MATH 2551 K - Dr. Hunter Lehmann

- Dr. Lehmann, Dr. H, Dr. Hunter, as you prefer

Daily Announcements & Reminders:

- Meet your neighbors
- Use the QR code to access Itempool
- Take the syllabus quiz on Canvas -> Modules

Goals for Today: Sections 12.1, 12.3, 12.4

- Set classroom norms
- Describe the big-picture goals of the class
- Review $\mathbb{R}^3$ and the dot product
- Introduce the cross product and its properties

Class Values/Norms:

- Mistakes are a learning opportunity
- Mathematics is collaborative
- Make sure everyone is included
- Criticize ideas, not people
- Be respectful of everyone
- Ask questions
- Participate actively
-

Big Idea: Extend differential & integral calculus.

What are some key ideas from these two courses?

Differential Calculus

- rates of change ✓
- ☆ max/min values of a function ✓
  - ↑ optimization
- related rates ✗
- limits ✓
- implicit differentiation ✓
- tangent line / linear approx. ✓
- concavity

Integral Calculus

- area under curve / volumes ✓
- convergence / divergence ✗
- integration techniques ?
- Riemann sums ✓
- polar coords ✓
- FTC ✓

Before: we studied **single-variable functions** $f : \mathbb{R} \to \mathbb{R}$ like $f(x) = 2x^2 - 6$.

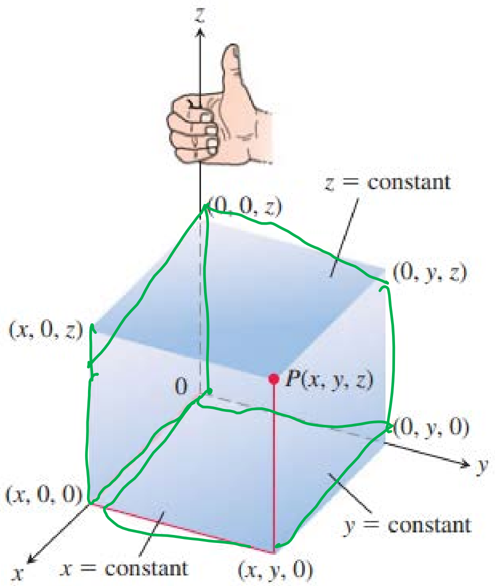
Now: we will study **multi-variable functions** $f : \mathbb{R}^n \to \mathbb{R}^m$: each of these functions is a rule that assigns one output vector with m entries to each input vector with n entries.

$f : \mathbb{R}^2 \to \mathbb{R}$ $\qquad f(x,y) = x^2 y + e^x + \sin(y)$

$\vec{r} : \mathbb{R} \to \mathbb{R}^2$ $\qquad \vec{r}(t) = \begin{bmatrix} 2t \\ t^2 \end{bmatrix} = \langle 2t, t^2 \rangle$

$g : \mathbb{R}^2 \to \mathbb{R}^2$ $\qquad g\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} xy \\ 2x + y \end{bmatrix}$

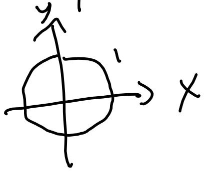
Section 12.1: Three-Dimensional Coordinate Systems

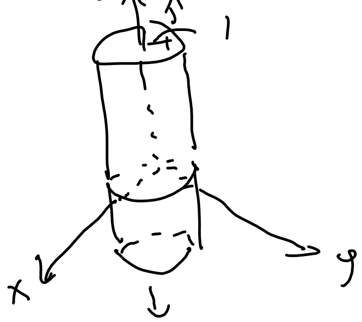


- Right-handed : (x, y, z)
- Coordinate planes:
  - xy-plane : $z=0$
  - xz-plane : $y=0$
  - yz-plane ; $x=0$

Question: What shape is the set of solutions $(x, y, z) \in \mathbb{R}^3$ to the equation $x^2+y^2 = 1$?

- cylinder : z is free
- circle : $x^2+y^2=1$





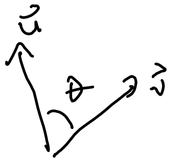
Section 12.3/4: Dot & Cross Products

Definition 1. The **dot product** of two vectors $\mathbf{u} = \langle u_1, u_2, \ldots, u_n \rangle$ and $\mathbf{v} = \langle v_1, v_2, \ldots, v_n \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \ldots + u_nv_n$$

This product tells us about the angle between $\vec{u}$ and $\vec{v}$.

- $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$



$$|\vec{u}| = \|\vec{u}\|$$
$$= \sqrt{\vec{u} \cdot \vec{u}}$$
$$= \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2}$$

In particular, two vectors are **orthogonal** if and only if their dot product is 0.

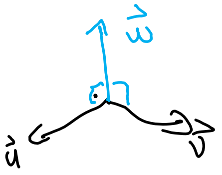
Example 2. Are $\mathbf{u} = \langle 1, 1 \rangle$ and $\mathbf{v} = \langle 2, -1 \rangle$ orthogonal?

$$\vec{u} \cdot \vec{v} = (1)(2) + (1)(-1) = 1 \neq 0$$

so $\vec{u}$ and $\vec{v}$ are not orthogonal

Goal: Given two vectors, produce a vector orthogonal to both of them in a "nice" way.

1. Be right-handed. $\vec{u} \times \vec{v} = \vec{w}$



2. Algebraically product: if $c \in \mathbb{R}$

$$c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v})$$

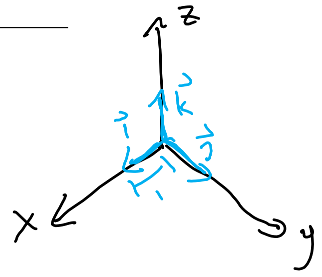
$$(\vec{u}_1 + \vec{u}_2) \times \vec{v} = (\vec{u}_1 \times \vec{v}) + (\vec{u}_2 \times \vec{v})$$

Definition 3. The **cross product** of two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ in $\mathbb{R}^3$ is

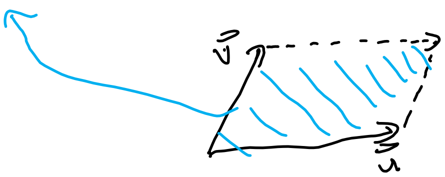
$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

(symbolic) 3x3 determinant

$\vec{i} = \langle 1,0,0 \rangle$
$\vec{j} = \langle 0,1,0 \rangle$
$\vec{k} = \langle 0,0,1 \rangle$



- $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$
- $|\vec{u} \times \vec{v}|$ is the area of the parallelogram



- $|(\vec{u} \times \vec{v}) \cdot \vec{w}|$ is the volume of the parallepiped formed by $\vec{u}, \vec{v}, \vec{w}$

$$= \left|\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}\right|$$

Example 4. Find $\langle 1, 2, 0\rangle \times \langle 3, -1, 0\rangle$. ← both in xy-plane, so an orthogonal vector to both should be in z-direction

$$\langle 1,2,0\rangle \times \langle 3,-1,0\rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ 3 & -1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 0 \\ -1 & 0 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} \vec{k}$$

$$= \big(2(0) - (0)(-1)\big)\vec{i} - \big((1)(0) - (0)(3)\big)\vec{j} + \big((1)(-1) - (2)(3)\big)\vec{k}$$

$$= -7\vec{k} = \langle 0, 0, -7\rangle = \begin{bmatrix} 0 \\ 0 \\ -7 \end{bmatrix}$$

- $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$ <u>antisymmetric</u>

Daily Announcements & Reminders:

- HW: 12.2, 12.3 due tonight at 10 pm
- No studio Monday
- Fill out office hour survey
- Do warmup Itempool

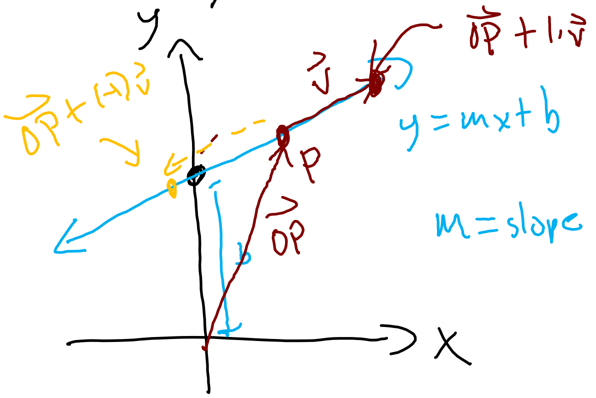
Goals for Today: Sections 12.5-12.6

- Apply the cross product to solve problems
- Learn the equations that describe lines, planes, and quadric surface in $\mathbb{R}^3$
- Solve problems involving the equations of lines and planes
- Sketch quadric surfaces in $\mathbb{R}^3$

Section 12.5 Lines & Planes

Lines in $\mathbb{R}^2$, a new perspective:

$y = mx + b$ $\qquad$ $y - y_0 = m(x - x_0)$

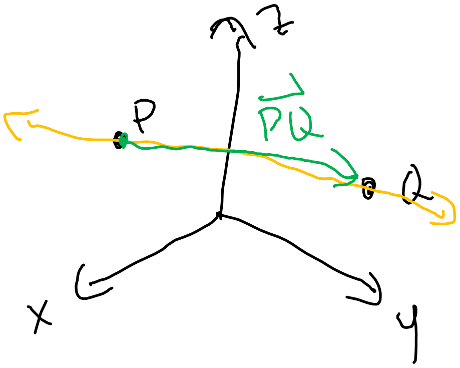


line is: all points of the form

vector equation for the line $\quad \ell(t) = \vec{OP} + t \cdot \vec{v}$

P is on the line $\qquad$ direction vector for the line

Example 5. Find a vector equation for the line that goes through the points $P = (1, 0, 2)$ and $Q = (-2, 1, 1)$.



Need: • point on the line : $P = (1,0,2)$ or $Q = (-2,1,1)$

- direction vector

$$\vec{v} = \vec{PQ} = \langle -2-1, 1-0, 1-2 \rangle = \langle -3, 1, -1 \rangle$$

vector eqn: $\ell(t) = \langle -3, 1, -1 \rangle t + \langle 1, 0, 2 \rangle$

$$= \langle -3t+1, t, -t+2 \rangle$$

$$-\infty \leq t \leq \infty$$

- Equation for a line is not unique

Parametric Equations for a Line: (cont. last ex)

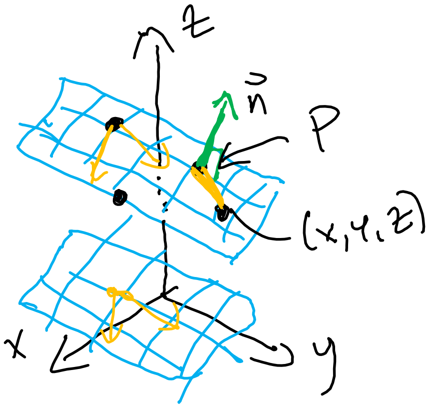
$x(t) = -3t + 1$

$y(t) = t$

$z(t) = 2 - t$

Planes in $\mathbb{R}^3$

Conceptually: A plane is determined by either three points in $\mathbb{R}^3$ or by a single point and a direction **n**, called the *normal vector*.



Need: a) 3 points in plane

$\boxed{\text{b) 1 point and normal vector}}$

c) 1 point and 2 vectors in plane

Algebraically: A plane in $\mathbb{R}^3$ has a *linear* equation (back to Linear Algebra! imposing a single restriction on a 3D space leaves a 2D linear space, i.e. a plane)

$$\boxed{ax + by + cz = d}$$

If $P = (x_0, y_0, z_0)$ is in the plane and $\vec{n} = \langle a, b, c\rangle$

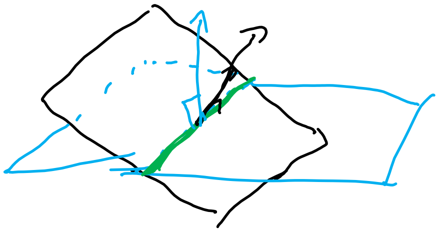
then

$$\langle a,b,c\rangle \cdot \langle x-x_0, y-y_0, z-z_0\rangle = 0$$

$$\boxed{a(x-x_0) + b(y-y_0) + c(z-z_0) = 0}$$

$$d = \vec{n} \cdot \overrightarrow{OP} = ax_0 + by_0 + cz_0$$

Example 6. Consider the planes $y - z = -2$ and $x - y = 0$. Show that the planes intersect and find an equation for the line passing through the point $P = (-8, 0, 2)$ which is parallel to the line of intersection of the planes.



Plane 1: $y - z = -2$ $\quad \vec{n}_1 = \langle \cancel{1, 0}\, 0\rangle = \langle 0, 1, -1\rangle$

Plane 2: $x - y = 0$ $\quad \vec{n}_2 = \langle 1, -1, 0\rangle$

- B/c $\vec{n}_1$ is not parallel to $\vec{n}_2$, the planes must meet.
- $(1, 1, 3)$ is on both planes

plane 2: $x = y$ plane 1 says: $z = y + 2$

- Goal: Find a line through $(-8, 0, 2)$ parallel to line of intersection

direction vector $\vec{v} = \vec{n}_1 \times \vec{n}_2$ b/c $\vec{v}$ lies in both planes

or $\vec{n}_2 \times \vec{n}_1$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ -1 & 0 \end{vmatrix} \vec{i} - \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} \vec{j} + \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} \vec{k}$$

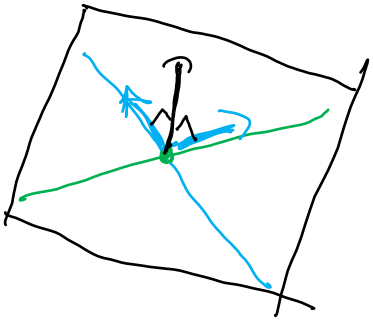
$$= (0 - 1)\vec{i} - (0 + 1)\vec{j} + (0 - 1)\vec{k}$$

$$= \langle -1, -1, -1\rangle$$

So $\boxed{\ell(t) = \langle -1, -1, -1\rangle t + \langle -8, 0, 2\rangle}$

Itempool: $\vec{n} = \vec{v}_1 \times \vec{v}_2 = \langle 1, 0, 1\rangle \times \langle 0, 1, -1\rangle = \langle -1, 1, 1\rangle$

$P = \langle 1, 0, 0\rangle$ ← point on both lines (but just needs on 1)



$$\boxed{-(x - 1) + 1(y - 0) + 1(z - 0) = 0}$$

Section 12.6 Quadric Surfaces

Definition 7. A **quadric surface** in $\mathbb{R}^3$ is the set of points that solve a quadratic equation in x, y, and z.

You know several examples already: $x^2+y^2+z^2=1$ (spheres)

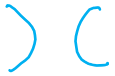
$x^2+y^2=1$ (cylinder)

- We are not going to worry about xy, yz, xz terms.

The most useful technique for recognizing and working with quadric surfaces is to examine their cross-sections.

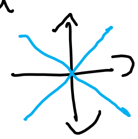
Example 8. Use cross-sections to sketch and identify the quadric surface $z = x^2 - y^2$.

If we fix $z = 1$: $1 = x^2 - y^2$ hyperbola

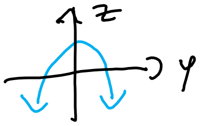


$z = C$: $C = x^2 - y^2$ hyperbola

except $0 = x^2 - y^2$

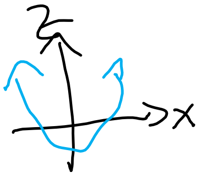


If we fix $x = 1$: $z = 1 - y^2$ parabola

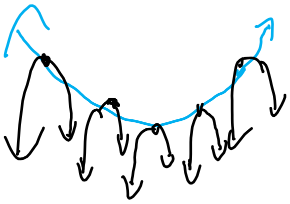


$x = C$: $z = C - y^2$

If we fix $y = 1$: $z = x^2 - 1$ parabola



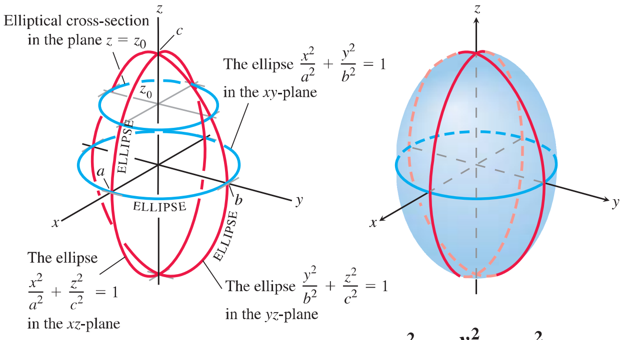
$y = C$: $z = x^2 - C$



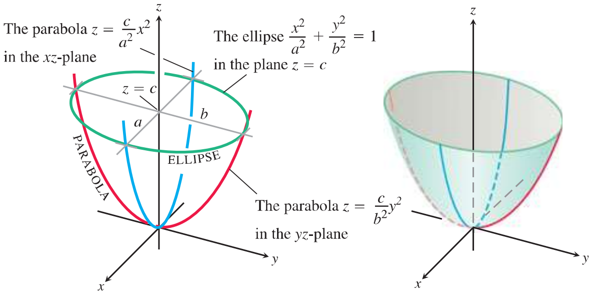
• <u>parabolic hyperboloid</u>

Example 9. Use cross-sections to sketch and identify the quadric surface $x = z^2 + y^2$.

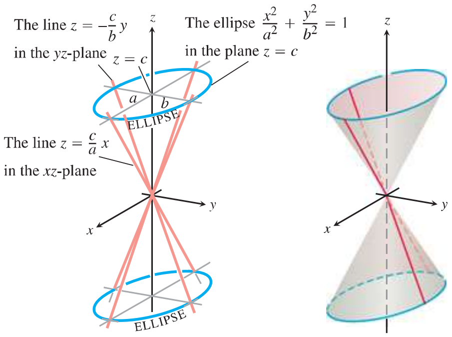
TABLE 12.1 Graphs of Quadric Surfaces



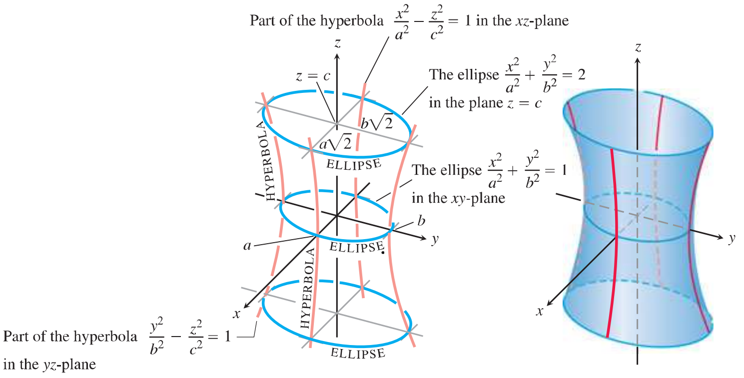
ELLIPSOID $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$



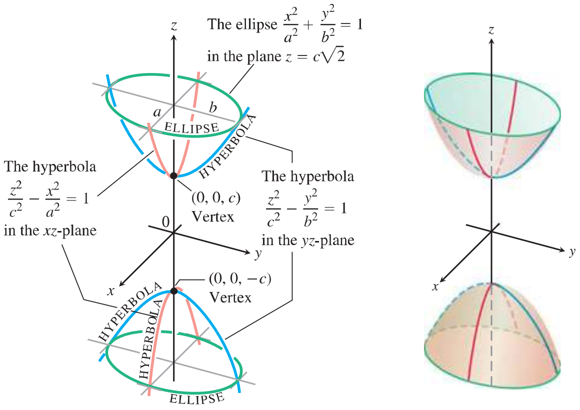
ELLIPTICAL PARABOLOID $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$



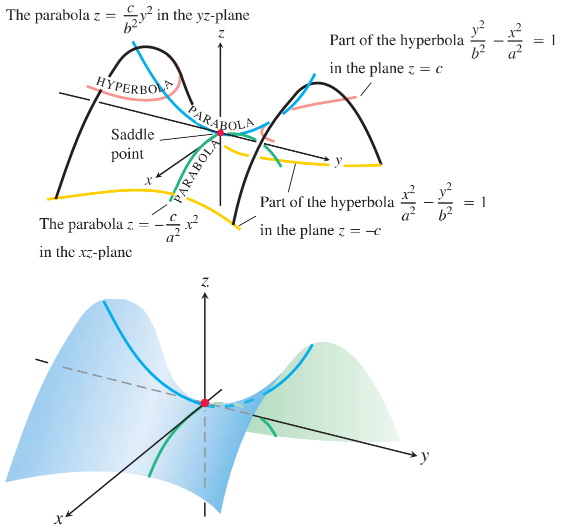
ELLIPTICAL CONE $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$



HYPERBOLOID OF ONE SHEET $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$



HYPERBOLOID OF TWO SHEETS $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



HYPERBOLIC PARABOLOID $\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}, c > 0$

Section 13.1 Curves in Space & Their Tangents

Daily Announcements & Reminders:

- HW 12.1, 12.4 due tonight at 10 pm
- Quiz 1 tomorrow in studio : 12.1 - 12.5
- Office hours are T 9-10 Skiles 218C
  R 12:30-1:30 Skiles 218C
  F 3-4 Zoom
- For studio attendance bonus you <u>must</u> attend the studio you are registered for

Goals for Today: Sections 13.1-13.2

- Introduce vector-valued functions
- Plot vector-valued functions and construct them from a graph
- Compute limits, derivatives, and tangent lines for vector-valued functions

- Last time "parabolic hyperboloid"
  ⇒ "hyperbolic paraboloid"

Warm-up problem: Let $\mathbf{r}(t) = \langle t, t^2 + 1\rangle$, where $t \geq 0$. $\vec{r}: \mathbb{R} \to \mathbb{R}^2$

↑ domain

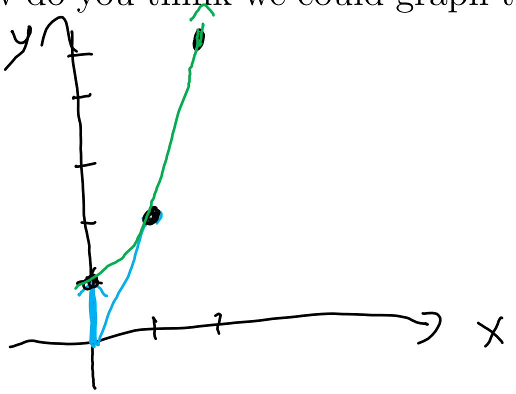
a) Find $\mathbf{r}(0), \mathbf{r}(1)$, and $\mathbf{r}(2)$.

$\vec{r}(0) = \langle 0, 1\rangle$

$\vec{r}(1) = \langle 1, 2\rangle$

$\vec{r}(2) = \langle 2, 5\rangle$

b) How do you think we could graph this function? What does the graph look like?



- half of an up-ward facing parabola

b/c $t \geq 0$ and $x = t$, $x \geq 0$

c) Is there a function $y = f(x)$ that has the same graph as $\mathbf{r}(t)$?

$y = x^2 + 1$, $x \geq 0$

Q: Can we take any $y = f(x)$ and write an equivalent function?

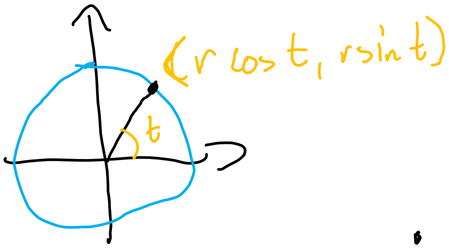
A: Yes! $\vec{r}(t) = \langle t, f(t)\rangle$ (with approp. domain)

The function $\mathbf{r}(t)$ in the warm-up problem is an example of a **vector-valued function**: its input is a real number t and its output is a vector. We graph a vector-valued function by plotting all of the terminal points of its output vectors, placing their initial points at the origin.

