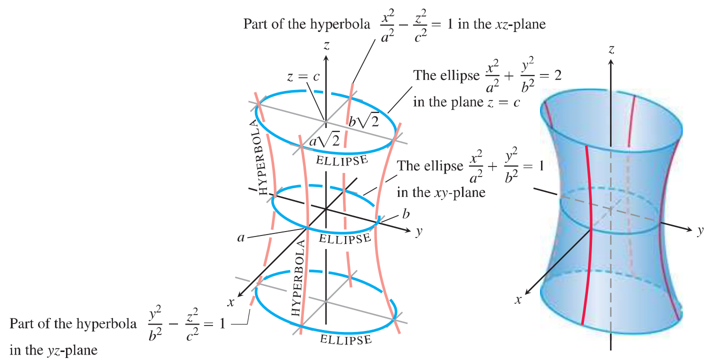
has linear term, same sign quadratics



ELLIPTICAL CONE $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$

no constant, no linear term, one sign different

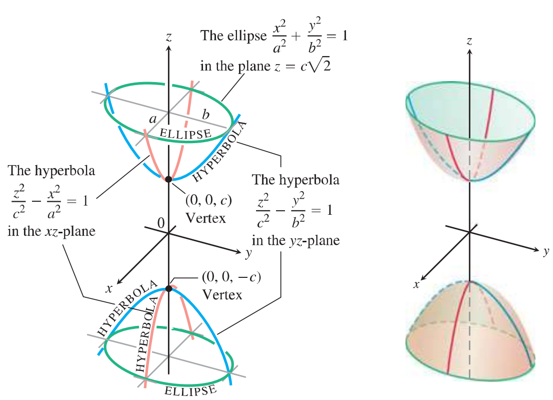
half cone: $z = \sqrt{x^2+y^2}$ or $z = -\sqrt{x^2+y^2}$



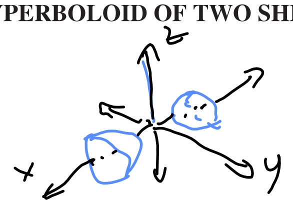
HYPERBOLOID OF ONE SHEET $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

have constant, no linear, 1 neg sign

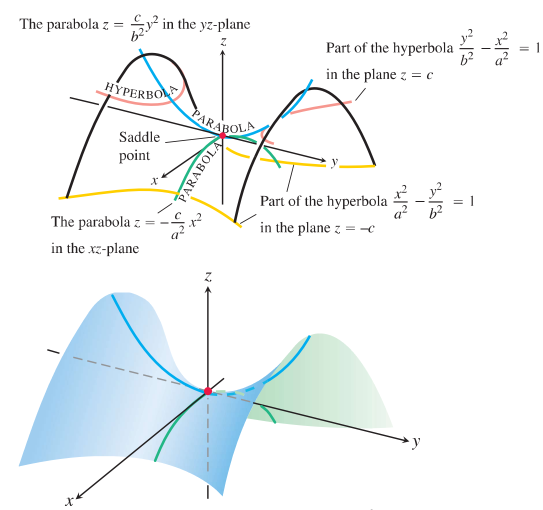
(last warmup rotated along x-axis



HYPERBOLOID OF TWO SHEETS $\quad \frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



have constant,
no linear,
2 neg sign



HYPERBOLIC PARABOLOID $\quad \frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}, c > 0$

has linear, diff. signs

Day 3 Lecture

Daily Announcements & Reminders:

- HW 12.1, 12.4 due tonight
- Quiz 1 in studio tomorrow; 12.1-12.5
  -20:61
- Do warmup poll on Ed →

Goals for Today:

Sections 13.1, 13.2

- Introduce vector-valued functions
- Plot vector-valued functions and construct them from a graph
- Compute limits, derivatives, and tangent lines for vector-valued functions
- Compute integrals of vector-valued functions and solve initial value problems

Example 12. Consider $\mathbf{r}_1(t) = \langle \cos(t), \sin(t), t\rangle$ and $\mathbf{r}_2(t) = \langle \cos(2t), \sin(2t), 2t\rangle$, each with domain $[0, 2\pi]$. What do you think the graph of each looks like? How are they similar and how are they different?