You have seen several examples already:

- lines in $\mathbb{R}^3$: $\vec{r}(t) = \langle 1, 2, 3\rangle t + \langle 0, 1, 0\rangle$
- Circle: $x^2+y^2=4$ in $\mathbb{R}^2$

  $\vec{r}(t) = \langle 2\cos(t), 2\sin(t)\rangle$ $0 \le t \le 2\pi$



- If we have $y=f(x)$ or $x=g(y)$

  $\vec{r}(t) = \langle t, f(t)\rangle$ $\vec{r}(t) = \langle g(t), t\rangle$

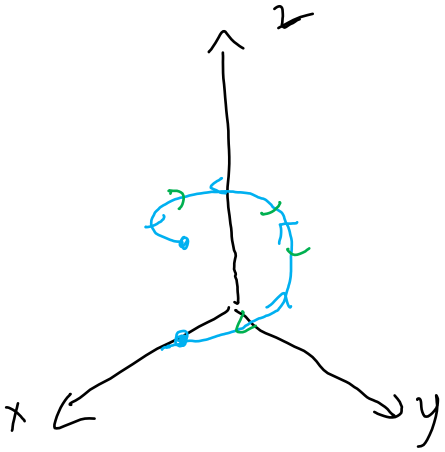
- A parameterization of a curve C is a vector-valued function whose graph is C.

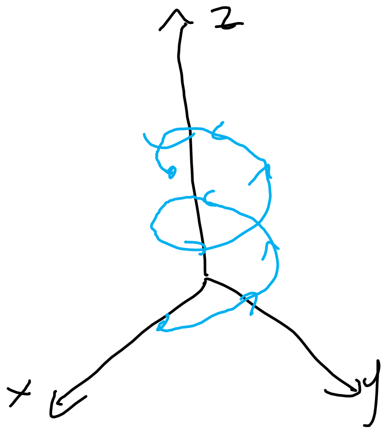
Example 10. Consider $\mathbf{r}_1(t) = \langle \cos(t), \sin(t), t\rangle$ and $\mathbf{r}_2(t) = \langle \cos(2t), \sin(2t), 2t\rangle$, each with domain $[0, 2\pi]$. What do you think the graph of each looks like? How are they similar and how are they different?

- wave
- ~~cylinder~~ (wrong dimension)
- spiral
- helix

- $\vec{r}_3(t) = \langle \cos(t), \sin(t), 0\rangle$

  circle of radius in xy-plane





- $\vec{r}_1$ & $\vec{r}_2$ parameterize different curves
  - if domain was $\mathbb{R}$, then same curve or change domain

    $\vec{r}_1(t)$ for $[0, 2\pi]$

    $\vec{r}_2(t)$ for $[0, \pi]$
- Does the direction (orientation) matter?
  - Yes!

Check your intuition

Calculus of vector-valued functions

Unifying theme: Do what you already know, componentwise.

This works with limits:

Example 11. Compute $\lim_{t\to e}\langle t^2, 2, \ln(t)\rangle$.

$$= \left\langle \lim_{t\to e} t^2, \lim_{t\to e} 2, \lim_{t\to e} \ln(t) \right\rangle$$

$$= \langle e^2, 2, 1 \rangle$$

- all limit laws apply

↑ $\langle t^2, 2, \ln(t)\rangle$ is continuous at $t=e$

And with continuity: $\vec{r}(t)$ is continuous at $t=a$ if $\vec{r}(a) = \lim_{t\to a} \vec{r}(t)$

Example 12. Determine where the function $\mathbf{r}(t) = t\mathbf{i} - \dfrac{1}{t^2-4}\mathbf{j} + \sin(t)\mathbf{k}$ is continuous.

- $x(t) = t$ is continuous for all $t \in \mathbb{R}$
- $y(t) = -\dfrac{1}{t^2-4} = \dfrac{-1}{(t-2)(t+2)}$ is cts for $(-\infty,-2) \cup (-2,2) \cup (2,\infty)$
- $z(t) = \sin(t)$ is continuous for all t

- $\vec{r}(t)$ is continuous where all 3 parts are cts

$\Rightarrow$ $\vec{r}(t)$ is cts on $(-\infty,-2) \cup (-2,2) \cup (2,\infty)$

And with derivatives:

Example 13. If $\mathbf{r}(t) = \langle 2t - \frac{1}{2}t^2 + 1, t - 1\rangle$, find $\mathbf{r}'(t)$. or $\frac{d\vec{r}}{dt}$ or $\vec{r}_t$ or $\dot{\vec{r}}$

$$\vec{r}\,'(t) = \left\langle \frac{d}{dt}\left(2t - \frac{1}{2}t^2 + 1\right), \frac{d}{dt}(t-1) \right\rangle$$
$$= \langle 2 - t, 1 \rangle$$

Interpretation: If $\mathbf{r}(t)$ gives the position of an object at time t, then

- $\mathbf{r}'(t)$ gives velocity of an object
- $|\mathbf{r}'(t)|$ gives speed of an object
- $\mathbf{r}''(t)$ gives acceleration of an object

$\|\vec{r}\,'(t)\|$ $\|$

$|x| = \sqrt{x^2}$

Ex 13)
$\vec{r}\,'(0) = \langle 2, 1 \rangle$ velocity

$|\vec{r}\,'(0)| = |\langle 2,1\rangle| = \sqrt{5}$
speed

Let's see this graphically

$\vec{r}\,'(t) = \langle 2 - t, 1 \rangle$

Example 14. Find an equation of the tangent line to $\mathbf{r}(t) = \langle 2t - \frac{1}{2}t^2 + 1, t - 1\rangle$ at time $t = 2$.

- $\vec{r}\,'(t_0)$ is tangent to the graph of $\vec{r}(t)$ at $t = t_0$ if we plot it with the base at $\vec{r}(t_0)$

Need:
- direction for tangent line → $\vec{r}\,'(t_0)$
- point on the line → $\vec{r}(t_0)$

tangent line: $\ell(s) = (\vec{r}\,'(t_0))s + \vec{r}(t_0)$

Here: At $t = 2$, $\vec{r}\,'(2) = \langle 0, 1 \rangle$
$\vec{r}(2) = \langle 4 - \frac{1}{2}(4) + 1, 2 - 1 \rangle = \langle 3, 1 \rangle$

So tangent line is $\ell(t) = \langle 0, 1 \rangle t + \langle 3, 1 \rangle$ $t \in \mathbb{R}$
$= \langle 3, 1 + t \rangle$

Itempool

Find parametric equations for the tangent line to the curve $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + t\mathbf{k}$ at the point $(1, 0, 2\pi)$.

$\vec{r}\,'(t) = (-\sin(t))\vec{i} + (\cos(t))\vec{j} + (1)\vec{k}$

At what t is $\vec{r}(t) = \langle 1, 0, 2\pi\rangle$

Next find $t_0 = 2\pi$:

$\cos(t_0) = 1$ $\quad \cos(2\pi) = 1$

$\sin(t_0) = 0$ $\quad \sin(2\pi) = 0$

$t_0 = 2\pi$

$\vec{r}\,'(2\pi) = \langle 0, 1, 1\rangle$

So $\ell(t) = \langle 0, 1, 1\rangle t + \langle 1, 0, 2\pi\rangle$

$= \langle 1, t, t + 2\pi\rangle$

Daily Announcements & Reminders:

Goals for Today: Sections 13.2, 13.3

- Compute integrals of vector-valued functions and solve initial value problems
- Compute arc lengths of curves using parameterizations
- Introduce the idea of an arc-length parameterization

Continuing from last time with integrals: Theme: work componentwise

Example 15. Find $\int_0^1 \langle t, e^{2t}, \sec^2(t)\rangle \, dt$.

$$\int_0^1 \langle t, e^{2t}, \sec^2(t)\rangle dt$$

$$= \langle \int_0^1 t\, dt, \int_0^1 e^{2t} dt, \int_0^1 \sec^2(t)\, dt \rangle$$

$$= \langle \tfrac{1}{2}t^2 \Big|_0^1, \tfrac{1}{2}e^{2t}\Big|_0^1, \tan(t)\Big|_0^1 \rangle$$

$$= \langle \tfrac{1}{2}, \tfrac{1}{2}(e^2-1), \tan(1) \rangle$$

- integral of a vector-valued fn is a vector
- $\int_a^b \vec{v}(t)\, dt$ = displacement btwn $t=a$ & $t=b$

$$\int_a^b \vec{r}\,'(t)\, dt = \vec{r}(b) - \vec{r}(a)$$

At this point we can solve initial-value problems like those we did in single-variable calculus:

Example 16. Wallace is testing a rocket to fly to the moon, but he forgot to include instruments to record his position during the flight. He knows that his velocity during the flight was given by

$$\mathbf{v}(t) = \langle -200\sin(2t), 200\cos(t), 400 - \frac{400}{1+t}\rangle \ m/s.$$

If he also knows that he started at the point $\mathbf{r}(0) = \langle 0,0,0\rangle$, use calculus to reconstruct his flight path.

Goal: Find $\vec{r}(t)$.

$u = 1+t$

1) Find antiderivative of $\vec{v}(t)$:

$$\vec{r}(t) = \int \langle -200\sin(2t), 200\cos(t), 400\left(1 - \frac{1}{1+t}\right)\rangle dt$$

$$= \langle 100\cos(2t) + C_1, 200\sin(t) + C_2, 400\left(t - \ln|1+t|\right) + C_3\rangle$$

$$= \langle 100\cos(2t), 200\sin(t), 400(t - \ln|1+t|)\rangle + \vec{c}$$

$$\vec{c} = \langle C_1, C_2, C_3\rangle$$

2) Apply initial condition to find $\vec{c}$:

$$\langle 0,0,0\rangle = \vec{r}(0) = \langle 100, 0, 400(0 - \overset{0}{\ln(1)})\rangle + \vec{c}$$

$$\langle 0,0,0\rangle = \langle 100,0,0\rangle + \vec{c}$$

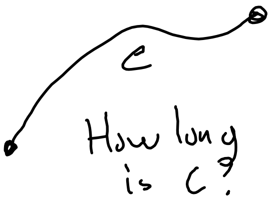
$$\langle -100,0,0\rangle = \vec{c} \iff C_1 = -100, \quad C_2 = C_3 = 0$$

$$\boxed{\vec{r}(t) = \langle 100\cos(2t) - 100, 200\sin(t), 400(t - \ln|1+t|)\rangle}$$

13.3 Arc length of curves

We have discussed motion in space using by equations like $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$.

Our next goal is to be able to measure distance traveled or arc length.

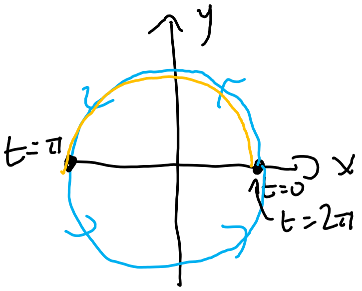


Motivating problem: Suppose the position of a fly at time t is

$$\mathbf{r}(t) = \langle \underline{2}\cos(t), \underline{2}\sin(t)\rangle,$$

where $0 \le t \le 2\pi$.

a) Sketch the graph of $\mathbf{r}(t)$. What shape is this?



↳ circle of radius 2

b) How far does the fly travel between $t = 0$ and $t = \pi$?

Yellow arc is half of circumference of the circle, so fly travels $\frac{1}{2}(2\pi\cdot 2)$
$= 2\pi$

c) What is the speed $|\mathbf{v}(t)|$ of the fly at time t?

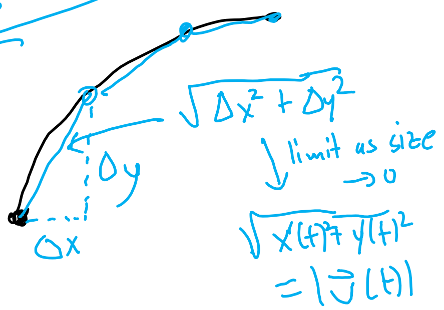
$$|\vec{v}(t)| = |\langle -2\sin(t), 2\cos(t)\rangle|$$
$$= \sqrt{4\sin^2(t)+4\cos^2(t)} = \sqrt{4(\underbrace{\sin^2(t)+\cos^2(t)}_{1})} = 2$$

d) Compute the integral $\int_0^\pi |\mathbf{v}(t)|\, dt$. What do you notice?

$$\int_0^\pi 2\, dt = 2t\Big|_0^\pi = 2\pi$$

1) $\text{dist} = \overset{\text{constant}}{\text{speed}}\cdot\text{time} \iff \int_a^b \underset{\text{non-constant speed}}{|\vec{v}(t)|}\, \underset{\text{small time}}{dt}$

2)



$\vec{r}\,'(t) \neq \langle 0,0,0\rangle$

Definition 17. We say that the **arc length** of a smooth curve $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ from $t=a$ to $t=b$ that is traced out exactly once is

$$L = \int_a^b |\vec{v}(t)|\,dt = \int_a^b |\vec{r}\,'(t)|\,dt$$

Example 18. Set up an integral for the arc length of the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ from the point $(1, 1, 1)$ to the point $(2, 4, 8)$.

Endpoints?: $t=a$: $(1,1,1) = \vec{r}(a) = \langle a, a^2, a^3\rangle \Rightarrow a=1,\ a^2=1,\ a^3=1 \Rightarrow a=1$

$t=b$: $(2,4,8) = \vec{r}(b) = \langle b, b^2, b^3\rangle \Rightarrow b=2,\ b^2=4,\ b^3=8 \Rightarrow b=2$

Speed: $\vec{r}\,'(t) = (1)\vec{i} + (2t)\vec{j} + (3t^2)\vec{k}$

$|\vec{r}\,'(t)| = \sqrt{1^2 + (2t)^2 + (3t^2)^2} = \sqrt{1+4t^2+9t^4}$

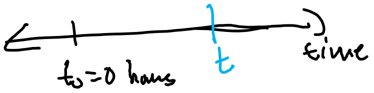
So, $\boxed{L = \int_1^2 \sqrt{1+4t^2+9t^4}\,dt}$

- these integrals can be complicated; look for perfect squares or u-subs.

Sometimes, we care about the distance traveled from a fixed starting time t_0 to an arbitrary time t, which is given by the **arc length function**.

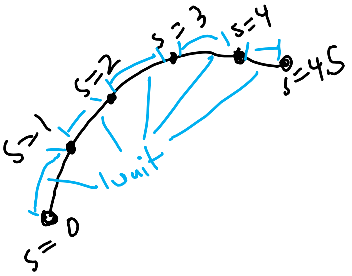
constant → t_0 ; t ← variable

$$s(t) = \int_{t_0}^{t} |\vec{v}(T)|\,dT$$



$s(t)$ = dist traveled along $\vec{r}(t)$ between $T=t_0$ and $T=t$

We can use this function to produce parameterizations of curves where the parameter s measures distance along the curve: the points where $s = 0$ and $s = 1$ would be exactly 1 unit of distance apart.



- like using mile markers to describe position on a highway

Example 19. Find an arc length parameterization of the circle of radius 4 about the origin in $\mathbb{R}^2$, $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t)\rangle, 0 \leq t \leq 2\pi$.

↖ <u>not</u> an arc-length parameterization

- All arc-length parameterization are unit speed: $|\vec{r}\,'(t)| = 1$

for this $\vec{r}(t)$: $\vec{r}\,'(t) = \langle -4\sin(t), 4\cos(t)\rangle$

$|\vec{r}\,'(t)| = \sqrt{16\sin^2(t) + 16\cos^2(t)} = 4$

1) Compute arc length function: $(t_0 = 0)$ • might be hard to integrate

$s(t) = \int_0^t 4\,dT = 4t$

• always possible, frequently difficult

2) Invert & solve for t:

$s = 4t \iff t = \frac{s}{4} = f(s)$

3) Plug in: $\vec{r}(s) = \vec{r}(f(s)) = \langle 4\cos(\frac{s}{4}), 4\sin(\frac{s}{4})\rangle$

$0 \leq \frac{s}{4} \leq 2\pi$

so $0 \leq s \leq 8\pi$

Daily Announcements & Reminders:

- HW 12.6, 13.1 due tonight
- Quiz 2 tomorrow on 12.6, 13.1, 13.2
- Do the warm up question
- Take advantage of PLUS sessions
  5-6 pm Sundays, 7-8 pm Tuesdays
  CULC 125

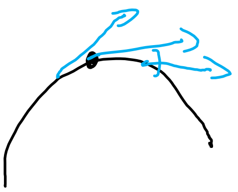
Goals for Today: Sections 13.3-13.4

- Define, interpret, and compute the curvature of a curve
- Compute the unit tangent and principal unit normal vectors of a curve
- Give examples of functions of multiple variables
- Find the domain of functions of two variables
- Graph functions of two variables

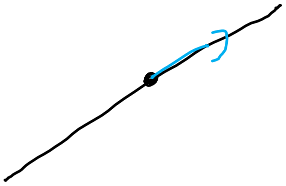
13.4 - Curvature, Tangents, Normals

The next idea we are going to explore is the curvature of a curve in space along with two vectors that orient the curve.

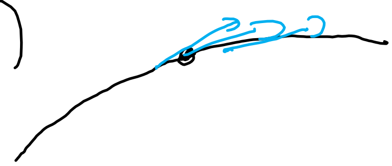
1)



2)



3)



Rank curvature of these curves at marked points.

$\boxed{1 > 3 > 2}$ $1 = 2 = 3$

- Curvature measures how fast direction of motion changes

First, we need the **unit tangent vector**, denoted **T**: $\vec{T}(t) = \vec{r}\,'(t)/|\vec{r}\,'(t)|$

$\vec{T}(s) = \vec{r}\,'(s)$

- In terms of an arc-length parameter s: $\vec{r}\,'(s)$
- In terms of any parameter t: $\vec{r}\,'(t)/|\vec{r}\,'(t)|$

This lets us define the **curvature**, $\kappa(s) =$ $|\vec{T}\,'(s)|$

Example 20. Last class we found an arc length parameterization of the circle of radius 4 centered at $(0, 0)$ in $\mathbb{R}^2$:

$$\mathbf{r}(s) = \left\langle 4\cos\left(\frac{s}{4}\right), 4\sin\left(\frac{s}{4}\right)\right\rangle, \qquad 0 \le s \le 8\pi.$$

Use this to find $\mathbf{T}(s)$ and $\kappa(s)$.

$$\vec{T}(s) = \vec{r}\,'(s) = \left\langle -4\sin\left(\frac{s}{4}\right)\cdot\frac{1}{4}, 4\cos\left(\frac{s}{4}\right)\cdot\frac{1}{4}\right\rangle$$

$$= \left\langle -\sin\left(\frac{s}{4}\right), \cos\left(\frac{s}{4}\right)\right\rangle$$

• unit vector!

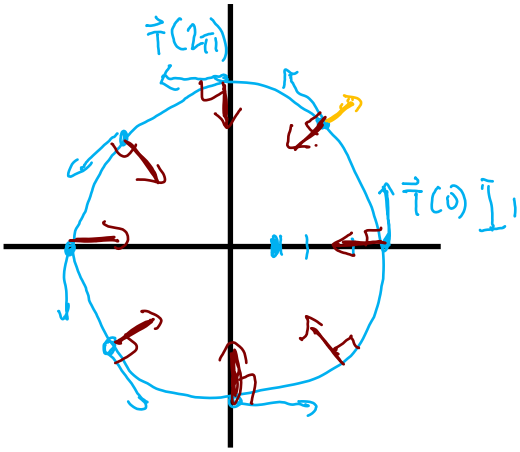
$$\kappa(s) = |\vec{T}\,'(s)|$$

$$= \left|\left\langle -\frac{1}{4}\cos\left(\frac{s}{4}\right), -\frac{1}{4}\sin\left(\frac{s}{4}\right)\right\rangle\right|$$

$$= \frac{1}{4}\left|\left\langle -\cos\left(\frac{s}{4}\right), -\sin\left(\frac{s}{4}\right)\right\rangle\right|$$

↑ $\vec{N}(s)$

$$= \frac{1}{4}$$



- independent of position for the circle
- curvature of any circle is 1/radius

Fun Fact: In $\mathbb{R}^3$ only lines, circles, and helices have constant curvatures

Question: In which direction is **T** changing? $\vec{T}\,'(s)$

← length 1

This is the direction of the **principal unit normal**, $\mathbf{N}(s) =$ $\vec{T}\,'(s)/|\vec{T}\,'(s)|$

↑ pointing in direction of motion

↑ ⊥ to curve

We said last time that it is often hard to find arc length parameterizations, so what do we do if we have a generic parameterization $\mathbf{r}(t)$?

- $\mathbf{T}(t) = \vec{r}\,'(t) / |\vec{r}\,'(t)|$
- $\mathbf{N}(t) = \vec{T}'(t) / |\vec{T}'(t)|$
- $\kappa(t) = |\vec{T}'(t)| / |\vec{r}\,'(t)|$ ← can be unpleasant or $\dfrac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$

Example 21. Find $\mathbf{T}, \mathbf{N}, \kappa$ for the helix $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), t - 1\rangle$.

First $\vec{r}\,'(t) = \langle -2\sin(t), 2\cos(t), 1\rangle$

$$|\vec{r}\,'(t)| = \sqrt{4\sin^2(t) + 4\cos^2(t) + 1} = \sqrt{5}$$

So $\boxed{\vec{T}(t) = \frac{1}{\sqrt{5}}\langle -2\sin(t), 2\cos(t), 1\rangle}$

$$\vec{N}(t) = \vec{T}'(t) / |\vec{T}'(t)|$$

and $\vec{T}'(t) = \frac{1}{\sqrt{5}}\langle -2\cos(t), -2\sin(t), 0\rangle$

$$|\vec{T}'(t)| = \frac{1}{\sqrt{5}} \cdot \sqrt{4\cos^2(t) + 4\sin^2(t) + 0^2} = \frac{2}{\sqrt{5}}$$

so $\vec{N}(t) = \frac{1}{2}\langle -2\cos(t), -2\sin(t), 0\rangle$

$$= \boxed{\langle -\cos(t), -\sin(t), 0\rangle}$$

$$\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}\,'(t)|} = \frac{2/\sqrt{5}}{1/\sqrt{5}} = \boxed{\frac{2}{5}}$$

- Frenet-Serre frame

14.1 Functions of Multiple Variables

Definition 22. A function of two variables is a rule that assigns to each pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$.

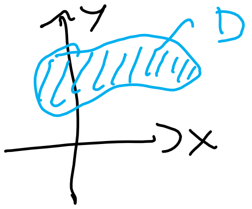
D is contained in $\mathbb{R}^2$

$$f : D \to \mathbb{R}, \text{ where } D \subseteq \mathbb{R}^2$$

name

domain is the set D (all pairs (x,y) where f is well-defined)

output (codomain) is a real number



- Intervals (-4, ∞) are not domains of functions of 2 variables.

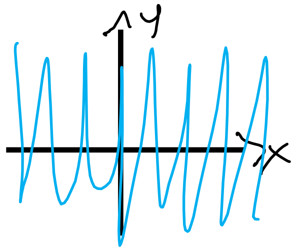
Example 23. Three examples are

$$f(x, y) = x^2 + y^2, \qquad g(x, y) = \ln(x + y), \qquad h(x, y) = \frac{1}{\sqrt{x + y}}.$$

Example 24. Find the domains of f, g, and h.