- coil (1 rotation vs 2 rotation) (same) helix
  shorter vs taller ← coils closer together yes!
- line (second steeper)

- $\vec{r}_2(t)$ moves twice as fast as $\vec{r}_1(t)$
- with domains $\mathbb{R}$ instead these give the same curve

$\boxed{\text{Domain matters}}$

Check your intuition

Section 13.1: Calculus of Vector-Valued Functions

Unifying theme: Do what you already know, componentwise.

This works with limits:

Example 13. Compute $\lim_{t\to e}\langle t^2, 2, \ln(t)\rangle$.

$$= \left\langle \lim_{t\to e} t^2, \lim_{t\to e} 2, \lim_{t\to e} \ln(t) \right\rangle$$

$$= \langle e^2, 2, 1 \rangle$$

And with continuity: $\lim_{x\to a} f(x) = f(a)$ $\qquad \lim_{t\to a} \vec{r}(t) = \vec{r}(a)$

Example 14. Determine where the function $\mathbf{r}(t) = t\mathbf{i} - \dfrac{1}{t^2-4}\mathbf{j} + \sin(t)\mathbf{k}$ is continuous.

- $x(t) = t$ is continuous on $\mathbb{R}$
- $y(t) = -\dfrac{1}{t^2-4}$ is cts on $\mathbb{R}\setminus\{-2,2\} = (-\infty,-2)\cup(-2,2)\cup(2,\infty)$

  "all real #s except −2 and 2"
- $z(t) = \sin(t)$ is cts on $\mathbb{R}$

So $\vec{r}(t)$ is cts on $\mathbb{R} \cap (\mathbb{R}\setminus\{-2,2\}) \cap (\mathbb{R})$

$$= \boxed{\mathbb{R}\setminus\{-2,2\}}$$

And with derivatives:

Example 15. If $\mathbf{r}(t) = \langle 2t - \frac{1}{2}t^2 + 1, t - 1\rangle$, find $\mathbf{r}'(t)$.

$\vec{r}\,'(t) = \langle x'(t), y'(t)\rangle$

$= \langle 2 - t, 1\rangle$

$\vec{r}\,''(t) = \langle -1, 0\rangle$

Interpretation: If $\mathbf{r}(t)$ gives the position of an object at time t, then

- $\mathbf{r}'(t)$ gives velocity
- $\|\mathbf{r}'(t)\|$gives speed
- $\mathbf{r}''(t)$ gives acceleration

• $\vec{r}\,'(t_0)$ is tangent to curve at $t = t_0$

Let's see this graphically

Example 16. Find an equation of the tangent line to $\mathbf{r}(t) = \langle 2t - \frac{1}{2}t^2 + 1, t - 1\rangle$ at time $t = 2$.

In S.V.C: tangent line to $y = f(x)$ at $x = a$ is $y = f(a) + f'(a)(x - a)$

$\vec{r}(a)$ $\vec{r}\,'(a)$

Tangent line to $\vec{r}(t)$ at $t = a$

is $\ell(t) = \vec{r}\,'(a)\cdot t + \vec{r}(a)$

$\vec{r}\,'(a)(t-a) + \vec{r}(a)$

$\ell(t) = \langle 2-2, 1\rangle \cdot t + \langle 3, 1\rangle$

$= \langle 0, 1\rangle t + \langle 3, 1\rangle$

And with integrals:

Example 17. Find $\int_0^1 \langle t, e^{2t}, \sec^2(t) \rangle \, dt$.