$f(x,y) = x^2 + y^2$

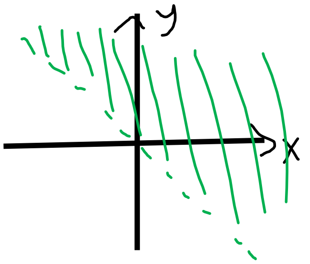
Domain is all of $\mathbb{R}^2 = \{(x,y) \mid x, y \in \mathbb{R}\}$



$g(x,y) = \ln(x+y)$

Domain is all (x,y) such that $x+y > 0$

$\{(x,y) \mid x+y > 0\}$



$x+y > 0$

$y > -x$

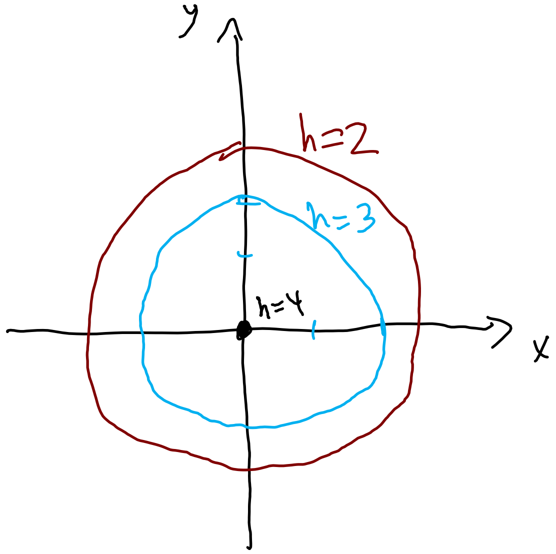
$h(x,y) = \frac{1}{\sqrt{x+y}}$

Domain is all

Definition 25. If f is a function of two variables with domain D, then the graph of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

Here are the graphs of the three functions above.

Example 26. Suppose a small hill has height $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$ m at each point (x, y). How could we draw a picture that represents the hill in 2D?



$4 = h \Rightarrow x = y = 0$

$3 = h(x,y) \Rightarrow 3 = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$

$\frac{1}{4}x^2 + \frac{1}{4}y^2 = 1$

$x^2 + y^2 = 4$

$2 = h(x,y) \Rightarrow x^2 + y^2 = 8$

In 3D, it looks like this.

Definition 27. The ____________ (also called ____________________) of a function f of two variables are the curves with equations ________________, where k is a constant (in the range of f). A plot of ____________ for various values of z is a ____________________(or ____________________________).

Some common examples of these are:

-
-
-

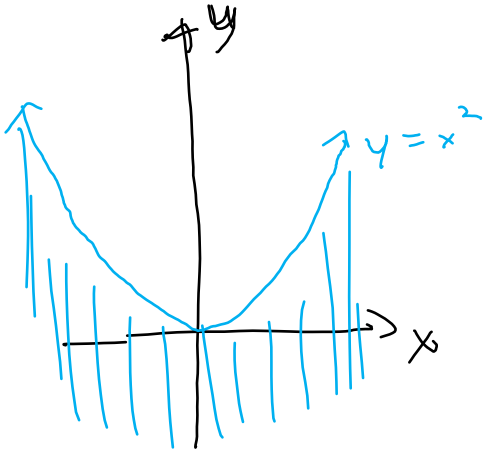
Itempool)

$f(x,y) = \sqrt{x^2 - y}$

Domain: Need $x^2 - y \geq 0$

$-y \geq -x^2$

~~$y \geq x^2$~~ $y \leq x^2$



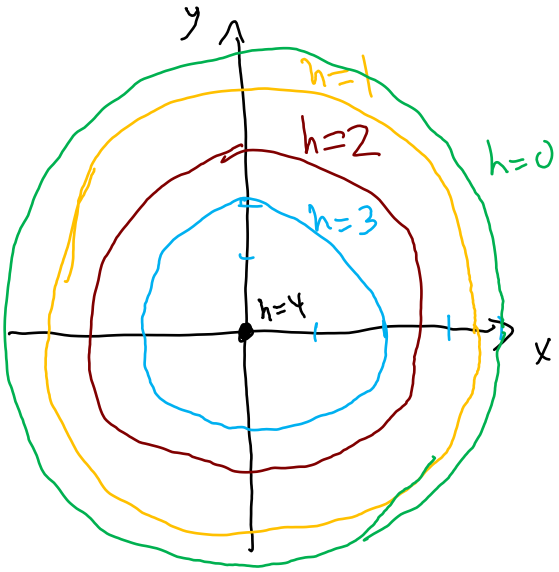
Daily Announcements & Reminders:

- HW 13.2 due tonight
- Exam 1 is on Tuesday 2/6 in lecture
  - more info about format / coverage via Canvas next week
- Do the warmup Itempool

Goals for Today: Section 14.1

- Introduce and sketch traces and contours of functions of two variables
- Find level surfaces of functions of three variables

Example 26. Suppose a small hill has height $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$ m at each point (x, y). How could we draw a picture that represents the hill in 2D?



$4 = h \Rightarrow x = y = 0$

$0 = -\frac{1}{4}x^2 - \frac{1}{4}y^2$

$3 = h(x,y) \Rightarrow 3 = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$

$\frac{1}{4}x^2 + \frac{1}{4}y^2 = 1$

$\boxed{x^2 + y^2 = 4}$

$2 = h(x,y) \Rightarrow x^2 + y^2 = 8$

$1 = h(x,y) = 1 = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$

$x^2 + y^2 = 12$

$0 = h(x,y) \Rightarrow x^2 + y^2 = 16$

In 3D, it looks like this.

Q: Are equally spaced heights giving equally spaced circles?

Definition 27. The contours (also called level curves) of a function f of two variables are the curves with equations $k = f(x, y)$, where k is a constant (in the range of f). A plot of contours for various values of z is a contour map (or level curve map).

Some common examples of these are:

- topographical maps
- temperature (precipitation / Pressure / etc.)
- voltage map?
- strength of electric/magnetic field maps
- phase diagrams

Example 28. Create a contour diagram of $f(x, y) = x^2 - y^2$

$4 = x^2 - y^2$, hyperbola

opening left/right

(no y-intercept! $4 = 0^2 - y^2$)

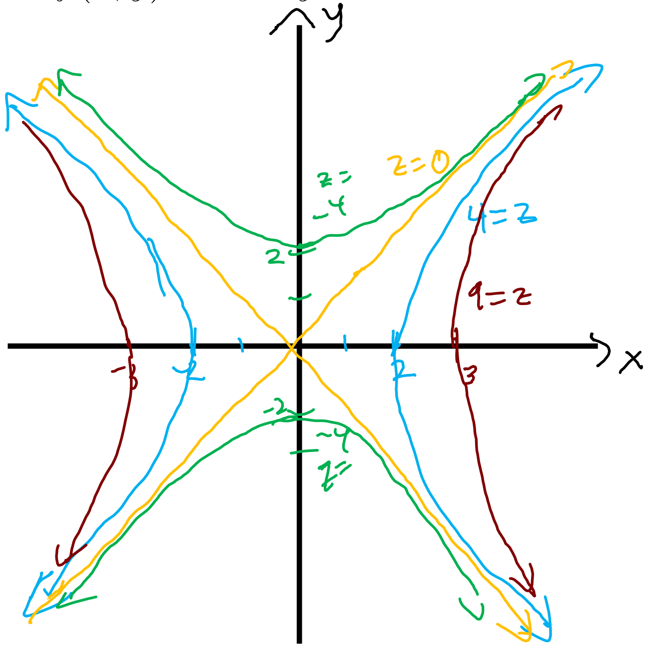
$0 = x^2 - y^2$

$x^2 = y^2$

$x = y$ or $x = -y$

$-4 = x^2 - y^2$

$4 = y^2 - x^2$, hyperbola



$9 = x^2 - y^2$ x-int: $9 = x^2 - 0^2$

Definition 29. The traces of a surface are the curves of intersection of the surface with planes parallel to the xz & yz-planes. i.e. planes of the form $y=k$ or $x=k$

Example 30. Use the traces and contours of $z = f(x, y) = 4 - 2x - y^2$ to sketch the portion of its graph in the first octant.

$x \geq 0$
$y \geq 0$
$z \geq 0$

$z=0$: $0=4-2x-y^2$
$x=2-\frac{1}{2}y^2$

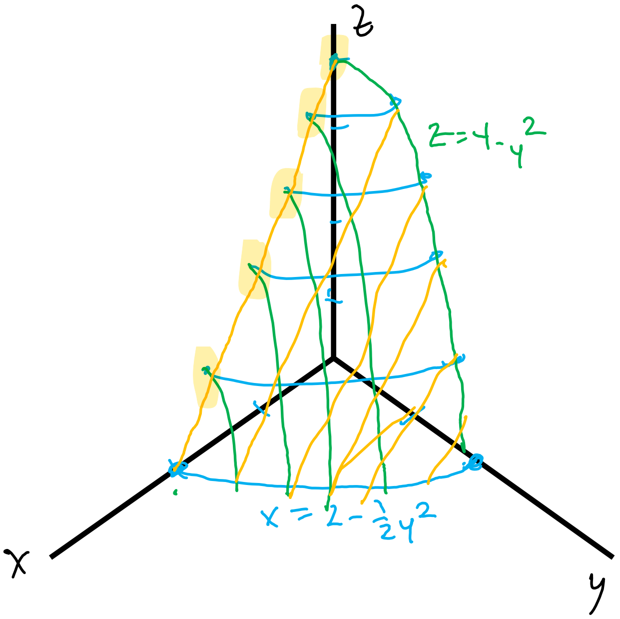
$z=k>0$: $k=4-2x-y^2$
$x=\frac{4-k}{2}-\frac{1}{2}y^2$

$x=0$: $z=4-0-y^2$
$=4-y^2$

$x=k>0$: $z=(4-2k)-y^2$

$y=0$: $z=4-2x-0^2$
$=4-2x$

$y=k>0$: $z=4-2x-k^2$
$=(4-k^2)-2x$



Let's check our work: https://tinyurl.com/math2551-2var-graph

Definition 31. A function of three variables is a rule that assigns to each triple of real numbers (x, y, z) in a set D a unique real number denoted by $f(x, y, z)$.

$$f : D \to \mathbb{R}, \text{ where } D \subseteq \mathbb{R}^3$$

We can still think about the domain and range of these functions. Instead of level curves, we get level surfaces.

Example 32. Describe the domain of the function $f(x, y, z) = \dfrac{1}{4 - x^2 - y^2 - z^2}$.

Domain is all (x,y,z) s.t. $4-x^2-y^2-z^2 \neq 0$

$\iff x^2+y^2+z^2 \neq 4$

i.e. all points in $\mathbb{R}^3$ except

the sphere of radius 2

centered at $(0,0,0)$

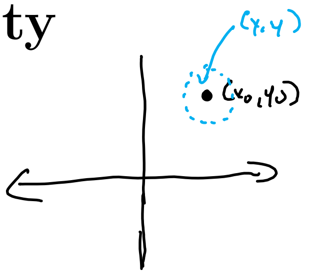
Example 33. Describe the level surfaces of the function $g(x, y, z) = 2x^2 + y^2 + z^2$.

$g = 1 = 2x^2+y^2+z^2$ is an ellipsoid

$g=k = 2x^2+y^2+z^2$ " " "

$k \geq 0$

Section 14.2 Limits & Continuity



$|f(x,y) - L| <$ small

Definition 34. What is a limit of a function of two variables?

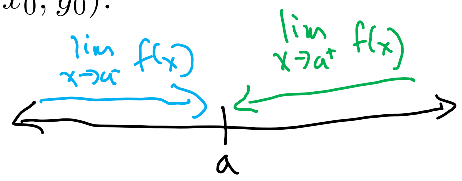
DEFINITION We say that a function $f(x, y)$ approaches the **limit** L as (x, y) approaches (x_0, y_0), and write

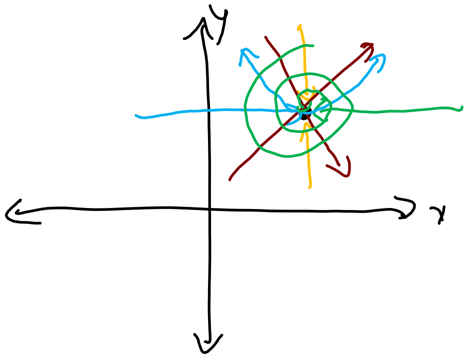
$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,

$$|f(x,y) - L| < \epsilon \quad \text{whenever} \quad 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta.$$

We won't use this definition much: the big idea is that $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ if and only if $f(x,y)$ approaches L regardless of how we approach (x_0, y_0).





Limits of functions of two variables work like limits you are used to in many ways:

THEOREM 1—Properties of Limits of Functions of Two Variables The following rules hold if L, M, and k are real numbers and

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M.$$

1. *Sum Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) + g(x,y)) = L + M$
2. *Difference Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) - g(x,y)) = L - M$
3. *Constant Multiple Rule:* $\lim_{(x,y)\to(x_0,y_0)} kf(x,y) = kL$ (any number k)
4. *Product Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) \cdot g(x,y)) = L \cdot M$
5. *Quotient Rule:* $\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}$, $M \neq 0$
6. *Power Rule:* $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer, and if n is even, we assume that $L > 0$.

Daily Announcements & Reminders:

- HW 13.3, 13.4 due tonight
- Quiz 3 on 13.3, 13.4, 14.1 tomorrow
- Exam 1 next T; Canvas announcement later today
- Do the warm-up Itempool

Goals for Today: Sections 14.2, 14.3

- Evaluate limits of functions of two variables
- Show that a limit does not exist using the two-path test
- Determine the set of points where a function is continuous
- Start to understand how we can measure how a function of two variables is changing

Definition 35. A function $f(x, y)$ is **continuous** at (x_0, y_0) if

1. $\lim_{(x,x)\to(x_0,y_0)} f(x,y)$ exists
2. $f(x_0,y_0)$ exists
3. these values are equal

"no holes"

Key Fact: Adding, subtracting, multiplying, dividing, or composing two continuous functions results in another continuous function.

- $\lim_{(x,y)\to(x_0,y_0)} x = x_0$
- $\lim_{(x,y)\to(x,y_0)} y = y_0$

$\Rightarrow$ $f(x,y) = x+y$ is continuous

$g(x,y) = x^2y + y^3 + x^5$ is cts

$h(x,y) = e^{x^2y}$ is continuous

$(x,y) \mapsto x^2y \mapsto e^{x^2y}$

$t \mapsto e^t$

Example 36. Evaluate $\lim\limits_{(x,y)\to(2,0)} \dfrac{\sqrt{2x-y}-2}{2x-y-4}$, if it exists.

Try to evaluate: $\dfrac{\sqrt{4-0}-2}{4-0-4} = \dfrac{0}{0}$

Try to simplify: $\lim\limits_{(x,y)\to(2,0)} \dfrac{(\sqrt{2x-y}+2)}{(\sqrt{2x-y}+2)} \cdot \dfrac{(\sqrt{2x-y}-2)}{2x-y-4}$

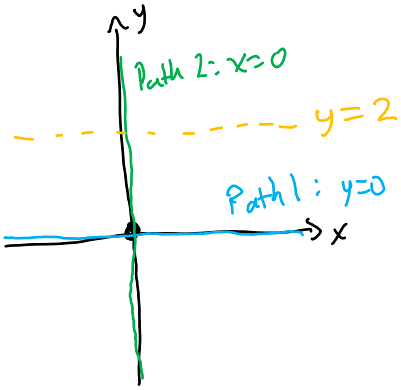
$$= \lim_{(x,y)\to(2,0)} \frac{\cancel{[(2x-y)-4]}^{\,1}}{(\sqrt{2x-y}+2)\cancel{(2x-y-4)}}$$

$$= \frac{1}{\sqrt{4-0}+2} = \boxed{\frac{1}{4}}$$

Sometimes, life is harder in $\mathbb{R}^2$ and limits can fail to exist in ways that are very different from what we've seen before.

Big Idea: Limits can behave differently along different paths of approach

Example 37. Evaluate $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2}{x^2+y^2}$, if it exists. Here is its graph.



$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along}\\ y=0}} \frac{x^2}{x^2+y^2} = \lim_{(x,0)\to(0,0)} \frac{x^2}{x^2+0^2} = \lim_{x\to0} \frac{x^2}{x^2}$$

$$= \lim_{x\to0} 1$$

$$= 1$$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along}\\ x=0}} \frac{x^2}{x^2+y^2} = \lim_{(0,y)\to(0,0)} \frac{0^2}{0^2+y^2}$$

$$= \lim_{y\to0} 0 = 0$$

Since these limits along different paths through (0,0) are different, the overall limit does not exist.

CAUTION: If we find two paths where the limit is the same, the overall limit may or may not exist.

This idea is called the **two-path test:**

If we can find two paths to (x_0, y_0) along which the limit of $f(x,y)$ takes on two different values, then the $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist.

Example 38. Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$$

does not exist.

$y=0$: $\frac{0}{x^4+0}$ $x=0$: $\frac{0}{0+y^2}$

Along $y=mx$:

$$\lim_{(x,mx)\to(0,0)} \frac{x^2(mx)}{x^4+(mx)^2}$$

$$= \lim_{x\to0} \frac{x^3m \;/x^2}{x^4+m^2x^2 \;/x^2}$$

$$= \lim_{x\to0} \frac{xm}{x^2+m^2} = \frac{0}{0+m^2} = 0$$

Along $y=x^2$:

$$\lim_{(x,x^2)\to(0,0)} \frac{x^2(x^2)}{x^4+(x^2)^2} = \lim_{x\to0} \boxed{\frac{x^4}{x^4+x^4}} = \lim_{x\to0} \frac{1}{2} = \frac{1}{2}$$

Because we found different values on two paths through (0,0) the limit does not exist.

Example 39. **[Challenge:]** Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^4y}{x^4+y^2}$$

does exist using the Squeeze Theorem.

Theorem 40 (Squeeze Theorem)**.** *If* $f(x,y) = g(x,y)h(x,y)$, *where* $\lim_{(x,y)\to(a,b)} g(x,y) = 0$ *and* $|h(x,y)| \le C$ *for some constant* C *near* (a,b), *then* $\lim_{(x,y)\to(a,b)} f(x,y) = 0$.

To use: 1) Recognize appropriate choices for g, h.

2) Bound h.

Ex 39: 1)

$$\frac{x^4y}{x^4+y^2} = \underbrace{y}_{g(x,y)} \cdot \underbrace{\frac{x^4}{x^4+y^2}}_{h(x,y)}$$

• $\lim_{(x,y)\to(0,0)} y = 0$ ✓

2) Bound h: $y^2 \ge 0$, so $x^4+y^2 \ge x^4$, so $1 \ge \frac{x^4}{x^4} \ge \frac{x^4}{x^4+y^2} \ge 0$

so the Squeeze Theorem says

$$\lim_{(x,y)\to(0,0)} \frac{x^4y}{x^4+y^2} = 0.$$

In general step 2 uses: $|\cos(t)| \le 1$ or $|\sin(t)| \le 1$

14.3: Partial Derivatives

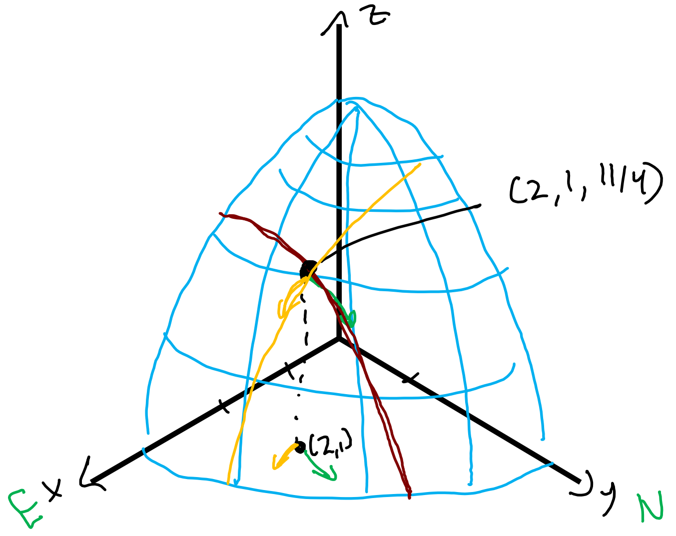
Goal: Describe how a function of two (or three, later) variables is changing at a point (a, b).

Example 41. Let's go back to our example of the small hill that has height

$$h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$$

meters at each point (x, y). If we are standing on the hill at the point with $(2, 1, 11/4)$, and walk due north (the positive y-direction), at what rate will our height change? What if we walk due east (the positive x-direction)?

Let's investigate graphically.



If we walk due north:

we have $(x,y) = (2,y)$:

$h(2,y) = 4 - 1 - \frac{1}{4}y^2 = 3 - \frac{1}{4}y^2$

How fast is this changing as y changes?

A: $\frac{d}{dy}\left(h(2,y)\right)\Big|_{y=1} = \frac{d}{dy}\left(3 - \frac{1}{4}y^2\right)\Big|_{y=1} = -\frac{1}{2}y\Big|_{y=1}$

$= -\frac{1}{2}$ m/m

height ↑ ↑ change in y

- The slope of the hill in the positive y-direction at $(2,1,\frac{11}{4})$ is $-\frac{1}{2}$.

In the negative y-direction: slope is $\frac{1}{2}$

This is the partial derivative of h w.r.t. y.

Notation: $\frac{\partial h}{\partial y}(2,1) = h_y(2,1)$

As x changes:

$\frac{\partial}{\partial x}\left(h(x,y)\right)\Big|_{(2,1)}$

$= \frac{\partial}{\partial x}\left(4 - \frac{1}{4}x^2 - \frac{1}{4}y^2\right)\Big|_{(2,1)}$

$= \frac{\partial}{\partial x}(4) - \frac{\partial}{\partial x}\left(\frac{1}{4}x^2\right) - \frac{\partial}{\partial x}\left(\frac{1}{4}y^2\right)\Big|_{(2,1)}$

$= 0 - \underbrace{\frac{1}{2}x}_{h_x(x,y)} - 0\Big|_{(2,1)} = \boxed{-1 \text{ m/m}}$

Ex] Compute f_x for $f(x,y) = x^2 y$.

$$\frac{\partial}{\partial x}\left(x^2 y\right) = 2xy$$

Example 38. Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$$

does not exist.

$y=0$: $\frac{0}{x^4+0}$ $x=0$: $\frac{0}{0+y^2}$

Along $y=mx$:

$$\lim_{(x,mx)\to(0,0)} \frac{x^2(mx)}{x^4+(mx)^2}$$

$$= \lim_{x\to0} \frac{x^3m \;/x^2}{x^4+m^2x^2 \;/x^2}$$

$$= \lim_{x\to0} \frac{xm}{x^2+m^2} = \frac{0}{0+m^2} = 0$$

Along $y=x^2$:

$$\lim_{(x,x^2)\to(0,0)} \frac{x^2(x^2)}{x^4+(x^2)^2} = \lim_{x\to0} \boxed{\frac{x^4}{x^4+x^4}} = \lim_{x\to0} \frac{1}{2} = \frac{1}{2}$$

Because we found different values on two paths through (0,0) the limit does not exist.

Example 39. **[Challenge:]** Show that the limit

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To use: 1) Recognize appropriate choices for g, h.
2) Bound h.

Ex 39: 1) $\frac{x^4y}{x^4+y^2} = \underbrace{y}_{g(x,y)} \cdot \underbrace{\frac{x^4}{x^4+y^2}}_{h(x,y)}$

- $\lim_{(x,y)\to(0,0)} y = 0$ ✓

2) Bound h: $y^2 \geq 0$, so $x^4+y^2 \geq x^4$, so $1 \geq \frac{x^4}{x^4} \geq \frac{x^4}{x^4+y^2} \geq 0$

so the Squeeze Theorem says

$$\lim_{(x,y)\to(0,0)} \frac{x^4y}{x^4+y^2} = 0.$$

In general step 2 uses: $|\cos(t)| \leq 1$
or $|\sin(t)| \leq 1$

Daily Announcements & Reminders:

- HW 14.1 due tonight, 14.2 on T
- Exam 1 on Tuesday, see Canvas
  - review in studio on Monday
- If you tried to come to office hours today:
  - come after class
  - come at 3-4 tomorrow on Zoom
  - email to find a time

Goals for Today: Section 14.3

- Learn how to compute partial derivatives of functions of multiple variables
- Learn how to compute higher-order partial derivatives
- Understand Clairaut's theorem
- Define the total derivative

Last time: we started to think about rates of changes of functions of two variables in the context of walking on this hill.

Definition 42. If f is a function of two variables x and y, its partial derivatives are the functions f_x and f_y defined by

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} \qquad f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

- Computationally: use derivative rules

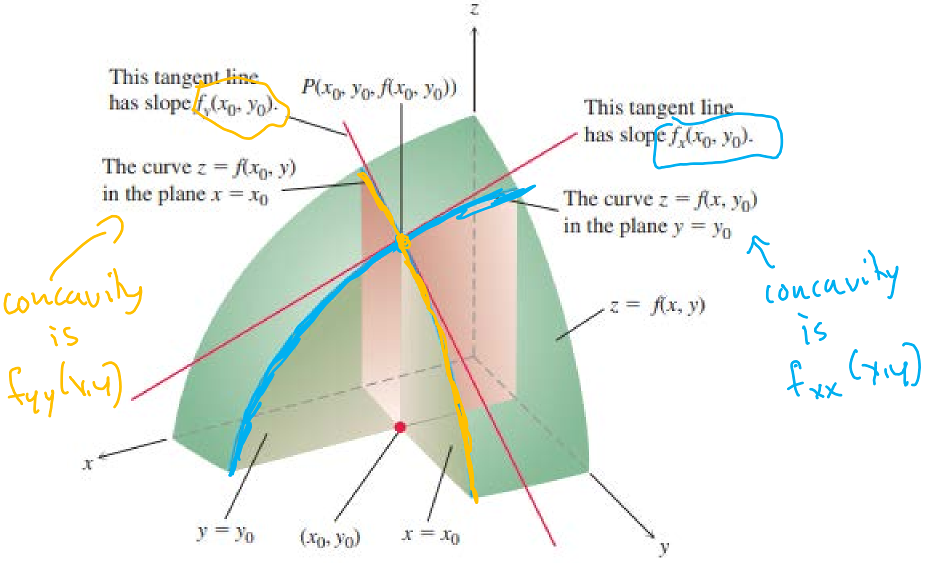
Notations:

$$f_x(x,y) = \frac{\partial}{\partial x}\{f(x,y)\} = \frac{\partial f}{\partial x}(x,y)$$

$$f_y(x,y) = \frac{\partial}{\partial y}(f(x,y)) = \frac{\partial f}{\partial y}(x,y)$$

Interpretations:

$f_{xy} = f_{yx}$ measure "twist"



concavity is $f_{yy}(x,y)$

concavity is $f_{xx}(x,y)$

- Computationally:

  - Treat variable(s) that we are not taking derivative wrt as constants.

Example 43. Find $f_x(1,2)$ and $f_y(1,2)$ of the functions below.

a) $f(x,y) = \sqrt{5x - y}$

$$f_x = \frac{\partial}{\partial x}\left(\sqrt{5x-y}\right)$$

$$= \frac{1}{2}(5x-y)^{-1/2} \cdot \boxed{\overset{5-0}{\frac{\partial}{\partial x}(5x-y)}}$$

$$= \frac{5}{2}(5x-y)^{-1/2}$$

$$f_y = \frac{\partial}{\partial y}\left(\sqrt{5x-y}\right)$$

$$= \frac{1}{2}(5x-y)^{-1/2} \cdot (-1)$$

$$f_x(1,2) = \frac{5}{2\sqrt{3}}$$

$$f_y(1,2) = \frac{-1}{2\sqrt{3}}$$

b) $f(x,y) = \tan(xy)$

$$f_x(x,y) = \sec^2(xy)\cdot y \qquad \boxed{f_x(1,2) = 2\sec^2(2)}$$

$$f_y(x,y) = \sec^2(xy)\cdot x \qquad \boxed{f_y(1,2) = \sec^2(2)}$$

Question: How would you define the second partial derivatives?

Take partial derivatives of partial derivatives

• inside-out

Notation:

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \boxed{\frac{\partial^2 f}{\partial y \partial x}}$$

$$f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$f_{yy} = (f_y)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

Example 44. Find f_{xx}, f_{xy}, f_{yx}, and f_{yy} of the functions below.

mixed partials

pure partials

a) $f(x, y) = \sqrt{5x - y}$

$f_x = \frac{5}{2}(5x-y)^{-1/2}$ $f_y = -\frac{1}{2}(5x-y)^{-1/2}$

$$f_{xx} = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}\left(\frac{5}{2}(5x-y)^{-1/2}\right) = -\frac{5}{4}(5x-y)^{-3/2}\cdot 5$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{5}{2}(5x-y)^{-1/2}\right) = -\frac{5}{4}(5x-y)^{-3/2}\cdot(-1)$$

$$f_{yx} = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}\left(-\frac{1}{2}(5x-y)^{-1/2}\right) = \frac{1}{4}(5x-y)^{-3/2}\cdot 5$$

=

$$f_{yy} = \frac{\partial}{\partial y}\left(-\frac{1}{2}(5x-y)^{-1/2}\right) = \frac{1}{4}(5x-y)^{-3/2}\cdot(-1)$$

b) $f(x, y) = \tan(xy)$

$f_x = y\sec^2(xy)$ $f_y = x\sec^2(xy)$

$$f_{xx} = \frac{\partial}{\partial x}\left(y\sec^2(xy)\right) = y\cdot 2\sec(xy)\cdot\sec(xy)\tan(xy)\cdot y = 2y^2\sec^2(xy)\tan(xy)$$

$$f_{xy} = \frac{\partial}{\partial y}\left(y\sec^2(xy)\right) = 1\cdot\sec^2(xy) + y\cdot 2\sec^2(xy)\tan(xy)\cdot x$$

=

$$f_{yx} = \frac{\partial}{\partial x}\left(x\sec^2(xy)\right) = 1\cdot\sec^2(xy) + x\cdot 2\sec^2(xy)\tan(xy)\cdot y$$

$$f_{yy} = \frac{\partial}{\partial y}\left(x\sec^2(xy)\right) = x\cdot 2\sec^2(xy)\tan(xy)\cdot x$$

What do you notice about f_{xy} and f_{yx} in the previous examples?

Theorem 45 (Clairaut's Theorem). *Suppose f is defined on a disk D that contains the point (a, b). If the functions $f, f_x, f_y, f_{xy}, f_{yx}$ are all continuous on D, then* $f_{xy}=f_{yx}$

- If all 1st, 2nd, & 3rd order partial derivs are cts on a disk around (a,b) then

$$f_{xxy}=f_{xyx}=f_{yxx}$$
$$\neq f_{yyx}$$

Example 46. What about functions of three variables? How many partial derivatives should $f(x, y, z) = 2xyz - z^2y$ have? Compute them.

$\uparrow 3$

$$f_x = \frac{\partial}{\partial x}(2xyz) - \frac{\partial}{\partial x}(z^2y) = 2yz - 0$$

$$f_y = 2xz - z^2$$

$$f_z = 2xy - 2zy$$

$$\boxed{f: \mathbb{R}^2 \to \mathbb{R}^3}$$

Example 47. How many rates of change should the function $f(s, t) = \begin{array}{c} f_1(s,t) \\ f_2(s,t) \\ f_3(s,t) \end{array} \begin{bmatrix} s^2 + t \\ 2s - t \\ st \end{bmatrix} \begin{array}{c} x(s,t) \\ y(s,t) \\ z(s,t) \end{array}$ have? Compute them.

6

$$\frac{\partial f_1}{\partial s} = \frac{\partial x}{\partial s} = 2s \qquad \frac{\partial f_1}{\partial t} = \frac{\partial x}{\partial t} = 1$$

$$\frac{\partial f_2}{\partial s} = \frac{\partial y}{\partial s} = 2 \qquad \frac{\partial f_2}{\partial t} = \frac{\partial y}{\partial t} = -1$$

$$\frac{\partial f_3}{\partial s} = \frac{\partial z}{\partial s} = t \qquad \frac{\partial f_3}{\partial t} = \frac{\partial z}{\partial t} = s$$

In the previous example, we computed $6=2\cdot 3$ partial derivatives. How might we **organize** this information?

For any function $f : \mathbb{R}^n \to \mathbb{R}^m$ having the form $f(x_1, \ldots, x_n) = \begin{bmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_m(x_1, \ldots, x_n) \end{bmatrix}$,

we have n inputs, m output, and $n \cdot m$ partial derivatives, which we can use to form the **total derivative**.

This is a linear map from $\mathbb{R}^n \to \mathbb{R}^m$, denoted $\boxed{Df,}$ and we can represent it with an $m \times n$ matrix, with one column per input and one row per output.

It has the formula $(Df)_{ij} = \frac{\partial}{\partial x_j}(f_i)$

- in row i take derivatives of the i^{th} component of f
- in col j take derivatives wrt the j^{th} variable

Example 48. Find the total derivatives of each function:

a) $f(x) = x^2 + 1$ $\quad f: \mathbb{R} \to \mathbb{R}$ $\quad Df(x) = [\,2x\,]$

b) $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$

$f: \mathbb{R} \to \mathbb{R}^3$

$$D\vec{r}(t) = \begin{bmatrix} -\sin(t) \\ \cos(t) \\ 1 \end{bmatrix} = \vec{r}\,'(t)$$

c) $f(x, y) = \sqrt{5x - y}$

$f: \mathbb{R}^2 \to \mathbb{R}$

$$Df(x,y) = \begin{array}{c} \left[\begin{array}{cc} \frac{5}{2\sqrt{5x-y}} & \frac{-1}{2\sqrt{5x-y}} \end{array}\right] \\ \begin{array}{cc} \quad f_x \qquad & \qquad f_y \end{array} \end{array}$$

d) $f(x, y, z) = 2xyz - z^2 y$

$f: \mathbb{R}^3 \to \mathbb{R}$

$$Df(x,y,z) = \begin{array}{c} \left[\begin{array}{ccc} 2yz & 2xz - z^2 & 2xy - 2zy \end{array}\right] \\ \begin{array}{ccc} \quad f_x \qquad & \qquad f_y \qquad & \qquad f_z \end{array} \end{array}$$

e) $\mathbf{f}(s, t) = \langle \underbrace{s^2 + t}_{}, 2s - t, st \rangle$

$\vec{f}: \mathbb{R}^2 \to \mathbb{R}^3$

$$D\vec{f}(s,t) = \begin{array}{cc} \begin{array}{cc} \frac{\partial}{\partial s} \swarrow & \swarrow \frac{\partial}{\partial t} \end{array} & \\ \begin{bmatrix} 2s & 1 \\ 2 & -1 \\ t & s \end{bmatrix} & \begin{array}{l} \leftarrow \text{deriv of } x(s,t) \\ \leftarrow \text{deriv of } y(s,t) \\ \leftarrow \text{deriv of } z(s,t) \end{array} \end{array}$$

What does it mean? In differential calculus, you learned that one interpretation of the derivative is as a slope. Another interpretation is that the derivative measures how a function transforms a neighborhood around a given point.

Check it out for yourself. (credit to samuel.gagnon.nepton, who was inspired by 3Blue1Brown.)

In particular, the (total) derivative of **any** function $f : \mathbb{R}^n \to \mathbb{R}^m$, evaluated at $\mathbf{a} = (a_1, \ldots, a_n)$, is the linear function that best approximates $f(\mathbf{x}) - f(\mathbf{a})$ at $\mathbf{a}$.

This leads to the familiar linear approximation formula for functions of one variable: $f(x) = f(a) + f'(a)(x - a)$.

Definition 49. The **linearization** or **linear approximation** of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$ at the point $\mathbf{a} = (a_1, \ldots, a_n)$ is

$$L(\mathbf{x}) = f(\vec{a}) + \boxed{Df(\vec{a})}(\vec{x} - \vec{a})$$

$$L(x,y) = f(a,b) + \underbrace{\begin{bmatrix} f_x(a,b) & f_y(a,b) \end{bmatrix}}_{1\times 2} \underbrace{\begin{bmatrix} x-a \\ y-b \end{bmatrix}}_{2\times 1}$$

1×1

Example 50. Find the linearization of the function $f(x, y) = \sqrt{5x - y}$ at the point $(1, 1)$. Use it to approximate $f(1.1, 1.1)$.

$$f(1,1) = \sqrt{5-1} = \sqrt{4} = 2$$

$$Df(1,1) = \left.\begin{bmatrix} \dfrac{5}{2\sqrt{5x-y}} & \dfrac{-1}{2\sqrt{5x-y}} \end{bmatrix}\right|_{(1,1)}$$

$$= \begin{bmatrix} \frac{5}{4} & -\frac{1}{4} \end{bmatrix}$$

$$L(x,y) = 2 + \begin{bmatrix} \frac{5}{4} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} x-1 \\ y-1 \end{bmatrix} = \boxed{2 + \frac{5}{4}(x-1) - \frac{1}{4}(y-1)}$$

$$f(1.1,1.1) \approx L(1.1,1.1) = 2 + \frac{5}{4}(.1) - \frac{1}{4}(0.1) = \boxed{2.1}$$

Question: What do you notice about the equation of the linearization?

This is a plane! Specifically it's the tangent plane to $z = f(x,y)$ at $(1,1)$

Daily Announcements & Reminders:

- HW 14.3 due tonight
- Do warmup itempool
- Exams will be returned on Gradescope next T

Goals for Today: Sections 14.4-14.6

- Learn the Chain Rule for derivatives of functions of multiple variables
- Be able to compute implicit partial derivatives
- Introduce the directional derivative of a function of multiple variables

At $\vec{x}=\vec{a}$ $L(\vec{x}) = f(\vec{a}) + Df(\vec{a})(\vec{x}-\vec{a})$ | $\boxed{f(a,b)} + \begin{bmatrix} f_x(a,b) & f_y(a,b) \end{bmatrix} \begin{bmatrix} x-a \\ y-b \end{bmatrix}$

Last time we introduced the total derivative and linearization. We say $f : \mathbb{R}^n \to \mathbb{R}$ is **differentiable** at **a** if its linearization is a good approximation of f near **a**.

error in approx. f w/ L

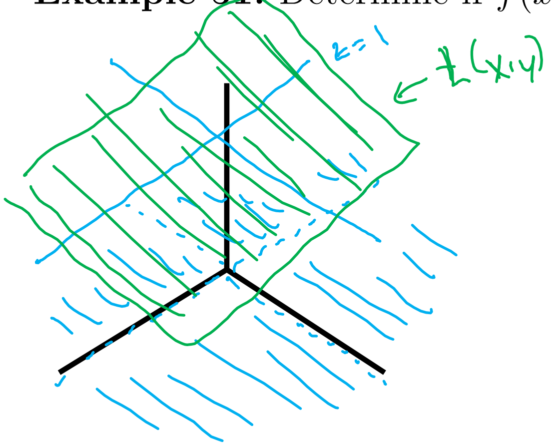
$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - L(x,y)}{|(x,y)-(a,b)|} = 0.$$

Along y=x:

$$\lim_{(x,x)\to(0,0)} \frac{0-1}{|(x,x)-(0,0)|} = \lim_{x\to 0} \frac{-1}{\sqrt{2}|x|} = \infty$$

In particular, if f is a function $f(x,y)$ of two variables, it is differentiable at (a,b) if it has a unique tangent plane at (a,b).

Example 51. Determine if $f(x,y) = \begin{cases} 1 & xy = 0 \\ 0 & xy \neq 0 \end{cases}$ is differentiable at $(0,0)$.



$f_x(0,0) = 0 = f_y(0,0)$

$\Rightarrow$ If f is differentiable: $L(x,y) = 1$

but this is not a good approx around (0,0)

so f is not differentiable even though Df(0,0) exists (the matrix)

The Chain Rule

Example 52. If $f(t) = \ln(t^2)$, then $\frac{df}{dt} = \frac{1}{t^2} \cdot 2t$

$$\frac{d}{dt}\left(f(g(t)\right) = \frac{df}{dt}\left(g(t)\right) \cdot \frac{dg}{dt}(t)$$

Similarly, the **Chain Rule** for functions of multiple variables says that if $f : \mathbb{R}^p \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ are both differentiable functions then

$$\begin{tikzcd}\mathbb{R}^n \arrow[r, "g"] \arrow[rr, bend right, "f\circ g"'] & \mathbb{R}^p \arrow[r, "f"] & \mathbb{R}^m\end{tikzcd}$$

$$D(f(g(\mathbf{x}))) = Df(g(\mathbf{x}))Dg(\mathbf{x}).$$

$$\underset{\text{matrix}}{(m \times p)} \cdot \underset{\text{matrix}}{(p \times n)}$$

Example 53. Suppose we are walking on our hill with height $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$ along the curve $\mathbf{r}(t) = \langle t + 1, 2 - t^2 \rangle$ in the plane. How fast is our height changing at time $t = 1$ if the positions are measured in meters and time is measured in minutes?

$f(t) =$ height at time t , $h(x,y) =$ height at (x,y) , $\vec{r}(t) = (x,y)$ at time t

$f(t) = h(\vec{r}(t))$

$$\begin{tikzcd}\mathbb{R} \arrow[r, "\vec{r}(t)"] \arrow[rr, bend right, "f(t)=h(\vec{r}(t))"'] & \mathbb{R}^2 \arrow[r, "h(t)"] & \mathbb{R}\end{tikzcd}$$

Need total derivatives:

$Dh(x,y) = \left[-\frac{1}{2}x \quad -\frac{1}{2}y\right]$

$D\vec{r}(t) = \begin{bmatrix} 1 \\ -2t \end{bmatrix}$

$h'(t)\Big|_{t=1} = Dh(\vec{r}(1))\, D\vec{r}(t=1)$

Need inner value: $\vec{r}(1) = \langle 2, 1 \rangle$

$\hookrightarrow = \left[-1 \quad -\frac{1}{2}\right] \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$= (-1)(1) + (-\frac{1}{2})(-2) = 0$

Example 54. Suppose that $W(s,t) = F(u(s,t), v(s,t))$, where F, u, v are differentiable functions and we know the following information.

$$u(1,0) = 2 \qquad v(1,0) = 3$$

$$u_s(1,0) = -2 \qquad v_s(1,0) = 5$$

$$u_t(1,0) = 6 \qquad v_t(1,0) = 4$$

$$F_u(2,3) = -1 \qquad F_v(2,3) = 10$$

$$g(s,t) = \begin{bmatrix} u(s,t) \\ v(s,t) \end{bmatrix}$$

$s=1$, $t=0$

Find $W_s(1,0)$ and $W_t(1,0)$.

i.e. find $DW(1,0)$

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "g"] & \mathbb{R}^2 \arrow[r, "F"] & \mathbb{R} \\ (s,t) \arrow[rr, bend right, "W(s,t)"'] & (u,v) & w \end{tikzcd}$$

$$DW(1,0) = \begin{bmatrix} W_s(1,0) & W_t(1,0) \end{bmatrix}$$

$$= DF(g(1,0))\, Dg(1,0)$$

$$= \begin{bmatrix} F_u(g(1,0)) & F_v(g(1,0)) \end{bmatrix} \begin{bmatrix} u_s(1,0) & u_t(1,0) \\ v_s(1,0) & v_t(1,0) \end{bmatrix}$$

$(2,3)$

$$= \begin{bmatrix} -1 & 10 \end{bmatrix} \begin{bmatrix} -2 & 6 \\ 5 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} (-1)(-2) + 10(5) & (-1)(6) + (10)(4) \end{bmatrix}$$

$$= \begin{bmatrix} 52 & 34 \end{bmatrix}$$

Application to Implicit Differentiation: If $F(x, y, z) = c$ is used to *implicitly* define z as a function of x and y, then the chain rule says:

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

Example 55. Compute $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for the sphere $x^2 + y^2 + z^2 = 4$.

$$F(x,y,z) = x^2 + y^2 + z^2$$

$$F_x = 2x \qquad F_y = 2y \qquad F_z = 2z$$

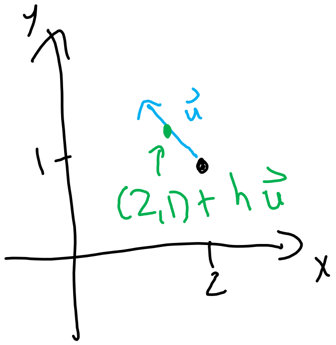
so $$\frac{\partial z}{\partial x}(x,y,z) = -\frac{2x}{2z} = -\frac{x}{z}$$

$$\frac{\partial z}{\partial y}(x,y,z) = -\frac{2y}{2z} = -\frac{y}{z}$$

Example 56. Recall that if $z = f(x, y)$, then f_x represents the rate of change of z in the x-direction and f_y represents the rate of change of z in the y-direction. What about other directions?

Let's go back to our hill example again, $\overset{f}{\cancel{h}}(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$. How could we figure out the rate of change of our height from the point $(2, 1)$ if we move in the direction $\langle -1, 1\rangle$? We know: $Dh(2,1) = \begin{bmatrix} -1 & -\frac{1}{2} \end{bmatrix}$

1) Direction → unit vector : $\vec{u} = \frac{\langle -1,1\rangle}{|\langle -1,1\rangle|} = \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle$



2) $$\lim_{h\to 0} \frac{f(2+h(-\frac{1}{\sqrt{2}}), 1+h(\frac{1}{\sqrt{2}})) - f(2,1)}{h} = \frac{1}{2\sqrt{2}}$$

- If we move in the $\langle -1,1\rangle$ direction, we gain height at an inst. rate of $\frac{1}{2\sqrt{2}}$ m per m travelled in $\langle -1,1\rangle$ dir.

Definition 57. The directional derivative of $f : \mathbb{R}^n \to \mathbb{R}$ at the point $\mathbf{p}$ in the direction of a unit vector $\mathbf{u}$ is

$$D_{\mathbf{u}}f(\mathbf{p}) = \lim_{h\to 0} \frac{f(\vec{p}+h\vec{u}) - f(\vec{p})}{h}$$

if this limit exists.

E.g. for our hill example from the last page we have:

$$D_{\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle} h(2,1) = \frac{1}{2\sqrt{2}}$$

Note that $D_{\mathbf{i}}f = f_x$ $D_{\mathbf{j}}f = f_y$ $D_{\mathbf{k}}f = f_z$

Definition 58. If $f : \mathbb{R}^n \to \mathbb{R}$, then the gradient of f at $\mathbf{p} \in \mathbb{R}^n$ is the vector function ∇f (or grad f) defined by

$$\nabla f(\mathbf{p}) = \langle f_{x_1}(\vec{p}), \ldots, f_{x_n}(\vec{p})\rangle$$
$$= (Df(\vec{p}))^T$$

$$\nabla h(x,y) = \langle -\tfrac{1}{2}x, -\tfrac{1}{2}y\rangle$$

Note: If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at a point $\mathbf{p}$, then f has a directional derivative at $\mathbf{p}$ in the direction of any unit vector $\mathbf{u}$ and

$$D_{\mathbf{u}}f(\mathbf{p}) = Df(\vec{p})\,\vec{u} = \nabla f(\vec{p})\cdot\vec{u}$$

Daily Announcements & Reminders:

- HW 14.4 due tonight
- Quiz 4 tomorrow: partial derivatives, total derivative, Chain Rule (not linear approx.)
- Exam 1 grades released tomorrow
- Midterm progress report:
  - S = 70% or higher on exam
  - U = not ↑

Goals for Today: Sections 14.4-14.6

- Learn to compute the rate of change of a multivariable function in any direction
- Investigate the connection between the gradient vector and level curves/surfaces
- Discuss tangent planes to surfaces, how to find them, and when they exist

Last time: If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at a point $\mathbf{p}$, then f has a directional derivative at $\mathbf{p}$ in the direction of any unit vector $\mathbf{u}$ and

$$D_{\mathbf{u}}f(\mathbf{p}) = Df(\vec{p})\,\vec{u} = \nabla f(\vec{p})\cdot\vec{u}$$

- $D_{\vec{u}}f(\vec{p})$ is the rate of change of f at the point $\vec{p}$ in the direction of the unit vector $\vec{u}$.
- If $\vec{u}$ is tangent to the contour $f(\vec{p})=c$, then $D_{\vec{u}}f(\vec{p})=0$.