$\int_a^b \vec{r}\,'(t)\,dt$ = displacement from $t=a$ to $t=b$

$= \langle \int_0^1 t\,dt, \int_0^1 e^{2t}\,dt, \int_0^1 \sec^2(t)\,dt \rangle$

$= \langle \frac{1}{2}t^2 \Big|_0^1, \frac{1}{2}e^{2t}\Big|_0^1, \tan(t)\Big|_0^1 \rangle$

$= \langle \frac{1}{2}, \frac{1}{2}(e^2-1), \tan(1) \rangle$

At this point we can solve initial-value problems like those we did in single-variable calculus:

Example 18. Wallace is testing a rocket to fly to the moon, but he forgot to include instruments to record his position during the flight. He knows that his velocity during the flight was given by

$t \geq 0$

$$\mathbf{v}(t) = \langle -200\sin(2t), 200\cos(t), 400 - \frac{400}{1+t} \rangle \; m/s.$$

If he also knows that he started at the point $\mathbf{r}(0) = \langle 0, 0, 0 \rangle$, use calculus to reconstruct his flight path.

Start here next time

1) Find antiderivative (with $+C$)

$\vec{r}(t) = \langle 100\cos(2t) + C_1, 200\sin(t) + C_2, 400t - 400\ln|1+t| + C_3 \rangle$

$= \langle 100\cos(2t), 200\sin(t), 400(t - \ln|1+t|) \rangle + \vec{C}$

2) Apply I.C. to find C

$\vec{r}(0) = \langle 0,0,0 \rangle$

$= \langle 100, 0, 0 \rangle + \vec{C}$

$\to \vec{C} = \langle -100, 0, 0 \rangle$

$\to$

$\boxed{\vec{r}(t) = \langle 100(\cos(2t)-1), 200\sin(t), 400(t - \ln(1+t)) \rangle}$

Day 4 Lecture

Daily Announcements & Reminders:

- HW 12.5 due tonight
- No studio on Monday - holiday
- Do warmup on Ed $\longrightarrow$

Goals for Today:

Sections 13.3, 13.4

- Compute arc lengths of curves using parameterizations
- Define and compute arc-length parameterizations
- Define, interpret, and compute the curvature of a curve
- Compute the unit tangent and principal unit normal vectors of a curve

Example 20. Set up an integral for the arc length of the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ from the point $(1, 1, 1)$ to the point $(2, 4, 8)$.

$$L = \int_a^b \|\vec{r}\,'(t)\|\, dt$$

1) Endpoints? $t = a$ & $t = b$

At $t=1$: $\vec{r}(1) = \langle 1,1,1\rangle$

$t=2$ $\vec{r}(2) = \langle 2,4,8\rangle$

2) Speed? $\vec{r}\,'(t) = \langle 1, 2t, 3t^2\rangle$

$$\|\vec{r}\,'(t)\| = \sqrt{1 + (2t)^2 + (3t^2)^2}$$

$$= \sqrt{1+4t^2+9t^4}$$

3) Substitute

$$\boxed{L = \int_1^2 \sqrt{1+4t^2+9t^4}\, dt}$$

Example 21. Find the distance traveled by a particle moving along the path

$$\mathbf{r}(t) = \langle \ln(t), \sqrt{2}t, \frac{1}{2}t^2 \rangle, \qquad t > 0$$

from $t = 1$ to $t = 2$.

Speed? $\vec{r}\,'(t) = \langle \frac{1}{t}, \sqrt{2}, t \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{\frac{1}{t^2} + 2 + t^2}$

dist $= \int_1^2 \sqrt{\frac{1}{t^2} + 2 + t^2}\, dt$

$= \int_1^2 \sqrt{\left(\frac{1}{t} + t\right)^2}\, dt$

$= \int_1^2 \frac{1}{t} + t \, dt$

$= \ln(t) + \frac{1}{2}t^2 \Big|_1^2$

$= (\ln 2 + 2) - (\ln(1) + \frac{1}{2}) = \boxed{\ln 2 + \frac{3}{2}}$

When is $\vec{r}(t) = \langle \ln(\sqrt{2}), 2, 1 \rangle$?