Example 59. Find the gradient vector and the directional derivative of each function at the given point **p** in the direction of the given vector **u**.

$$D_{\vec{u}}f(\vec{p}) = Df(\vec{p})\,\vec{u}$$
$$= \nabla f(\vec{p}) \cdot \vec{u}$$

a) $f(x,y,z) = z\ln(x^2+y^2), \mathbf{p} = (-1,1,0), \mathbf{u} = \left\langle \frac{1}{3}, \frac{-2}{3}, \frac{2}{3} \right\rangle$

$$Df = \left[\frac{z \cdot 2x}{x^2+y^2} \quad \frac{2yz}{x^2+y^2} \quad \ln(x^2+y^2) \right]$$

$$\nabla f = \begin{bmatrix} \frac{2xz}{x^2+y^2} \\ \frac{2yz}{x^2+y^2} \\ \ln(x^2+y^2) \end{bmatrix}$$

$$Df(-1,1,0) = [\,0 \quad 0 \quad \ln 2\,]$$

$$\nabla f(-1,1,0) = \langle 0, 0, \ln 2 \rangle$$

$$|\vec{u}| = \sqrt{\frac{1}{9}+\frac{4}{9}+\frac{4}{9}} = \sqrt{9/9} = 1$$

$$D_{\langle \frac{1}{3}, \frac{-2}{3}, \frac{2}{3} \rangle} f(-1,1,0) = [\,0 \quad 0 \quad \ln 2\,] \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix} = \boxed{\frac{2}{3}\ln 2}$$

1) Find $Df(\vec{p})$

2) Check unit vector ✓

3) Multiply

b) $g(x,y,z) = x^2 + 4xy^2 + z^2$, $\mathbf{p} = (1,2,1)$, **u** the unit vector in the direction of $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

1) $\nabla g(x,y,z) = \langle g_x, g_y, g_z \rangle = \langle 2x + 4y^2, 8xy, 2z \rangle$

$\nabla g(1,2,1) = \langle 18, 16, 2 \rangle$

- ∇g is a vector in $\mathbb{R}^n$ ($n=3$ here)

2) unit vector: $\vec{u} = \dfrac{\vec{i}+2\vec{j}-\vec{k}}{|\vec{i}+2\vec{j}-\vec{k}|} = \dfrac{1}{\sqrt{1+4+1}}(\vec{i}+2\vec{j}-\vec{k}) = \dfrac{1}{\sqrt{6}}(\vec{i}+2\vec{j}-\vec{k})$

3) $D_{\vec{u}}g(1,2,1) = \langle 18, 16, 2 \rangle \cdot \frac{1}{\sqrt{6}}\langle 1, 2, -1 \rangle = \frac{1}{\sqrt{6}}(18 + 32 - 2) = \boxed{\frac{48}{\sqrt{6}}}$

c) $h(x,y) = e^{xy} - x^2, \mathbf{p} = (1,1)$, **u** which makes an angle of $\pi/3$ with the positive x-axis.

Post class

1) $\nabla h(x,y) = \langle ye^{xy} - 2x, xe^{xy} \rangle$; $\nabla h(1,1) = \langle e-2, e \rangle$

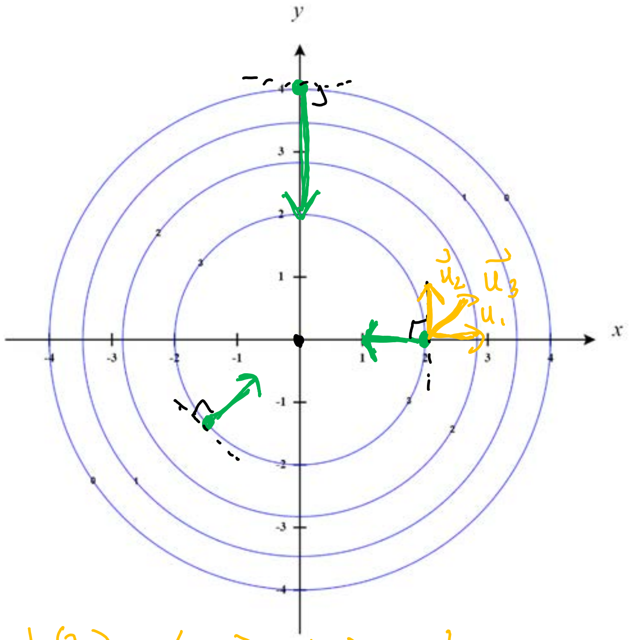
2) unit vector: $\langle \cos(\pi/3), \sin(\pi/3) \rangle = \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$

3) $D_{\vec{u}}h(1,1) = \langle e-2, e \rangle \cdot \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle = \frac{1}{2}e - 1 + \frac{\sqrt{3}e}{2} = \boxed{\frac{1+\sqrt{3}}{2}e - 1}$

Example 60. If $h(x,y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$, the contour map is given below. Find and draw ∇h on the diagram at the points $(2,0), (0,4)$, and $(-\sqrt{2}, -\sqrt{2})$. At the point $(2,0)$, compute $D_{\mathbf{u}}h$ for the vectors $\mathbf{u}_1 = \mathbf{i}, \mathbf{u}_2 = \mathbf{j}, \mathbf{u}_3 = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$.

$\nabla h = \langle h_x, h_y \rangle = \langle -\frac{1}{2}x, -\frac{1}{2}y \rangle$

(a,b)	$\nabla h(a,b)$
$(2,0)$	$\langle -1, 0 \rangle$
$(0,4)$	$\langle 0, -2 \rangle$
$(-\sqrt{2},-\sqrt{2})$	$\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$



$D_{u_1}h(2,0) = \langle -1,0 \rangle \cdot \langle 1,0 \rangle = -1$

$D_{u_2}h(2,0) = \langle -1,0 \rangle \cdot \langle 0,1 \rangle = 0$

$D_{u_3}h(2,0) = \langle -1,0 \rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = -\frac{1}{\sqrt{2}}$

- At $\vec{p}$, $\nabla f(\vec{p})$ is $\perp$ to level set containing $\vec{p}$
- $\nabla f(\vec{p})$ points in direction of greatest increase of f.
- Q: What is the significance of $|\nabla f(\vec{p})|$?

A: $D_{\vec{u}}f(\vec{p}) = \boxed{\nabla f(\vec{p}) \cdot \vec{u}}$

$D_{\vec{u}}f(\vec{p}) = |\nabla f(\vec{p})| \overset{1}{\cancel{|\vec{u}|}} \cos\theta$ ≤ 1

↳ $|\nabla f(\vec{p})|$ = max. rate of change of f at $\vec{p}$

achieved when $\vec{u} = \frac{\nabla f(\vec{p})}{|\nabla f(\vec{p})|}$

Note that the gradient vector is orthogonal to level curves.

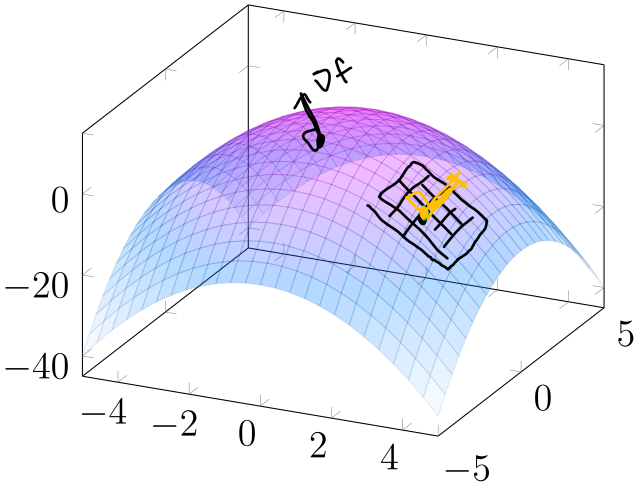
Similarly, for $f(x,y,z)$, $\nabla f(a,b,c)$ is orthogonal to level surfaces

Tangent planes to level surfaces

$x^2+y^2+z^2=4$

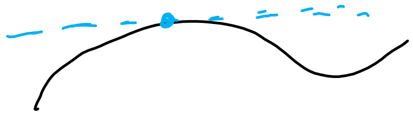
Suppose S is a surface with equation $F(x,y,z)=k$. How can we find an equation of the tangent plane of S at $P(x_0,y_0,z_0)$?

$z=f(x,y) \iff \underbrace{f(x,y)-z}_{F(x,y,z)}=0$



$x^2+y^2+z=10, P=(-1,3,0)$

In calc I:
tangent line



For tangent plane:
- need point: use given point of tangency
- need $\vec{n}$: use $\nabla\vec{F}$ at given point

1) Identify F for which our surface is a level surface

$F(x,y,z)=x^2+y^2+z$ w/ $F(x,y,z)=10$ OR $G(x,y,z)=x^2+y^2+z-10$ w/ $G(x,y,z)=0$

ALL VARIABLES MUST BE ON SAME SIDE

2) point: $(-1,3,0)$

3) normal: $\nabla\vec{F}=\langle 2x,2y,1\rangle$

$\nabla\vec{F}(-1,3,0)=\langle -2,6,1\rangle$

So, our tangent plane is $\boxed{-2(x+1)+6(y-3)+1(z-0)=0}$

Example 61. Find the equation of the tangent plane at the point $(-2, 1, -1)$ to the surface given by

$$z = 4 - x^2 - y$$

$f_x = -2x$

$f_y = -1$

1) Identify $F = c$

$x^2 + y + z = 4$

$F(x,y,z) = x^2 + y + z$

2) Point: $(-2,1,-1)$ $4-(-2)^2-1 = 4-4-1 = -1$ ✓

3) Normal: $\nabla\vec{F} = \langle 2x, 1, 1\rangle$

$\nabla\vec{F}(-2,1,-1) = \langle -4, 1, 1\rangle$

Tangent plane: $\boxed{-4(x+2) + 1(y-1) + 1(z+1) = 0}$

$$z = \underset{\overset{\uparrow}{f(-2,1)}}{-1} + \underset{\overset{\uparrow}{f_x(-2,1)}}{4}(x+2) - \underset{\overset{\uparrow}{f_y(-2,1)}}{1}(y-1) = L(x,y) \text{ at } (-2,1)$$

$x^2+y^2+z^2=4$ is not in the special case ↓

Special case: if we have $z = f(x, y)$ and a point $(a, b, f(a, b))$, the equation of the tangent plane is

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

i.e.

the linearization $L(x,y)$ at (a,b).

This should look familiar: it's ______________________________

Daily Announcements & Reminders:

- HW 14.5 due tonight, 14.6 on Tuesday
- Exam 1 grades are out
  - regrade requests open until 9am W
  - mean 76.5 %, median 80 %
- Do warmup Itempool

Goals for Today: Section 14.7

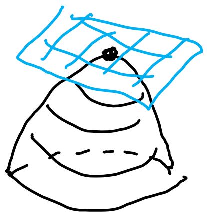
- Define local & global extreme values for functions of two variables
- Learn how to find local extreme values for functions of two variables
- Learn how to classify critical points for functions of two variables
- Learn how to find global extreme values on a closed & bounded domain

Last time: If $f(x, y)$ is a function of two variables, we said $\nabla f(a, b)$ points in the direction of greatest change of f.

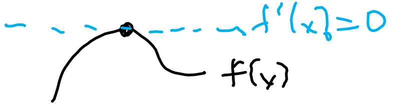
Back to the hill $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$! What should we expect to get if we compute $\nabla h(0, 0)$? Why? What does the tangent plane to $z = h(x, y)$ at $(0, 0, 4)$ look like?

$z = h(x_0, y_0) + \nabla h(x_0, y_0) \langle x - x_0, y - y_0 \rangle$

- $\nabla h(0,0) = \langle -\frac{1}{2}x, -\frac{1}{2}y \rangle \big|_{(0,0)} = \langle 0, 0 \rangle$ b/c $h(0,0) = 4$ is the largest height possible, so there is no direction to increase

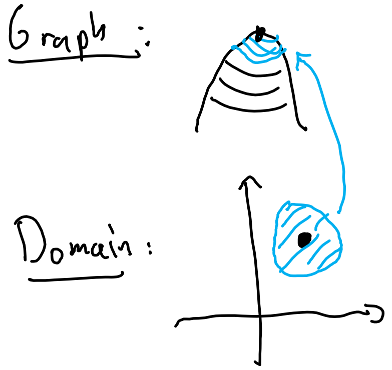


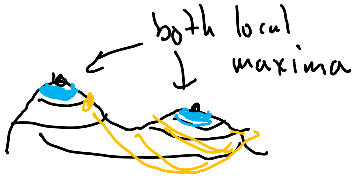
- $z = 4$ is the tangent plane & is horizontal.

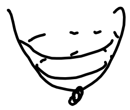


Definition 62. Let $f(x, y)$ be defined on a region containing the point (a, b). We say

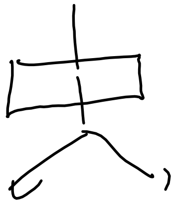
- $f(a, b)$ is a local maximum value of f if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in a disk centered at (a, b)
- $f(a, b)$ is a local minimum value of f if $f(a, b) \leq f(x, y)$ for all domain points (x, y) in a disk centered at (a, b)







Ex: $z=4$



every point is a local max & min.

In $\mathbb{R}^3$, another interesting thing can happen. Let's look at $z = x^2 - y^2$ (a hyperbolic paraboloid!) near $(0, 0)$.

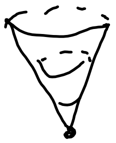
This is called a saddle point

Notice that in all of these examples, we have a horizontal tangent plane at the point in question, i.e.

$$Df(a,b) = [0 \;\; 0]$$

OR

$$Df(a,b) = \langle 0, 0 \rangle$$



Df does not exist.

Definition 63. If $f(x, y)$ is a function of two variables, a point (a, b) in the domain of f with $Df(a, b) =$ $[0 \;\; 0]$ or where $Df(a, b)$ fails to exist is called a Critical points of f.

Example 64. Find the critical points of the function $f(x, y) = x^3 + y^3 - 3xy$.

1) Find Df and set it equal to $[0 \;\; 0]$

$$Df(x,y) = \begin{bmatrix} 3x^2 - 3y & 3y^2 - 3x \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

- For this f, Df exists everywhere

2) Solve resulting system

$$\begin{cases} 3x^2 - 3y = 0 \\ 3y^2 - 3x = 0 \end{cases}$$

Divide by 3 $\Rightarrow$

$$\begin{cases} x^2 - y = 0 & ① \\ y^2 - x = 0 & ② \end{cases}$$

Eqn ① gives $\underline{y = x^2}$

Plug this into eqn ②.

$$\Rightarrow \quad (x^2)^2 - x = 0 \quad \Rightarrow \quad x^4 - x = 0$$

$$x(x^3 - 1) = 0$$

$x = 0$

Plug into $y = x^2$

$y = 0$

$x^3 - 1 = 0$

$x^3 = 1$

$x = 1$

Plug into $y = x^2$

$y = 1$

The critical points of $f(x,y)$ are $(0,0)$ & $(1,1)$.

To classify critical points, we turn to the **second derivative test** and the **Hessian matrix**.

The **Hessian matrix** of $f(x, y)$ at (a, b) is

$$Hf(a,b) = \begin{bmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{bmatrix}$$

Theorem 65 (2nd derivative test)**.** *Suppose (a, b) is a critical point of $f(x, y)$ and $Hf(a, b)$ exists. Then we have:*

$\swarrow$ or $f_{yy}(a,b)$

- *If* $\det(Hf(a, b)) > 0$ *and* $f_{xx}(a, b) > 0$, $f(a, b)$ *is a local minimum*
- *If* $\det(Hf(a, b)) > 0$ *and* $f_{xx}(a, b) < 0$, $f(a, b)$ *is a local maximum*
- *If* $\det(Hf(a, b)) < 0$, f *has a saddle point at* (a, b)
- *If* $\det(Hf(a, b)) = 0$, *the test is inconclusive.*

$$\det(Hf(a,b)) = f_{xx}(a,b) f_{yy}(a,b) - f_{yx}(a,b) f_{xy}(a,b)$$

$\det(Hf(a,b)) > 0$ means that f behaves the same in all directions

$\updownarrow$

is changing in the same way

$\det(Hf(a,b)) < 0$ means that there are two directions where f is changing in different ways

Example 66. Classify the critical points of $f(x, y) = x^3 + y^3 - 3xy$ from Example 64.

$Df = [\,3x^2 - 3y \quad 3y^2 - 3x\,]$ crit pts: $(0,0)$ & $(1,1)$

1) Find Hf: $\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6x & -3 \\ -3 & 6y \end{bmatrix}$

At $(0,0)$: $Hf(0,0) = \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix}$

$\det Hf(0,0) = 0 - (-3)(-3) = -9 < 0$

By the 2nd deriv. test, f has a saddle point at $(0,0)$.

At $(1,1)$: $Hf(1,1) = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}$

$\det Hf(1,1) = (6)(6) - (-3)(-3) = 27 > 0$

and $f_{xx}, f_{yy} > 0$, so

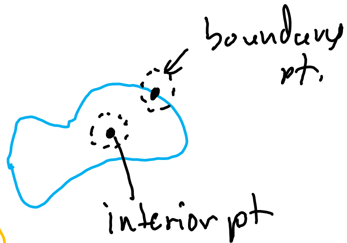
By the 2nd deriv. test, f has a local min at $(1,1)$

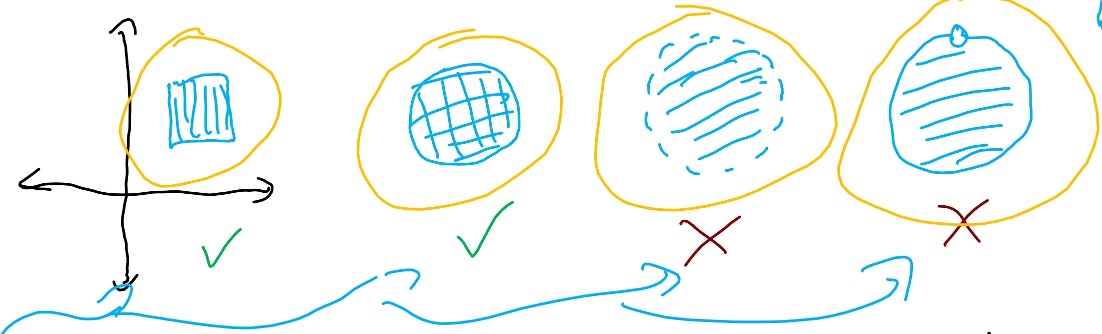
Two Local Maxima, No Local Minimum: The function $g(x, y) = -(x^2 - 1)^2 - (x^2y - x - 1)^2 + 2$ has two critical points, at $(-1, 0)$ and $(1, 2)$. Both are local maxima, and the function never has a local minimum!

A global maximum of $f(x, y)$ is like a local maximum, except we must have $f(a, b) \geq f(x, y)$ for **all** (x, y) in the domain of f. A global minimum is defined similarly.

Theorem 67. *On a closed & bounded domain, any continuous function $f(x, y)$ attains a global minimum & maximum.*

Closed: The set contains all of its boundary points.



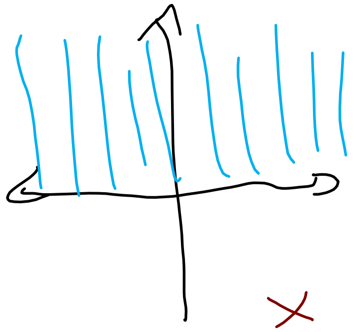


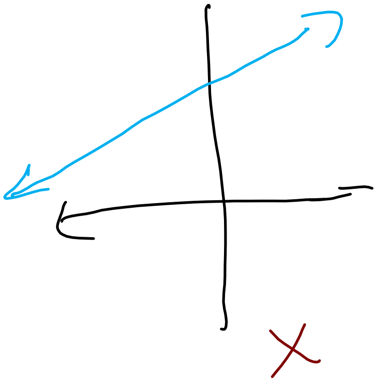
All of $\mathbb{R}^2$ ✓

Not bounded

Bounded: The set fits in a large enough circle.

All of these





Daily Announcements & Reminders:

- HW 14.6 due tonight
- Quiz 5 in studio tomorrow
  - linearization, gradients/directional derivs, local min/max
- Do warmup Itempool

Goals for Today: Sections 14.7, 14.8

- Find global extreme values of continuous functions of two variables on closed & bounded domains
- Apply the method of Lagrange multipliers to find extreme values of functions of two or more variables subject to one or more constraints

Last time: A set is:

closed if it contains all its boundary points

bounded if it fits in a large enough disk

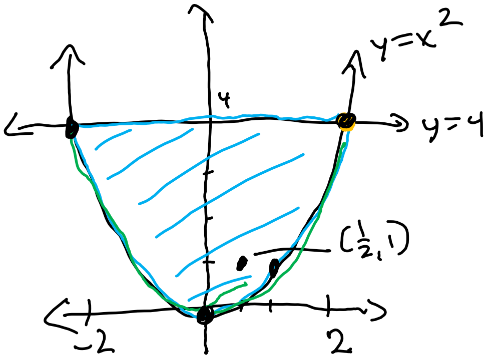
Strategy for finding global min/max of $f(x,y)$ on a closed & bounded domain R

0. Draw a picture

• Analogous to Extreme Value Theorem

1. Find all critical points of f inside R.
2. Find all critical points of f on the boundary of R
3. Evaluate f at each critical point as well as at any endpoints on the boundary.
4. The smallest value found is the global minimum; the largest value found is the global maximum.

Example 69. Find the global minimum and maximum of $f(x,y) = 4x^2 - 4xy + 2y$ on the closed region R bounded by $y = x^2$ and $y = 4$.



1. Find crit pts inside R

$$Df = [0 \quad 0]$$

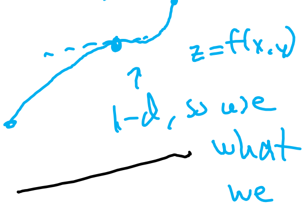
$$[8x - 4y \quad -4x + 2] = [0 \quad 0]$$

$$\begin{cases} 8x - 4y = 0 & ① \\ -4x + 2 = 0 & ② \end{cases} \longrightarrow 4x = 2 \rightarrow x = \frac{1}{2}$$

Plug into ①:

$4 - 4y = 0 \rightarrow y = 1$

2. Find crit pts on boundary of R



1-d, so use what we know

a) On $y=4$: $(x,4)$ for some $x \in [-2, 2]$

so $f(x,y)$ becomes -

$$f(x,4) = \underline{4x^2 - 4x(4) + 2(4)}$$

$$g(x) = 4x^2 - 16x + 8$$

• look for x-values where $g'(x) = 0$ & endpoints of domain

$$g'(x) = 8x - 16 = 0$$

$$8x = 16$$

$$x = 2$$

Add $(2,4)$ & endpoints $(-2,4)$ & $(2,4)$

3) Evaluate

Test pts	f
$(\frac{1}{2},1)$	1
$(2,4)$	$\boxed{-8}$
$(-2,4)$	$\boxed{56}$
$(0,0)$	0
$(1,1)$	2

4) Conclude

- Global min of f on R is -8 attained at $(2,4)$
- Global max of f on R is 56 attained at $(-2,4)$

Most useful $\longrightarrow$
for
curves like circles/ellipses

b) On $y=x^2$: Parameterize the boundary curve & plug it into f

$\vec{r}(t) = \langle t, t^2 \rangle \quad -2 \le t \le 2$

$f(\vec{r}(t)) = 4(t)^2 - 4(t)(t^2) + 2(t^2)$

$h(t) = 6t^2 - 4t^3$

$h'(t) = 12t - 12t^2 = 0$

$12t(1-t) = 0$

$t=0 \quad \text{or} \quad t=1$

$(0,0) \quad \text{or} \quad (1,1)$

$-2 \le t \le 2$

& endpoints

$(-2,4)$ & $(2,4)$

Constrained Optimization

Goal: Maximize or minimize $f(x, y)$ or $f(x, y, z)$ subject to a *constraint*, $g(x, y) = c$.