$\ln(t) = \ln(\sqrt{2})$

$\sqrt{2}\,t = 2$

$\frac{1}{2}t^2 = 1$

↳ so $t = \sqrt{2}$

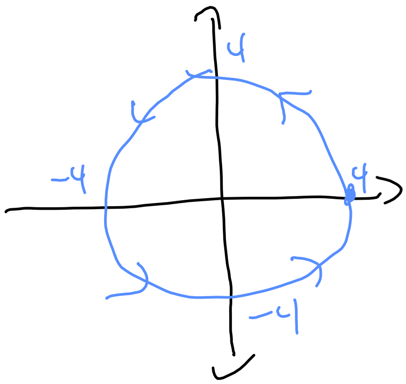
Sometimes, we care about the distance traveled from a fixed starting time t_0 to an arbitrary time t, which is given by the **arc length function**.

$$s(t) = \int_{t_0}^{t} \|\vec{r}\,'(\tau)\|\, d\tau$$

We can use this function to produce parameterizations of curves where the parameter s measures distance along the curve: the points where $s = 0$ and $s = 1$ would be exactly 1 unit of distance apart.

arc length parameterization $\Longleftrightarrow$ unit speed parameterization

Example 22. Find an arc length parameterization of the circle of radius 4 about the origin in $\mathbb{R}^2$, $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t)\rangle, 0 \le t \le 2\pi$.



1) Find $s(t) = \int_{t_0}^{t} \|\vec{r}\,'(T)\|\, dT$ ← maybe hard to integrate

Take $t_0 = 0$

$$\vec{r}\,'(t) = \langle -4\sin(t), 4\cos(t)\rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{16\sin^2(t) + 16\cos^2(t)}$$

$$= \sqrt{16(\sin^2(t) + \cos^2(t))}$$

$$= 4$$

$$s(t) = \int_0^t 4\, dT = 4t$$

2) Solve for $t = f(s)$: $s = 4t$ ← maybe hard to solve

$$\to t = \frac{s}{4}$$

3) Substitute $t = f(s)$ in $\vec{r}(t)$:

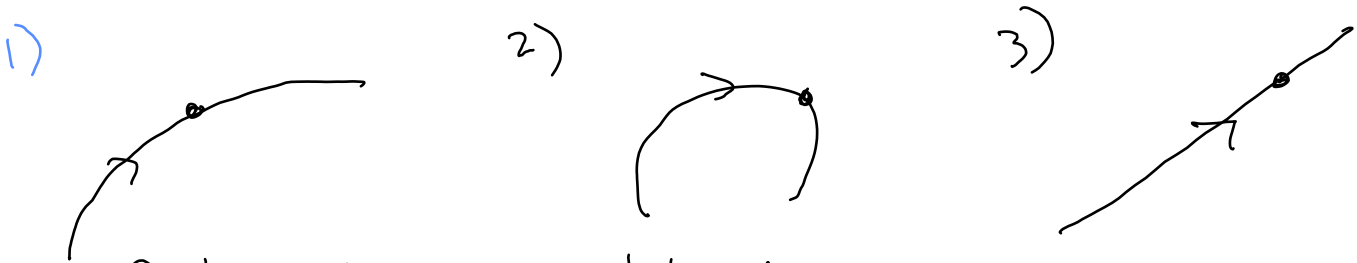
$$\vec{r}_2(s) = \vec{r}(f(s)) = \vec{r}\left(\frac{s}{4}\right) = \left\langle 4\cos\left(\frac{s}{4}\right), 4\sin\left(\frac{s}{4}\right)\right\rangle$$

$$0 \le \frac{s}{4} \le 2\pi$$

$$\iff 0 \le s \le 8\pi$$

13.4 - Curvature, Tangents, Normals

The next idea we are going to explore is the curvature of a curve in space along with two vectors that orient the curve.



Rank curvature from most to least :

First, we need the **unit tangent vector**, denoted **T**:

- In terms of an arc-length parameter s: $\vec{r}\,'(s)$
- In terms of any parameter t: $\vec{r}\,'(t) \,/\, \|\vec{r}\,'(t)\|$

This lets us define the **curvature**, $\kappa(s) = \|\vec{T}\,'(s)\|$

Example 23. Earlier, we found an arc length parameterization of the circle of radius 4 centered at $(0,0)$ in $\mathbb{R}^2$:

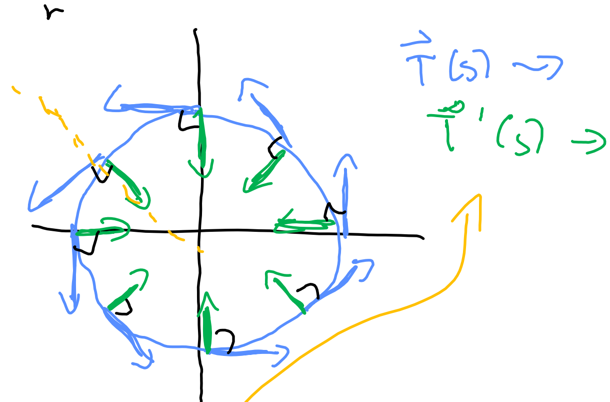
$$\mathbf{r}(s) = \left\langle 4\cos\left(\frac{s}{4}\right), 4\sin\left(\frac{s}{4}\right)\right\rangle, \qquad 0 \leq s \leq 8\pi.$$

Use this to find $\mathbf{T}(s)$ and $\kappa(s)$.

- $\vec{T}(s) = \vec{r}\,'(s) = \left\langle -4\sin\left(\frac{s}{4}\right)\cdot\frac{1}{4}, 4\cos\left(\frac{s}{4}\right)\cdot\frac{1}{4}\right\rangle$
  $= \left\langle -\sin\left(\frac{s}{4}\right), \cos\left(\frac{s}{4}\right)\right\rangle$

- $\kappa(s) = \|\vec{T}\,'(s)\|$
  $= \left\|\left\langle -\frac{1}{4}\cos\left(\frac{s}{4}\right), -\frac{1}{4}\sin\left(\frac{s}{4}\right)\right\rangle\right\|$
  $= \left|\frac{1}{4}\right| \cdot \underbrace{\left\|\left\langle -\cos\left(\frac{s}{4}\right), -\sin\left(\frac{s}{4}\right)\right\rangle\right\|}_{1 \text{ by Pythagorean Thm}}$
  $= \frac{1}{4}$

- every circle has constant curvature $\frac{1}{r}$



Question: In which direction is $\mathbf{T}$ changing?

This is the direction of the **principal unit normal**, $\mathbf{N}(s) =$ $\vec{T}\,'(s) / \|\vec{T}\,'(s)\|$

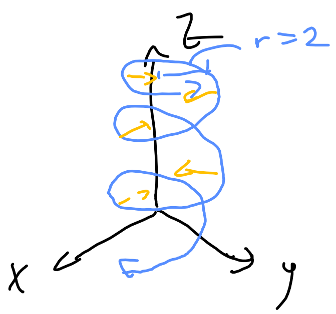
in direction of motion $\qquad \|\vec{N}\|=1 \qquad \vec{T}\cdot\vec{N}=0$

We said that it is often hard to find arc length parameterizations, so what do we do if we have a generic parameterization $\mathbf{r}(t)$?