Example 70. A new hiking trail has been constructed on the hill with height $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$, above the points $\boxed{y = -0.5x^2 + 3}$ in the xy-plane. What is the highest point on the hill on this path?

Objective function: • the thing we are optimizing

$$h(x,y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$$

Constraint equation:

$$\underbrace{y + \frac{1}{2}x^2}_{g(x,y)} = 3$$

• Know ∇g is orthogonal to this curve

Goal: Find (x,y) where $\nabla h = \overset{\lambda \in \mathbb{R}}{\lambda} \nabla g$ & $g(x,y) = 3$

$$\downarrow$$

$$\langle -\tfrac{1}{2}x, -\tfrac{1}{2}y \rangle = \lambda \langle x, 1 \rangle$$

$$\begin{cases} -\frac{1}{2}x = \lambda x & ① \\ -\frac{1}{2}y = \lambda & ② \\ y + \frac{1}{2}x^2 = 3 & ③ \end{cases}$$

→ start w/ ①: $x(\lambda + \frac{1}{2}) = 0$

$x = 0$ OR $\lambda = -\frac{1}{2}$

$x = 0$ ↓ Plug into ③

$y + 0 = 3$

$(0,3)$ is a possible location of the max

$\lambda = -\frac{1}{2}$ ↓ Plug into ②

$-\frac{1}{2}y = -\frac{1}{2}$

$y = 1$

↓ Plug into ③

$1 + \frac{1}{2}x^2 = 3$

$\frac{1}{2}x^2 = 2$

$x^2 = 4$

$x = \pm 2$

(x,y)	h
$(0,3)$	$7/4$ ←
$(2,1)$	$11/4$
$(-2,1)$	$11/4$

So, the max height is 11/4 meters.

Method of Lagrange Multipliers: To find the maximum and minimum values attained by a function $f(x, y, z)$ subject to a constraint $g(x, y, z) = c$, find all points where $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = c$ and compute the value of f at these points.

If we have more than one constraint $g(x, y, z) = c_1, h(x, y, z) = c_2$, then find all points where $\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$ and $g(x, y, z) = c_1, h(x, y, z) = c_2$.

$\nabla f \in \text{Span}\{\nabla g, \nabla h\}$

Example 71. Find the points on the surface $z^2 = xy + 4$ that are closest to the origin.

Objective: $d(x,y,z) = \sqrt{x^2+y^2+z^2}$ | $f(x,y,z) = x^2+y^2+z^2$

Constraint: $g(x,y,z) = z^2 - xy = 4$

Lagrange multiplier system: $\nabla f = \lambda \nabla g$ & $g = 4$: $\langle 2x, 2y, 2z\rangle = \lambda\langle -y, -x, 2z\rangle$

$z^2 - xy = 4$

After class: we have

$$\begin{cases} 2x = -\lambda y & ① \\ 2y = -\lambda x & ② \\ 2z = 2\lambda z & ③ \\ z^2 - xy = 4 & ④ \end{cases}$$

From ③, we have $z - \lambda z = 0 \Rightarrow z(1-\lambda) = 0$

either $z = 0$ or $\lambda = 1$

Case 1: $z = 0$

Then ④ becomes

$-xy = 4 \Rightarrow y = \frac{4}{-x}$

Plugging this into ① & ② gives

$$\begin{cases} 2x = \frac{4\lambda}{x} \\ -\frac{8}{x} = -\lambda x \end{cases}$$

$$\Rightarrow \begin{cases} x^2 = 2\lambda \\ 8 = \lambda x^2 \end{cases} \Rightarrow \lambda = \frac{8}{x^2}$$

so

$x^4 = 16$, i.e. $x = \pm 2$

Thus $(2, -2, 0)$ & $(-2, 2, 0)$ are possibilities

Case 2: $\lambda = 1$

Plugging this into ① & ② gives

$$\begin{cases} 2x = -y \\ 2y = -x \quad \circledast \\ z^2 - xy = 4 \end{cases}$$

if $y = -2x$, then we have via $\circledast$

$-4x = -x$, so $x = 0$ and $y = 0$.

Then ④ becomes

$z^2 = 4$, so $z = \pm 2$.

So $(0,0,2)$ & $(0,0,-2)$ are possibilities

Conclusion

(x,y,z)	dist.
$(\pm 2, \mp 2, 0)$	$\sqrt{4+4+0} = \sqrt{8}$
$(0,0,\pm 2)$	$\sqrt{0+0+4} = 2$

So the points on the surface closest to the origin are $(0,0,\pm 2)$.

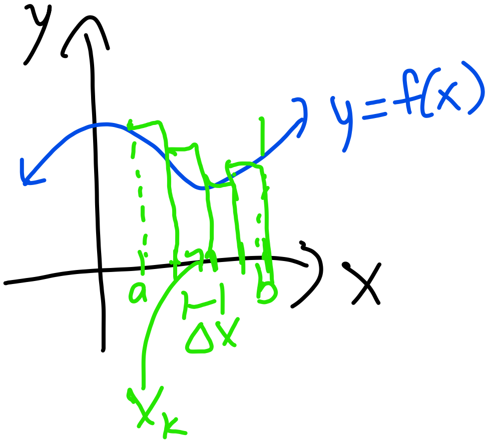
Daily Announcements & Reminders:

- HW 14.7, 14.8 due tonight
- Exam 2 in two weeks

Goals for Today: Sections 15.1, 15.2

- Introduce double and iterated integrals for functions of two variables on rectangles
- Use Fubini's Theorem to change the order of integration of a iterated integral
- Be able to set up & evaluate a double integral over a general region
- Change the order of integration for general regions

Recall: Riemann sum and the definite integral from single-variable calculus.



$$\text{area} \approx \sum_{k=1}^{n} \underset{\overset{\uparrow}{\text{height}}}{f(x_k)} \; \underset{\overset{\uparrow}{\text{width}}}{\Delta x}$$

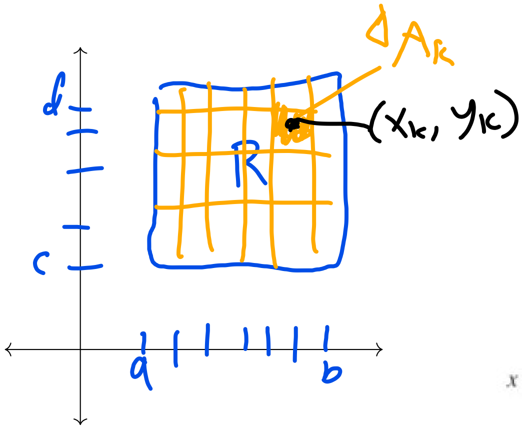
$$\text{area} = \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$$

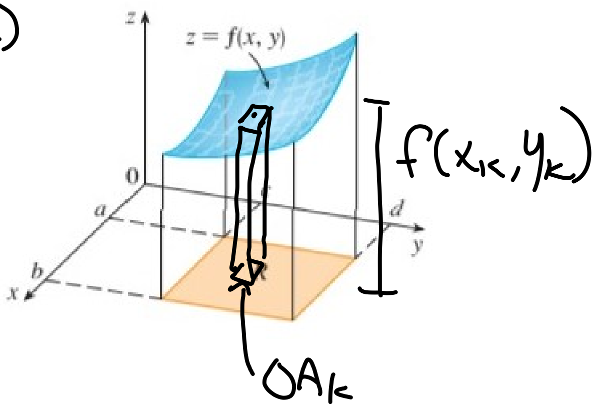
Double Integrals

Volumes and Double integrals Let R be the closed rectangle defined below:

$$R = [a,b] \times [c,d] = \{(x,y) \in \mathbb{R}^2 | a \leq x \leq b, c \leq y \leq d\}$$

Let $f(x,y)$ be a function defined on R such that $f(x,y) \geq 0$. Let S be the solid that lies above R and under the graph f.





Question: How can we estimate the volume of S?

$$\text{Volume}(S) \approx \sum_{k=1}^{n} \underset{\uparrow \ \text{height}}{f(x_k, y_k)} \ \underset{\uparrow \ \text{area of base}}{\Delta A_k}$$

Definition 72. The double integral of $f(x, y)$ over a rectangle R is

$$\iint_R f(x,y)\ dA = \lim_{|P|\to 0} \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k$$

↑ region of integration

↑ $|P|$ is the biggest size of all rectangles in your subdivision

if this limit exists.

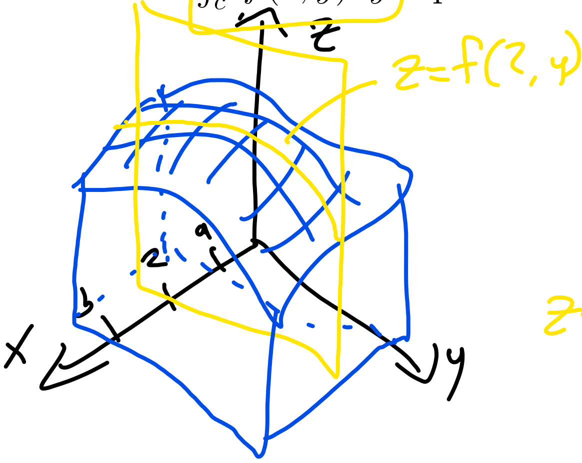
- $\iint_R f(x,y)\,dA$ = signed volume between $z=f(x,y)$ & xy-plane above R
- If f is cts on R, the limit exists
- Some discontinuous functions are integrable

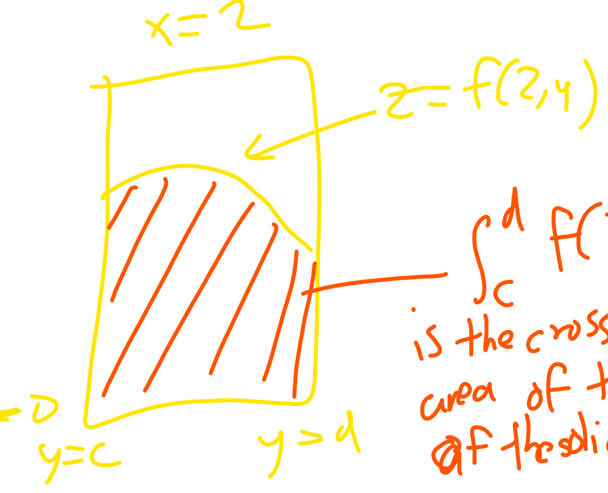
Question: How can we compute a double integral?

Answer: Iterated Integrals

Suppose that f is a function of two variables that is integrable on the rectangle $R = [a, b] \times [c, d]$.

What does $\int_c^d f(2, y)dy$ represent?





$\int_c^d f(2,y)\,dy$ is the cross-sectional area of the slice of the solid at x=2

What about $\int_c^d f(x,y)dy$? (for $a \le x \le b$)

This is the area of the cross-section of the solid at the value x

Let $A(x) = \int_c^d f(x,y)dy$. Then,

$$\text{Volume of solid} = \int_a^b A(x)dx = \int_a^b \left(\underbrace{\int_c^d f(x,y)\,dy}_{\text{treat } x \text{ as constant}} \right) dx$$

This is called an iterated integral.

$$\overset{?}{=} \int_3^4 \int_1^2 6x^2y\,dx\,dy$$

$$\neq \int_1^2 \int_3^4 6x^2y\,dx\,dy$$

Example 73. Evaluate $\int_1^2 \int_3^4 6x^2y\,dy\,dx$.

$$= \int_1^2 \left(3x^2y^2 \Big|_{y=3}^{y=4} \right) dx$$

$$= \int_1^2 3x^2\cdot 16 - 3x^2\cdot 9\,dx$$

$$= \int_1^2 21x^2\,dx = 7x^3\Big|_1^2 = 56-7$$

$$= \boxed{49}$$

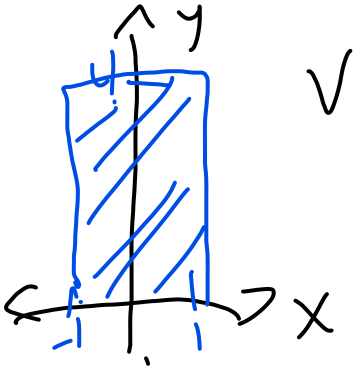
- Volume of the solid under $z = 6x^2y$ above $[1,2]\times[3,4]$ is 49 units^3.

Theorem 74 (Fubini's Theorem). *If f is continuous on the rectangle $R = [a,b] \times [c,d]$, then*

$$\int_a^b \left(\int_c^d f(x,y)\,dy \right) dx = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy = \iint_R f(x,y)\,dA$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Example 75. Compute $\iint_R xe^{e^{e^y}}\,dA$, where R is the rectangle $[-1,1]\times[0,4]$.



$$V = \int_{-1}^{1}\int_0^4 x e^{e^{e^y}}\,dy\,dx \quad \text{HARD}$$

$$V = \int_0^4 \int_{-1}^{1} x e^{e^{e^y}}\,dx\,dy$$

$$= \int_0^4 \frac{1}{2}x^2 e^{e^{e^y}}\Big|_{x=-1}^{x=1} dy$$

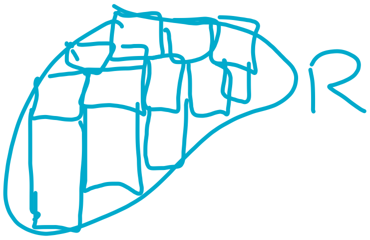
$$= \int_0^4 \frac{1}{2}e^{e^{e^y}} - \frac{1}{2}e^{e^{e^y}}\,dy = \int_0^4 0\,dy$$

$$= 0$$

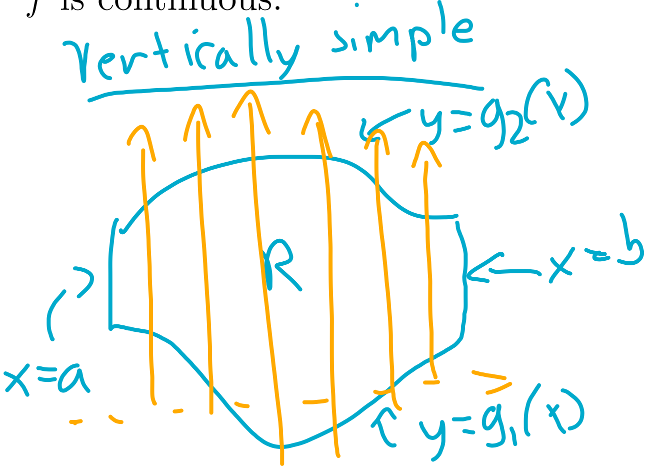
- region is symmetric around x

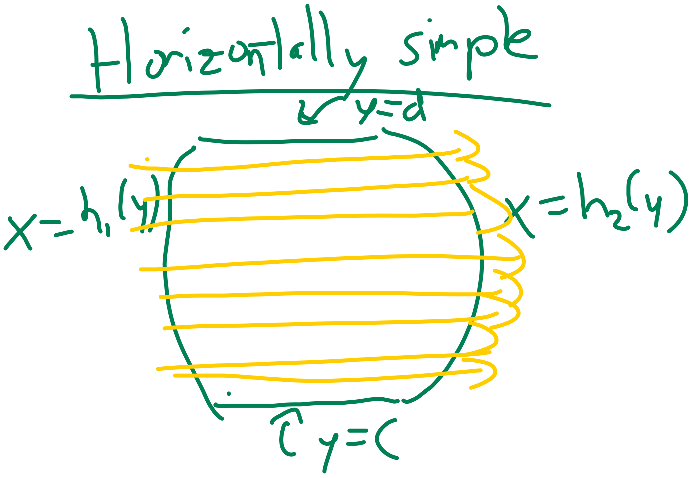
& $f(x,y) = -f(-x,y)$
(odd fn w.r.t x)

Question: What if the region R we wish to integrate over is not a rectangle?



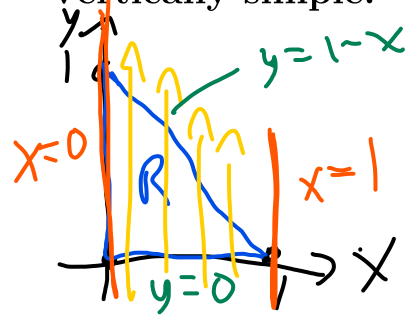
Answer: Repeat same procedure - it will work if the boundary of R is smooth and f is continuous.

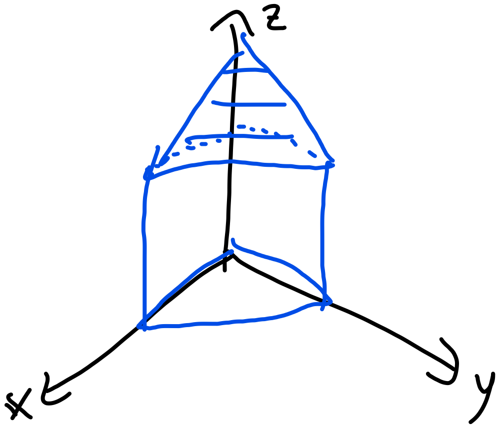




Example 76. Compute the volume of the solid whose base is the triangle with vertices $(0,0),(0,1),(1,0)$ in the xy-plane and whose top is $z = 2 - x - y$.

Vertically simple:





$$V = \iint_R 2-x-y \, dA$$

$$= \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} 2-x-y \, dy \, dx$$

$$= \int_0^1 (2-x)y - \frac{y^2}{2} \Big|_{y=0}^{y=1-x} dx$$

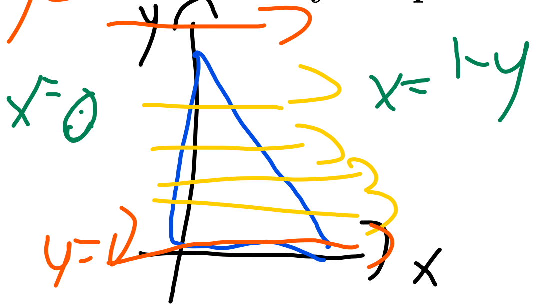
$$= \int_0^1 (2-x)(1-x) - \frac{(1-x)^2}{2} \, dx$$

$$= \int_0^1 \frac{3}{2} - 2x + \frac{1}{2}x^2 \, dx$$

$$= \frac{3}{2}x - x^2 + \frac{1}{6}x^3 \Big|_0^1 = \frac{3}{2} - 1 + \frac{1}{6}$$

$$= \boxed{\frac{2}{3}}$$

Horizontally simple:



$$V = \int_{y=0}^{y=1} \int_{x=0}^{x=1-y} 2-x-y \, dx \, dy$$

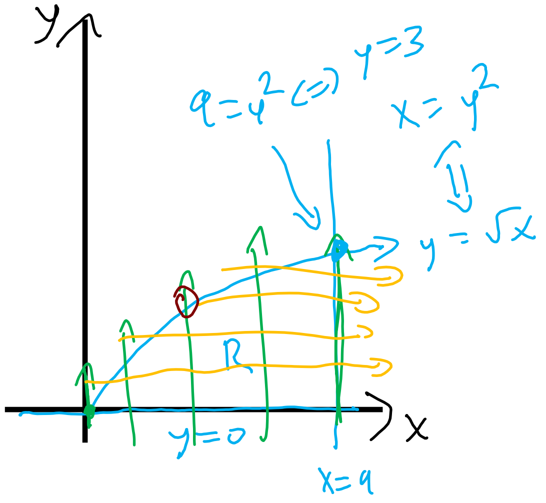
Daily Announcements & Reminders:

- HW 15.1 due tonight
- Quiz 6 on 14.8, 15.1, 15.2 tomorrow
- Exam 2 on 3/7 next week: 14.3-14.8, 15.1-15.4
- Do warmup

Goals for Today: Sections 15.2, 15.3

- Be able to set up & evaluate a double integral over a general region
- Change the order of integration for general regions
- Compute areas of general regions in the plane
- Compute the average value of a function of two variables

Example 77. Write the two iterated integrals for $\iint_R 1\ dA$ for the region R which is bounded by $y = \sqrt{x}, y = 0$, and $x = 9$.



Vertically simple? ✓ ← functions of x only

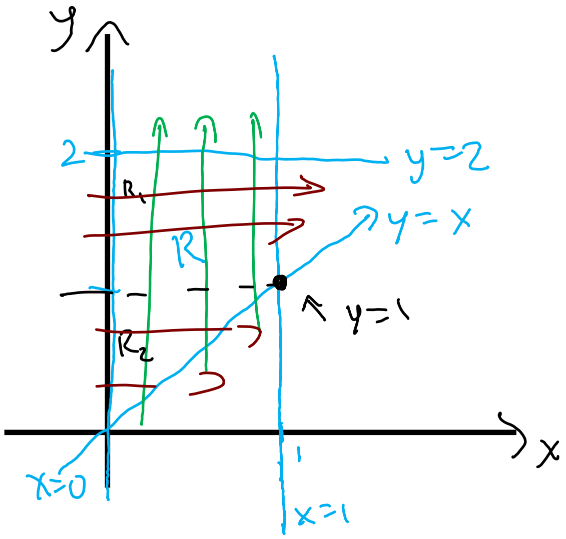
$$\iint_R 1\,dA = \int_{x=0}^{x=9}\int_{y=0}^{y=\sqrt{x}} 1\ dy\ dx$$

↑ AWAYS CONSTANTS!

Horizontally simple? ✓

$$\iint_R 1\,dA = \int_{y=0}^{y=3}\int_{x=y^2\quad x=0}^{x=9} 1\ dx\ dy$$

Example 78. Set up an iterated integral to evaluate the double integral $\iint_R 6x^2y\ dA$, where R is the region bounded by $x = 0, x = 1, y = 2$, and $y = x$.



H.S? ✗ V.S? ✓

Easier order dy dx:

$$\int_0^1\int_x^2 6x^2y\ dy\ dx$$

What about dx dy? Split up region.

- If $R = R_1 \cup R_2$, $\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$

$$\overset{R_1}{\int_1^2\int_0^1 6x^2y\ dx\ dy} + \overset{R_2}{\int_0^1\int_0^y 6x^2y\ dx\ dy}$$

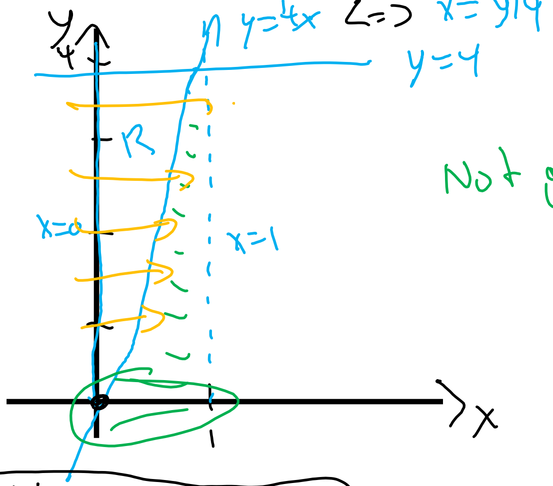
Example 79. Sketch the region of integration for the integral

$$\int_0^1 \int_{4x}^4 f(x,y)\;dy\;dx.$$

Then write an equivalent iterated integral in the order $dx\;dy$.

$4x \leq y \leq 4$

$0 \leq x \leq 1$



Not green! Unbounded below.

$$\boxed{\int_0^4 \int_0^{y/4} f(x,y)\;dx\;dy}$$

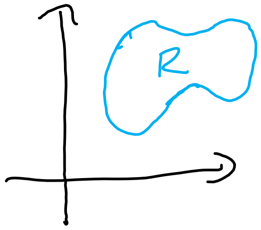
- Setup of bounds is independent of function to integrate.