- $\mathbf{T}(t) = \vec{r}'(t) / \|\vec{r}'(t)\|$
- $\mathbf{N}(t) = \vec{T}'(t) / \|\vec{T}'(t)\|$
- $\kappa(t) = \|\vec{T}'(t)\| / \|\vec{r}'(t)\|$ or $\dfrac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$ ← only in $\mathbb{R}^3$

↑ don't use if $\vec{T}'(t)$ is hard to compute

Example 24. Find $\mathbf{T}, \mathbf{N}, \kappa$ for the helix $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), t-1 \rangle$. ← $t \in \mathbb{R}$



$\vec{r}'(t) = \langle -2\sin(t), 2\cos(t), 1 \rangle$

$\|\vec{r}'(t)\| = \sqrt{\underbrace{4\sin^2(t) + 4\cos^2(t)}_{=4} + 1}$

$= \sqrt{5}$

$$\boxed{\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \left\langle -\frac{2}{\sqrt{5}}\sin(t), \frac{2}{\sqrt{5}}\cos(t), \frac{1}{\sqrt{5}} \right\rangle}$$

$\vec{T}'(t) = \left\langle -\frac{2}{\sqrt{5}}\cos(t), \frac{-2}{\sqrt{5}}\sin(t), 0 \right\rangle$

$\|T'(t)\| = \sqrt{\frac{4}{5}\cos^2(t) + \frac{4}{5}\sin^2(t) + 0} = \frac{2}{\sqrt{5}}$

$$\boxed{\vec{N}(t) = \vec{T}'(t) / \|\vec{T}'(t)\| = \langle -\cos(t), -\sin(t), 0 \rangle}$$

$\kappa(t) = \|\vec{T}'(t)\| / \|\vec{r}'(t)\| = \dfrac{2}{\sqrt{5}} \Big/ \sqrt{5} = \boxed{\dfrac{2}{5}}$

Day 5 Lecture

Daily Announcements & Reminders:

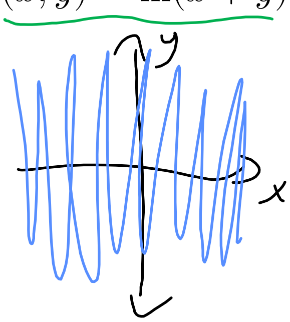
- HW 12.6 & 13.1 due tonight
- Quiz 2 in studio tomorrow if normal operations
  - 12.6, 13.1, 13.2 / L.O. G2/G4
- Do warmup on Ed ⟶

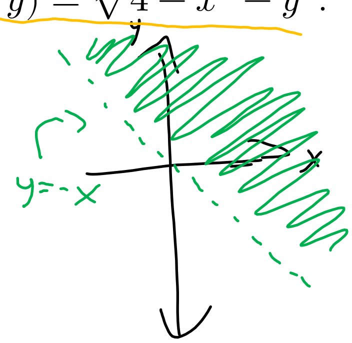
Goals for Today:

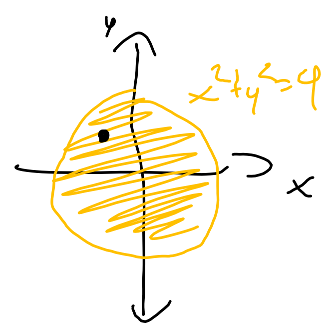
Sections 13.4-14.1

- Compute the unit tangent and principal unit normal vectors of a curve
- Give examples of functions of multiple variables
- Find the domain of functions of two variables
- Introduce and sketch traces and contours of functions of two variables
- Graph functions of two variables

In the pre-lecture video, we discussed the domains of the functions $f(x, y) = x^2 + y^2$, $g(x, y) = \ln(x + y)$, and $h(x, y) = \sqrt{4 - x^2 - y^2}$.



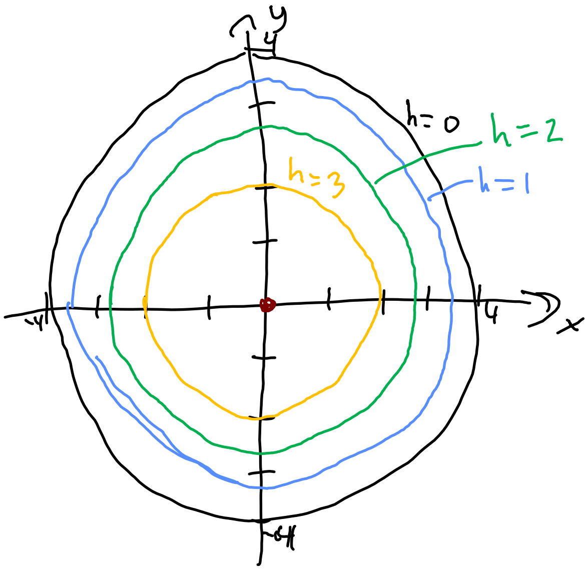




Definition 28. If f is a function of two variables with domain D, then the graph of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

Here are the graphs of the three functions above.

Example 29. Suppose a small hill has height $z = h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$ m at each point (x, y). How could we draw a picture that represents the hill in 2D?



Domain: $z \geq 0$: $x^2+y^2 \leq 16$

- Draw curves of all pts in domain with a fixed height

h=0 m: $0 = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2 \Rightarrow x^2+y^2=16$

h=1 m: $1 = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2 \Rightarrow 4 = 16 - x^2 - y^2$
$x^2+y^2 = 12$

h=2 m: $2 = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2 \Rightarrow x^2+y^2=8$

h=3 m: $3 = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2 \Rightarrow x^2+y^2=4$

h=4 m: $4 = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2 \Rightarrow x^2+y^2=0$

In 3D, it looks like this.

Definition 30. The contours (also called level curves or level sets) of a function f of two variables are the curves with equations $k = f(x,y)$, where k is a constant (in the range of f). A plot of contours for various values of z is a contour plot/map (or level curve plot).

Some common examples of these are:

- topographical maps
- field lines / equipotential lines
- blueprint / slicing for 3D printing
- heat/weather maps
- electron density plot

$\{(x,y) \in \mathbb{R}^2 \mid x,y \in \mathbb{R}\} = \mathbb{R}^2$

Example 31. Create a contour diagram of $f(x, y) = x^2 - y^2$

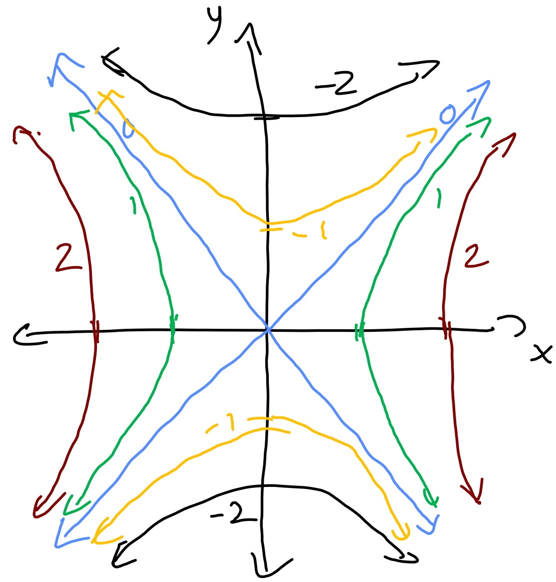
$0 = x^2 - y^2 \Rightarrow x^2 = y^2 \Rightarrow x = y$
$x = -y$

$1 = x^2 - y^2 \to$ hyperbola, $x \neq 0$
$y = 0 \to x = \pm 1$

$-1 = x^2 - y^2$
$\Leftrightarrow 1 = y^2 - x^2 \to$ hyperbola, $y \neq 0$
$x = 0 \to y = \pm 1$

$4 = x^2 - y^2 \to$ hyperbola, $x \neq 0$

$-4 = x^2 - y^2$



Example 32. Create a contour diagram of $g(x, y) = \sqrt{16 - 4x^2 - y^2}$.

Definition 33. The ________________ of a surface are the curves of ________________________ of the surface with planes parallel to the ____________________________.

Example 34. Use the traces and contours of $z = f(x, y) = 4 - 2x - y^2$ to sketch the portion of its graph in the first octant.

Let's check our work: https://tinyurl.com/math2551-2var-graph