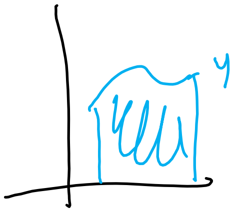
Area & Average Value

Two other applications of double integrals are computing the area of a region in the plane and finding the average value of a function over some domain.

Area: If R is a region bounded by smooth curves, then

$$\text{Area}(R) = \iint_R 1 \cdot dA$$





$y=f(x)$, $A = \int_a^b f(x)\,dx$

$\iint_R 1\,dA =$ volume of a solid cylinder of height 1 with base R

Example 80. Find the area of the region R bounded by $y = \sqrt{x}$, $y = 0$, and $x = 9$.

$$\text{area} = \iint_R 1\,dA = \int_0^9 \boxed{\int_0^{\sqrt{x}} dy}\,dx = \int_0^9 y\Big|_{y=0}^{y=\sqrt{x}} dx$$

$$= \boxed{\int_0^9 \sqrt{x}\,dx}$$

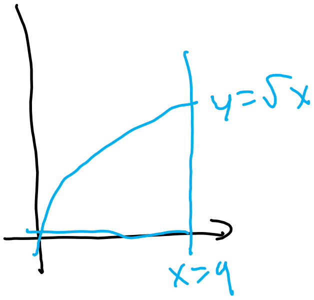
$$= \frac{2}{3}x^{3/2}\Big|_0^9 = \frac{2}{3}(27-0)$$

$$= 18$$

Average Value: The average value of $f(x, y)$ on a region R contained in $\mathbb{R}^2$ is

$$f_{avg} = \frac{\iint_R f(x,y)\,dA}{\iint_R 1\,dA}$$

Example 81. Find the average temperature on the region R in the previous example if the temperature at each point is given by $T(x,y) = 4xy^2$. °C



Area(R) = 18

$$T_{avg} = \frac{\iint_R T(x,y)\,dA}{\text{Area}(R)}$$

$$= \frac{1}{18}\int_0^9\int_0^{\sqrt{x}} 4xy^2\,dy\,dx$$

$$= \frac{1}{18}\int_0^9 \frac{4}{3}xy^3\Big|_{y=0}^{y=\sqrt{x}}\,dx$$

$$= \frac{1}{18}\int_0^9 \frac{4}{3}x\left(x^{3/2}\right) - \frac{4}{3}x(0)\,dx$$

$$= \frac{1}{18}\int_0^9 \frac{4}{3}x^{5/2}\,dx$$

$$= \frac{1}{\cancel{18}\,9}\cdot\frac{4}{3}\cdot\frac{\overset{1}{\cancel{2}}}{7}x^{7/2}\Big|_0^9$$

$$= \frac{4\cdot 3^7}{9\cdot 3\cdot 7} = \frac{4\cdot 3^4}{7} = \boxed{\frac{324}{7}\ °C}$$

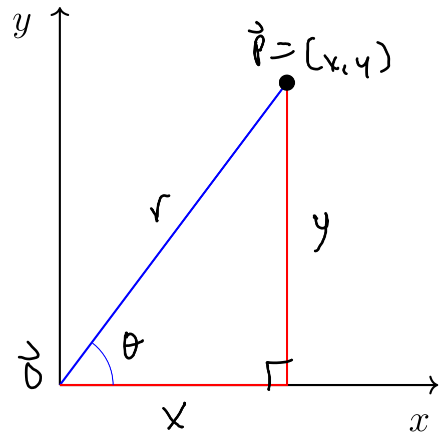
Daily Announcements & Reminders:

- HW 15.2 due tonight
- Exam 2 next Th. ; Canvas announcement tonight

Goals for Today: Sections 15.4, 15.5

- Introduce the polar coordinate system
- Convert double integrals to iterated polar integrals
- Compute iterated polar integrals
- Define triple integrals and compute basic triple integrals

Polar Coordinates:



Cartesian coordinates: Give the distances in $\vec{i}$ and $\vec{j}$ directions from (0,0)

Polar coordinates:

- r = distance from (0,0) to (x,y)
- θ = angle between the ray $\overrightarrow{OP}$ and the positive x-axis (measured CCW from x axis)

We can use trigonometry to go back and forth.

$(r, \theta) \to (x,y)$

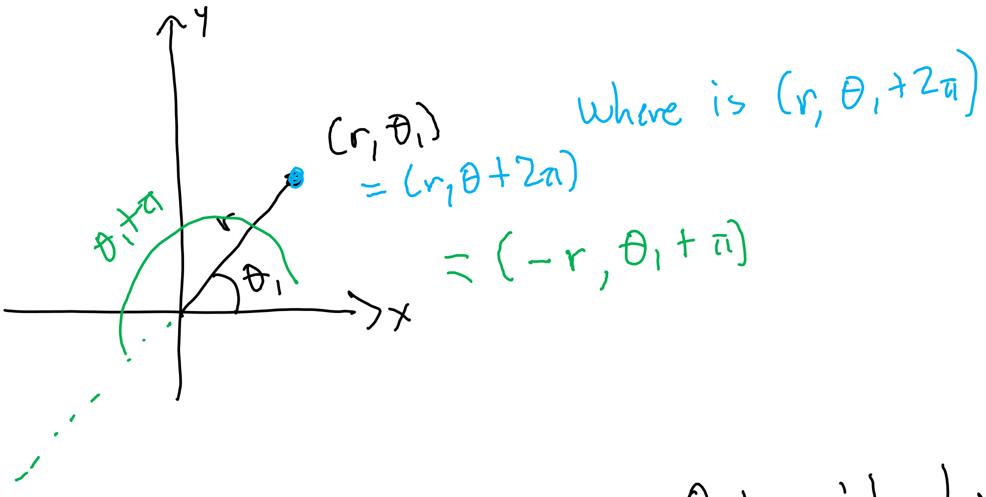
Polar to Cartesian:

$$x = r\cos(\theta) \qquad y = r\sin(\theta)$$

Cartesian to Polar: $(x,y) \to$ multiple possibilities for (r,θ)

$$r^2 = x^2 + y^2 \qquad \tan(\theta) = \frac{y}{x}$$

- not unique



(r, θ_1)
$= (r, \theta + 2\pi)$

where is $(r, \theta_1 + 2\pi)$

$= (-r, \theta_1 + \pi)$

To fix non-uniqueness: $r \geq 0$, θ in an interval of length 2π
e.g. $[0, 2\pi)$

Example 82. a) Find a set of polar coordinates for the point $(x, y) = (1, 1)$.

$x^2+y^2=r^2 \to 2=r^2 \to r=\sqrt{2}$

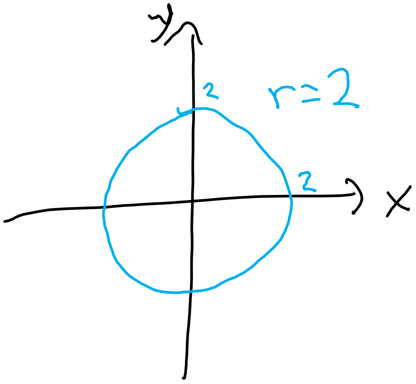
$\tan\theta=\frac{y}{x} \to \tan\theta=1 \to \theta=\pi/4$

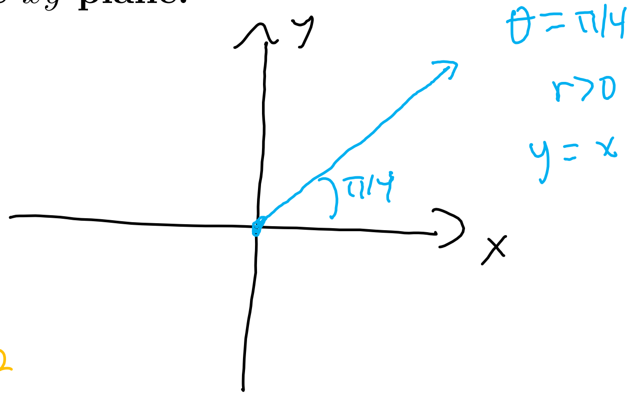
For $(x,y)=(-1,-1)$

$2=r^2 \to r=\sqrt{2}$

$\tan\theta=1 \quad \theta=5\pi/4$

b) Graph the set of points (x, y) that satisfy the equation $r = 2$ and the set of points that satisfy the equation $\theta = \pi/4$ **in the xy-plane.**





c) Write the function $f(x, y) = \sqrt{\overset{r^2}{\boxed{x^2 + y^2}}}$ in polar coordinates.

$f(r,\theta)=\sqrt{(r\cos\theta)^2+(r\sin\theta)^2}=\sqrt{r^2\cos^2\theta+r^2\sin^2\theta}=\sqrt{r^2(\cos^2\theta+\sin^2\theta)}$

$=\sqrt{r^2}$

$=r$

d) [Itempool] Write a Cartesian equation describing the points that satisfy $r = 2\sin(\theta)$.

$r=2\sin\theta \to r^2=2r\sin\theta$

$\cancel{r^2=4\sin^2\theta}$

$r=\frac{2y}{r}$

$r^2=2y$

$\boxed{x^2+y^2=2y}$

$x^2+y^2-2y+\left(\frac{-2}{2}\right)^2=1$

$x^2+(y^2-2y+1)=1$

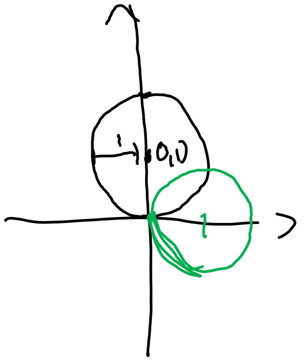
$x^2+(y-1)^2=1$

$x=r\cos\theta$

$y=r\sin\theta$

$x^2+y^2=r^2$

$\tan(\theta)=\frac{y}{x}$

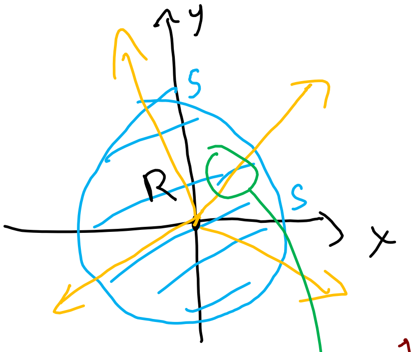


Q: $r=2\cos\theta$

15.4: Double Integrals in Polar Coordinates

Goal: Given a region R in the xy-plane described in polar coordinates and a function $f(r,\theta)$ on R, compute $\iint_R f(r,\theta)\, dA$.

Example 83. Compute the area of the disk of radius 5 centered at $(0,0)$.



$0 \le r \le 5$

$0 \le \theta \le 2\pi$

- radially simple

$$\text{Area} = \iint_R 1 \cdot dA$$

WRONG

$$= \int_0^{2\pi} \int_0^5 \boxed{dr\, d\theta}$$

$$= \int_0^{2\pi} r\Big|_0^5 d\theta$$

$$= 5 \int_0^{2\pi} d\theta$$

$$= 5\theta\Big|_0^{2\pi}$$

$$= 10\pi$$

$A = \pi r^2$

$= 25\pi$

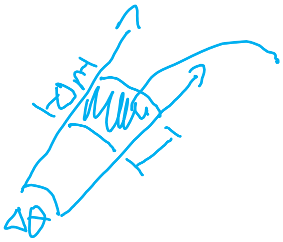
In polar coords $dA \neq dr\, d\theta$, instead it is $dA = r\, dr\, d\theta$

$$\text{Area} = \int_0^{2\pi} \int_0^5 r\, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} r^2 \Big|_0^5 d\theta$$

$$= \int_0^{2\pi} \frac{25}{2}\, d\theta$$

$$= \boxed{25\pi}$$

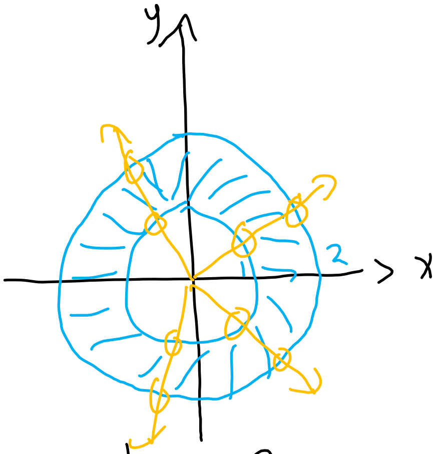


$\Delta A \neq \Delta\theta\, \Delta r$

$\Delta A = r \cdot \Delta\theta\, \Delta r$

Remember: In polar coordinates, the area form $dA =$ $r\, dr\, d\theta$

Example 84. Compute $\iint_D e^{-(x^2+y^2)}\,dA$ on the washer-shaped region $1 \le x^2+y^2 \le 4$.



$1 \le r \le 2$

$0 \le \theta \le 2\pi$

$$\iint_D e^{-(x^2+y^2)}\,dA$$

$$= \int_0^{2\pi}\int_1^2 e^{-r^2}\cdot r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_{-1}^{-4} \frac{e^u}{-2}\,du\,d\theta$$

$$= \int_0^{2\pi} -\frac{1}{2}\left(e^{-4}-e^{-1}\right)d\theta$$

$$= \boxed{\pi\left(e^{-1}-e^{-4}\right)}$$

- $e^{-(x^2+y^2)}$ has no elementary antiderivative wrt x or y

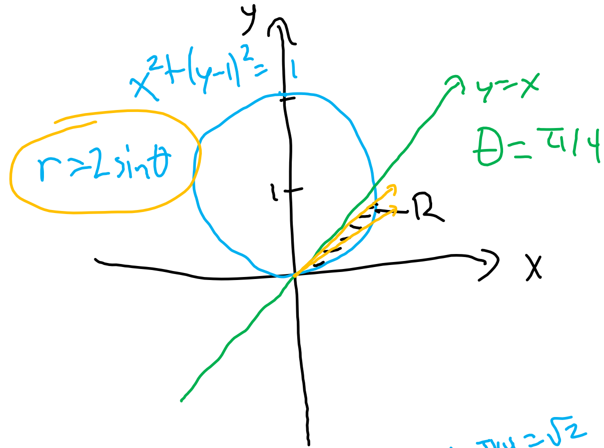
$u=-r^2$

$du=-2r\,dr$

Q: Do we have Fubini's Thm?

A: Yes, but.

Example 85. Compute the area of the smaller region bounded by the circle $x^2 + (y-1)^2 = 1$ and the line $y = x$.

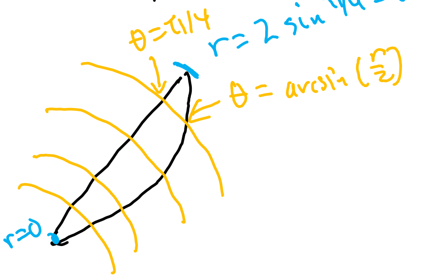


$$\text{area} = \iint_R 1\,dA$$

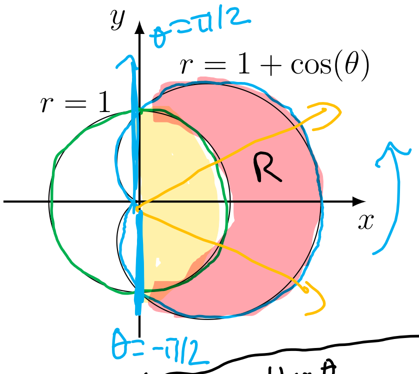
$$= \int_0^{\pi/4}\int_0^{2\sin\theta} r\,dr\,d\theta$$

$$= \int_0^{\sqrt{2}}\int_{\arcsin(\frac{r}{2})}^{\pi/4} r\,d\theta\,dr$$

$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$



Example 86 (Itempool)**.** Write an integral for the volume under $z = x$ on the region between the cardioid $r = 1 + \cos(\theta)$ and the circle $r = 1$, where $x \geq 0$.



$\pi/2$ to $3\pi/2$ ← doesn't match

$-\pi/2$ to $\pi/2$

$$V = \iint_R x\,dA = \boxed{\int_{-\pi/2}^{\pi/2}\int_{1}^{1+\cos\theta} (r\cos\theta)\cdot r\,dr\,d\theta}$$

Daily Announcements & Reminders:

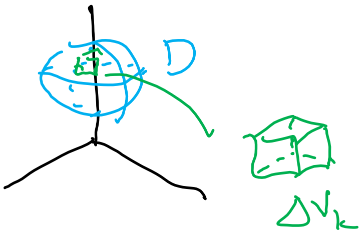
- HW 15.3 due tonight, 15.4 due Th
- Exam in class Thursday, see Canvas announcement
- Office Hours 12:30 -1:30 on Th

Goals for Today: Sections 15.5, 15.6

- Learn how to write triple integrals as iterated integrals.
- Compute triple iterated integrals
- Change the order of integration in a triple iterated integral.
- Apply our work to find the mass and center of mass of objects in $\mathbb{R}^2$ and $\mathbb{R}^3$

15.5 Triple Integrals

Idea: Suppose D is a solid region in $\mathbb{R}^3$. If $f(x,y,z)$ is a function on D, e.g. mass density, electric charge density, temperature, etc., we can approximate the total value of f on D with a Riemann sum.



$$\sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta V_k,$$

by breaking D into small rectangular prisms ΔV_k. Taking the limit gives a

triple integral : $\iiint_D f(x,y,z)\,\underline{dV}$

$$f_{avg} \text{ on } D = \frac{\iiint_D f(x,y,z)\,dV}{\text{volume}(D)}$$

Important special case:

$$\iiint_D 1\,dV = \text{volume of } D$$

Again, we have Fubini's theorem to evaluate these triple integrals as iterated integrals.

Computationally, this is straightforward.

Example 87. Compute $\int_0^1 \int_0^2 \underbrace{\int_0^3 dz}\,dy\,dx$ and interpret your answer.

$$\int_0^1 \int_0^2 z\Big|_{z=0}^{z=3} dy\,dx = \int_0^1 \int_0^2 3\,dy\,dx$$

$$= \int_0^1 6\,dx$$

$$= 6$$

volume of
$0 \le z \le 3$
$0 \le y \le 2$
$0 \le x \le 1$
is 6

← outer bounds constants

Example 88. 1. **Mechanics:** Compute $\int_0^1 \int_0^{2-x} \int_0^{2-x-y} dz\, dy\, dx.$ ← inner bounds cannot have inner variable

↑ middle bounds can have only outer variable

$$= \int_0^1 \int_0^{2-x} 2-x-y \, dy \, dx$$

$$= \int_0^1 (2-x)y - \frac{y^2}{2} \Big|_0^{2-x} dx$$

$$= \int_0^1 (2-x)^2 - \frac{(2-x)^2}{2} - 0 + 0 \, dx$$

$$= \int_0^1 \frac{1}{2}(2-x)^2 \, dx = -\frac{1}{6}(2-x)^3 \Big|_0^1 = -\frac{1}{6}(1-8) = \boxed{\frac{7}{6}}$$

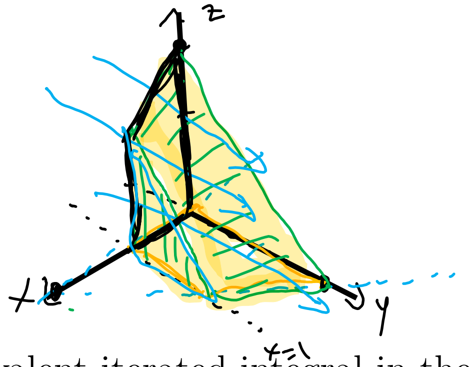
$u=2-x$

2. **Interpretation:** What shape is this the volume of?

$0 \le z \le 2-x-y$

$0 \le y \le 2-x$

$0 \le x \le 1$



$z = 2-x-y$

$x+y+z=2$

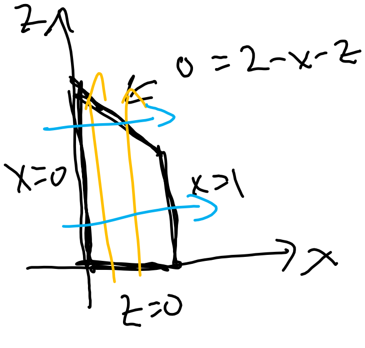
↓

$y = 2-x-z$

3. **Rearrange:** Write an equivalent iterated integral in the order $dy\, dz\, dx$.

$$\int_{x=0}^{x=1} \int_{z=0}^{z=2-x} \int_{y=0}^{y=2-x-z} dy \, dz \, dx$$

- For inner bounds, draw arrows.
- For others: sketch shadow



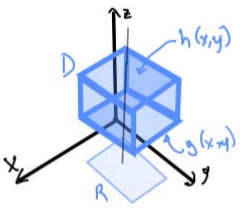
$0 = 2-x-z \to z = 2-x$

We will think about converting triple integrals to iterated integrals in terms of the shadow of D on one of the coordinate planes.

Case 1: z-**simple**) region. If R is the shadow of D on the xy-plane and D is bounded above and below by the surfaces $z = h(x, y)$ and $z = g(x, y)$, then

$$\iiint_D f(x,y,z)\, dV = \iint_R \left(\int_{g(x,y)}^{h(x,y)} f(x,y,z)\, dz \right)\, dy\, dx$$

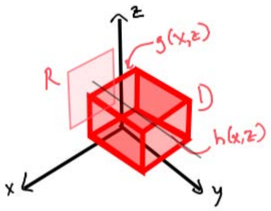
$dx\, dy$



Case 2: y-**simple**) region. If R is the shadow of D on the xz-plane and D is bounded right and left by the surfaces $y = h(x, z)$ and $y = g(x, z)$, then

$$\iiint_D f(x,y,z)\, dV = \iint_R \left(\int_{g(x,z)}^{h(x,z)} f(x,y,z)\, dy \right)\, dz\, dx$$

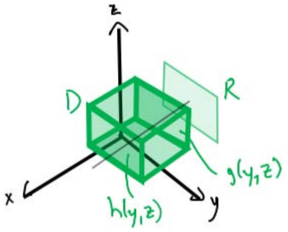
$dx\, dz$



Case 3: x-**simple**) region. If R is the shadow of D on the yz-plane and D is bounded front and back by the surfaces $x = h(y, z)$ and $x = g(y, z)$, then

$$\iiint_D f(x,y,z)\, dV = \iint_R \left(\int_{g(y,z)}^{h(y,z)} f(x,y,z)\, dx \right)\, dz\, dy$$

$dy\, dx$



Example 89. Write an integral for the volume of the solid in the first octant bounded by $z = 3 - x^2 - y^2$ and $z = 2y$ treating the solid as a) z-simple and b) x-simple. Is the solid also y-simple?

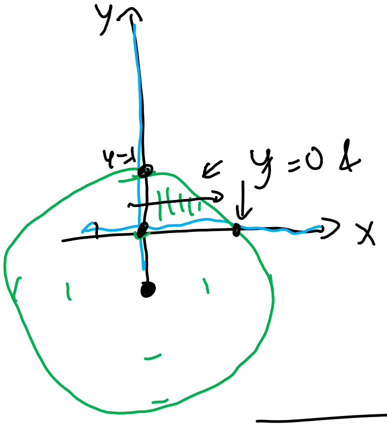
$x \geq 0$
$y \geq 0$
$z \geq 0$

a) z-simple: $\int_0^1 \int_0^{\sqrt{4-(y+1)^2}} \int_{2y}^{3-x^2-y^2} 1 \, dz \, dx \, dy$

← arrow leaving at paraboloid

↑ z-arrow enters at plane

- Sketch shadow in xy-plane



↙ $y=0$ & $x^2+1^2=4$

$x=\sqrt{3}$

set z-bounds equal:

$2y = 3 - x^2 - y^2$

$x^2+y^2+2y+1 = 3+1$

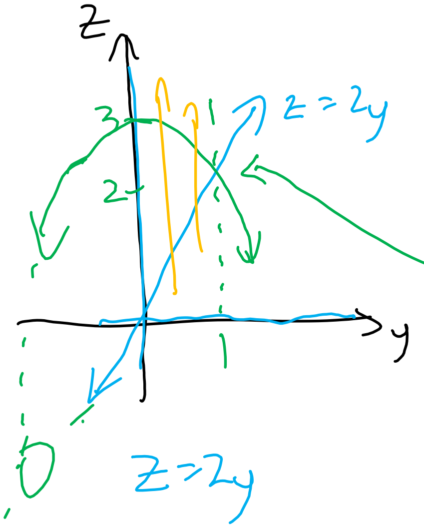
$x^2+(y+1)^2 = 4$

$x = \sqrt{4-(y+1)^2}$

---

b) x-simple: $\int_{y=0}^{y=1} \int_{z=2y}^{z=3-y^2} \int_{x=0}^{x=\sqrt{3-y^2-z}} 1 \, dx \, dz \, dy$

$z = 3 - x^2 - y^2 \Rightarrow$ solve for x: $x^2 = 3 - y^2 - z$



$0 = \sqrt{3-y^2-z}$

$\underline{z = 3 - y^2}$

$2y = 3 - y^2$

$y^2 + 2y - 3 = 0$

$(y+3)(y-1) = 0$

$y = 1$

Example 89 (cont.)

Rules for Triple Integrals for the Sketching Impaired (credit to Wm. Douglas Withers)

Basic

Rule 1: Choose a variable appearing exactly twice for the next integral.

Rule 2: After setting up an integral, cross out any constraints involving the variable just used.

Rule 3: Create a new constraint by setting the lower limit of the preceding integral less than the upper limit.

Intermediate

Rule 4: A square variable counts twice.

Rule 5: The region of integration of the next step must lie within the domain of any function used in previous limits.

Rule 6: If you do not know which is the upper limit and which is the lower, take a guess - but be prepared to backtrack.

Advanced

Rule 7: When forced to use a variable appearing more than twice, choose the most restrictive pair of constraints.

Rule 8: When unable to determine the most restrictive pair of constraints, set up the integral using each possible most restrictive pair and add the results.

Example 90. Set up an integral for the volume of the region D defined by

Typo: should be y^2

$$x + y^2 \leq 8, \quad y + 2z^2 \leq x, \quad y \geq 0$$

x appears twice

y appears 4 times

z appears twice

$$V = \int_0^{-} \int_{-\sqrt{4-\frac{1}{2}y-\frac{1}{2}y^2}}^{\sqrt{4-\frac{1}{2}y-\frac{1}{2}y^2}} \int_{y+2z^2}^{8-y^2} dx \quad dz \quad d$$

By Rule 1 integrate x first: $x \leq 8 - y^2$

$y + 2z^2 \leq x$

Rule 2: cross out these two constraints: $y \geq 0$

Rule 3: make a new constraint $y + 2z^2 \leq 8 - y^2$

$y + y^2 + 2z^2 \leq 8$

Outer bounds: $y \geq 0$, $y + y^2 + 2z^2 \leq 8$

y appears 3 times

z appears 2 times

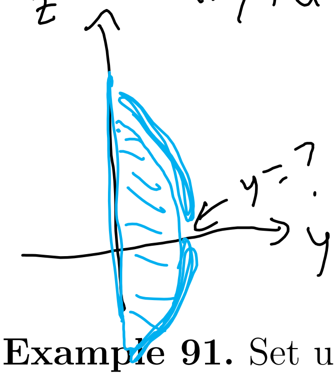
By Rule 1: choose z.

$$2z^2 \leq 8 - y - y^2$$

$$z^2 \leq 4 - \tfrac{1}{2}y - \tfrac{1}{2}y^2$$

$$-\sqrt{4 - \tfrac{1}{2}y - \tfrac{1}{2}y^2} \leq z \leq \sqrt{4 - \tfrac{1}{2}y - \tfrac{1}{2}y^2}$$

$$0 = 4 - \tfrac{1}{2}y - \tfrac{1}{2}y^2 \Rightarrow y^2 + y - 8 = 0$$



Example 91. Set up a triple iterated integral for the triple integral of $f(x, y, z) = x^3y$ over the region D bounded by

$$x^2 + y^2 = 1, \quad \cancel{z = 0}, \quad \cancel{x + y + z = 2}.$$

Rule 4: x appears 3 times, y appears 3 times, z appears twice

Rule 1: Start with z, since it appears twice.

$$\int \int \int_{2-x-y}^{0} x^3y \, dz \, d \, d$$ ← WRONG!

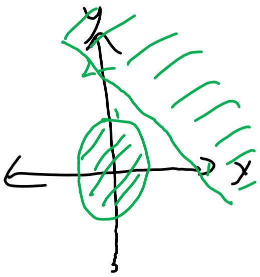
$z = 0$ $z = 2 - x - y$

Rule 6: Guess order!

Rule 2: Cross used constraints

Rule 3: Combine limits to get new constraint: $2 - x - y < 0$

Now in xy-plane: $x^2 + y^2 = 1$ $2 - x - y < 0$ or $y > 2 - x$



Problem! No overlap! So our guess was wrong!

Backtrack:

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{0}^{2-x-y} x^3y \, dz \, dy \, dx$$

⇓

Rule 2: Cross used constraints

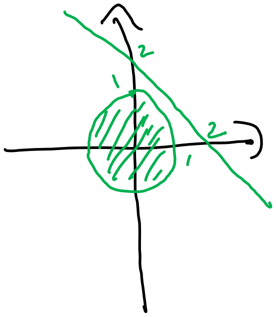
Rule 3: Get $0 < 2 - x - y$

In xy-plane: $x^2 + y^2 = 1$

$y < 2 - x$

Use dy dx: $-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$

$-1 \leq x \leq 1$



Ex 90 (Fixed) D: $\cancel{x + y^2 \le 8}$, $\cancel{y^2 + 2z^2 \le x}$, $y \ge 0$

Set up a triple integral for volume of D.

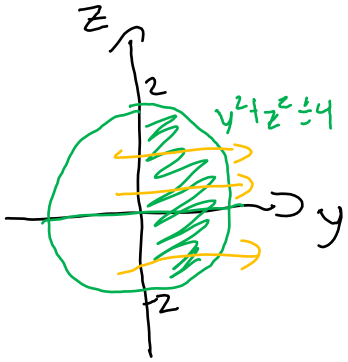
- Rule 4: x appears twice, y appears 5 times, z appears twice
- Rule 1: Choose x for first integral (appears exactly twice)

$$V = \int_{-2}^{2} \int_{0}^{\sqrt{4-z^2}} \int_{y^2+2z^2}^{8-y^2} dx\, dy\, dz$$

$x \le 8 - y^2$, $y^2 + 2z^2 \le x$

- Rule 2: Cross out used constraints.
- Rule 3: Create new constraint from limits: $y^2 + 2z^2 \le 8 - y^2$

$y^2 + z^2 \le 4$, $y \ge 0$



- dy dz looks slightly easier:

$0 \le y \le \sqrt{4-z^2}$

$-2 \le z \le 2$

Daily Announcements & Reminders:

pushing this back

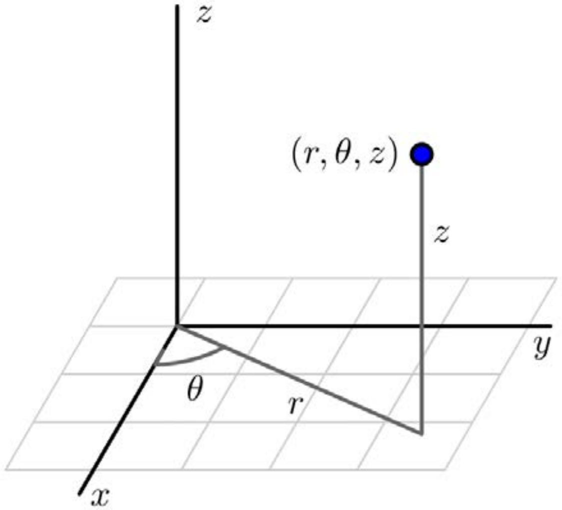
- HW 15.5, 15.6 due tonight
- 15.6 is just particular formulas for using integrals to compute physical properties
  - the list of formulas is on Canvas w/ lecture notes
- Quiz 7 tomorrow on 15.5 only
- Exam 2 grades released, full results back Th, deadline to withdraw is tomorrow at 4pm

Goals for Today:

Section 15.7

- Be able to convert between Cartesian, cylindrical, and spherical coordinate systems in $\mathbb{R}^3$
- Compute triple integrals expressed in cylindrical coordinates
- Compute triple integrals expressed in spherical coordinates

Cylindrical Coordinate System



Cylindrical to Cartesian:

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad z = z$$

Cartesian to Cylindrical:

$$r^2 = x^2 + y^2, \quad \tan(\theta) = \frac{y}{x}, \quad z = z$$

For uniqueness:

- $r \geq 0$, θ lies in an interval of length 2π, e.g. $[0, 2\pi]$ or $[-\pi, \pi]$

Example 91. a) Find cylindrical coordinates for the point with Cartesian coordinates $(-1, \sqrt{3}, 3)$.

$x \quad y \quad z$

$r^2 = x^2 + y^2 = 1 + 3 = 4$

$r = 2$

$(r, \theta, z) = (2, \frac{2\pi}{3}, 3)$

$\tan\theta = \frac{\sqrt{3}}{-1} = -\sqrt{3}$

$\theta = 2\pi/3$

b) Find Cartesian coordinates for the point with cylindrical coordinates $(2, 5\pi/4, 1)$.

$r \quad \theta \quad z$

$x = 2\cos(5\pi/4)$

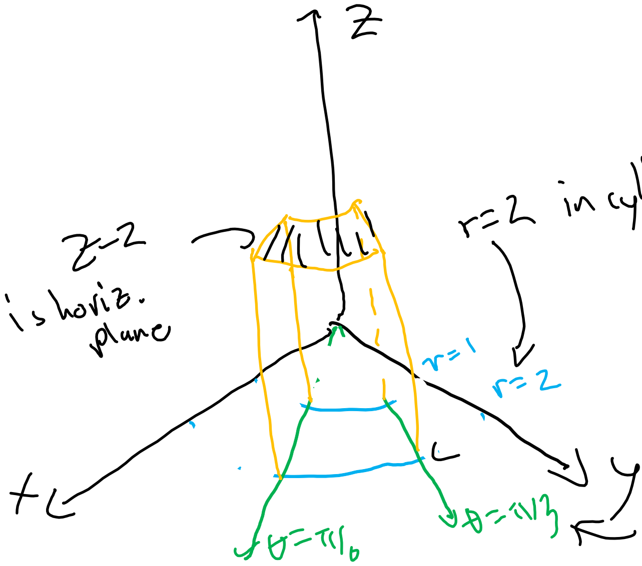
$y = 2\sin(5\pi/4)$

$(x, y, z) = (-\frac{2}{\sqrt{2}}, -\frac{2}{\sqrt{2}}, 1)$

Example 92. In xyz-space sketch the *cylindrical box*

- paraboloids, cones, spheres ok too

$$B = \{(r, \theta, z) \mid \underline{1} \le r \le \underline{2},\ \underline{\pi/6} \le \theta \le \underline{\pi/3},\ \underline{0} \le z \le \underline{2}\}.$$



r=2 in cylindrical coords is a cylinder

θ=π/3 is a vertical plane

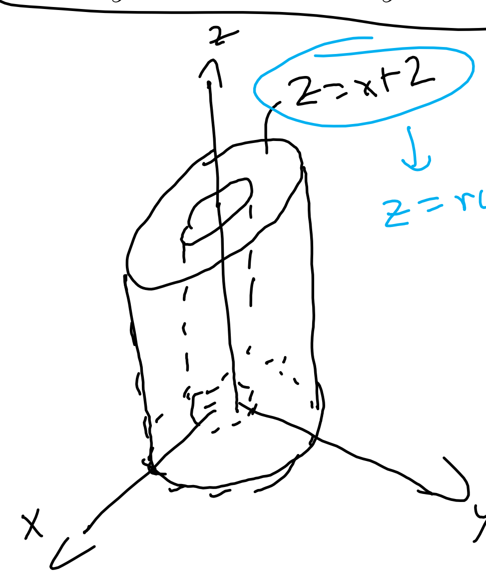
Triple Integrals in Cylindrical Coordinates

In polar: $dA = r\,dr\,d\theta$

We have $dV =$ $r\,dz\,dr\,d\theta$

Example 93. Set up a iterated integral in cylindrical coordinates for the volume of the region D lying below $z = x + 2$, above the xy-plane, and between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

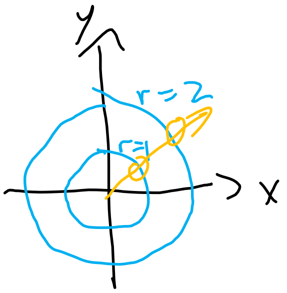
$z=0$



$z = r\cos\theta + 2$

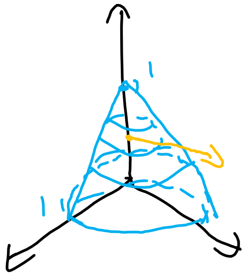
$$V = \iiint_D 1\,dV$$

$$= \int_0^{2\pi} \int_1^2 \int_0^{r\cos\theta + 2} 1 \cdot r\,dz\,dr\,d\theta$$



Example 94 (Itempool)**.** Suppose the density of the cone defined by $r = 1 - z$ with $z \geq 0$ is given by $\delta(r, \theta, z) = z$. Set up an iterated integral in cylindrical coordinates that gives the mass of the cone.

$$\text{mass} = \iiint_D (\text{density})\, dV$$

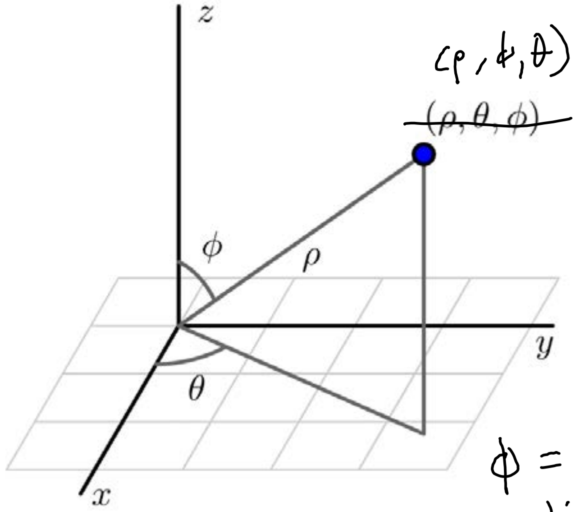


$$= \int_0^{2\pi} \int_0^1 \int_0^{1-r} \underline{zr\, dz}\, dr\, d\theta \quad ✓$$

$$= \boxed{\int_0^{2\pi} \int_0^1} \int_0^{1-z} zr\, dr\, dz\, d\theta$$

$$= \int_0^1 \int_0^{1-r} \int_0^{2\pi} zr\, d\theta\, dz\, dr$$

Spherical Coordinate System



$\phi = \varphi$

phi

For uniqueness:

$\rho \geq 0, \quad \varphi \in [0, \pi], \quad \theta \in [0, 2\pi]$

Spherical to Cartesian:

$$x = \rho \sin(\varphi) \cos(\theta)$$
$$y = \rho \sin(\varphi) \sin(\theta)$$
$$z = \rho \cos(\varphi)$$

Cartesian to Spherical:

$$\rho^2 = x^2 + y^2 + z^2$$
$$\tan(\theta) = \frac{y}{x}$$
$$\tan(\varphi) = \frac{\sqrt{x^2 + y^2}}{z}$$

Example 95. a) Find spherical coordinates for the point with Cartesian coordinates $(-2, 2, \sqrt{8})$.

$\quad x \quad y \quad z$

$$\rho = \sqrt{(-2)^2 + (2)^2 + (\sqrt{8})^2} = \sqrt{4 + 4 + 8} = 4$$

$$\tan\theta = \frac{2}{-2} = -1 \qquad \theta = 3\pi/4$$

$$\tan\varphi = \frac{\sqrt{8}}{\sqrt{8}} = 1 \qquad \varphi = \pi/4$$

b) Find Cartesian coordinates for the point with spherical coordinates $(2, \pi/2, \pi/3)$.

$\rho \quad \varphi \quad \theta$

$$x = 2 \sin\frac{\pi}{2} \cos\frac{\pi}{3} = 1$$
$$y = 2 \sin\frac{\pi}{2} \sin\frac{\pi}{3} = \sqrt{3}$$
$$z = 2 \cos\frac{\pi}{2} = 0$$

Example 96. In xyz-space sketch the *spherical box*

$$B = \{(\rho, \varphi, \theta) \mid 1 \le \rho \le 2,\ 0 \le \varphi \le \pi/4,\ \pi/6 \le \theta \le \pi/3\}.$$

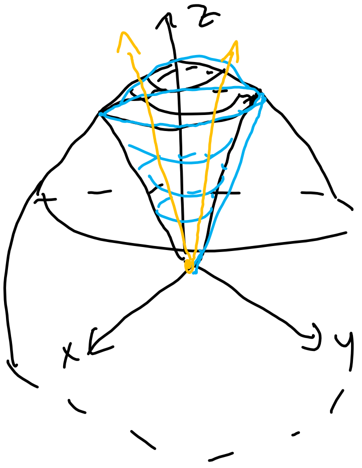
- spheres / cones / planes through z-axis work well in spherical coords

Triple Integrals in Spherical Coordinates

We have $dV =$ $\rho^2 \sin(\varphi)\, d\rho\, d\varphi\, d\theta$

$\sqrt{x^2+y^2} = \rho \sin \varphi$

Example 97. Write an iterated integral for the volume of the "ice cream cone" D bounded above by the sphere $x^2+y^2+z^2 = 1$ and below by the cone $z = \sqrt{3}\sqrt{x^2+y^2}$.



Convert bounds:

$\rho^2 = 1$

$\boxed{\rho = 1}$

$\frac{1}{\sqrt{3}} = \frac{\sqrt{x^2+y^2}}{z}$

$= \tan(\varphi)$

$\boxed{\varphi = \pi/6}$

OR

$\rho \cos \varphi = \sqrt{3}\sqrt{\rho^2 \sin^2\varphi \cos^2\theta + \rho^2 \sin^2\varphi \sin^2\theta}$

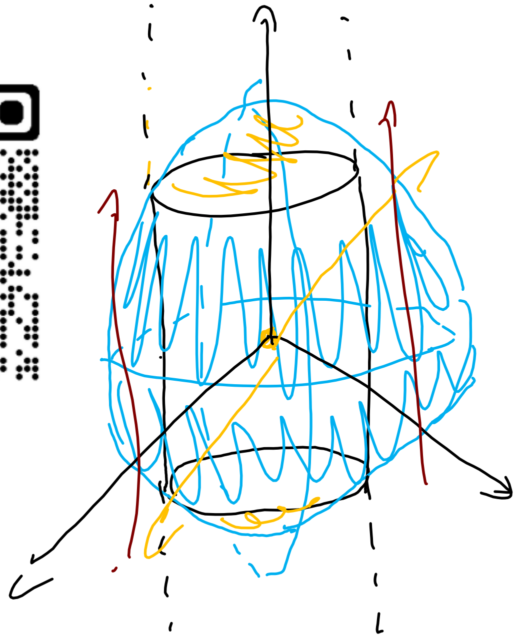
$= \sqrt{3}\sqrt{\rho^2 \sin^2\varphi (1)}$

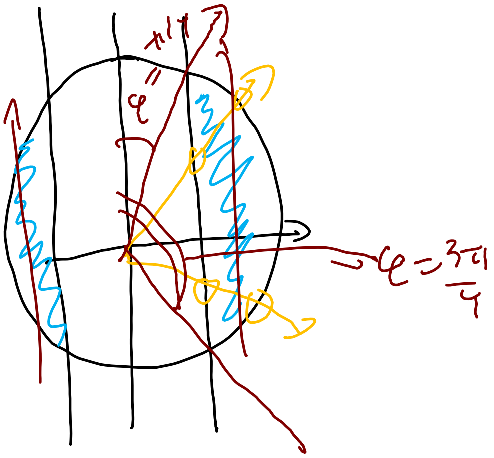
$= \sqrt{3} \cdot \underline{\rho \sin \varphi}$

$\frac{1}{\sqrt{3}} = \tan \varphi$

$$V = \int_0^{2\pi} \int_0^{\pi/6} \int_0^1 1 \cdot \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

Example 98 (Itempool)**.** Write an iterated integral for the volume of the region that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and outside the cylinder $x^2 + y^2 = 1$.



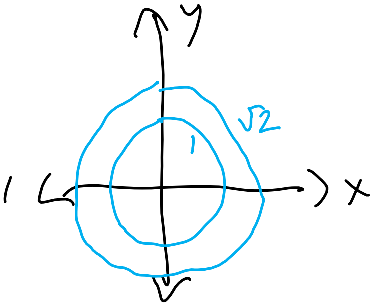


- Simple in both spherical & cylindrical coords.

Cylindrical

$$V = \int_0^{2\pi}\int_1^{\sqrt{2}}\int_{z=\sqrt{2-x^2-y^2}}^{z=\sqrt{2-x^2-y^2}} r\,dz\,dr\,d\theta$$

z bounded below by lower hemisphere
above by upper hemisphere



Post class: Spherical

$$V = \int_0^{2\pi}\int_{\pi/4}^{3\pi/4}\int_{\csc\varphi}^{\sqrt{2}} \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

From our picture we see
cylinder < ρ < sphere

$x^2+y^2=1$
$\Rightarrow \rho^2\sin^2\varphi = 1$
$\Rightarrow \rho\sin\varphi = 1$
$\Rightarrow \rho = \csc\varphi$

$x^2+y^2+z^2=2$
$\Rightarrow \rho^2 = 2$
$\Rightarrow \rho = \sqrt{2}$

To get φ bounds, set ρ-bounds equal:

$\csc\varphi = \sqrt{2}$
$\sin\varphi = \frac{1}{\sqrt{2}}$
$\varphi = \pi/4, 3\pi/4$

Daily Announcements & Reminders:

- HW 15.7 due tonight
- Exam 2 released on Gradescope
- In class polls will be in Piazza for rest of semester
- Enjoy break!

Goals for Today: Section 15.8

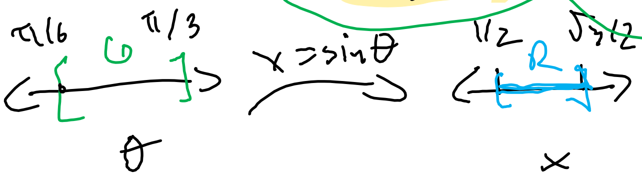
- Change variables in multiple integrals
- Identify choices for changing variables in a given integration problem

Change bounds

convert both!

Thinking about single variable calculus: Compute $\int_{1/2}^{\sqrt{3}/2} \frac{1}{\sqrt{1-x^2}}\,dx$

We use the substitution: ① $x = \sin\theta$ ② $dx = \cos\theta\, d\theta$



③ Find new region of integration: $\frac{1}{2} = \sin\theta$ $\quad \frac{\sqrt{3}}{2} = \sin\theta$

Goal

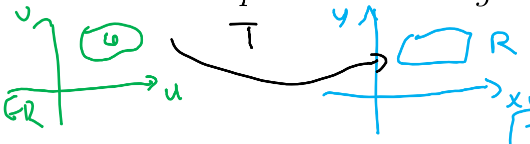
easier

$$\int_{1/2}^{\sqrt{3}/2} \frac{1}{\sqrt{1-x^2}}\,dx = \int_{\pi/6}^{\pi/3} \frac{1}{\sqrt{1-\sin^2\theta}} \cos\theta\, d\theta$$

④ $= \cos\theta$

$$= \int_{\pi/6}^{\pi/3} 1\, d\theta = \boxed{\pi/6}$$

Theorem 99 (Substitution Theorem). *Suppose* $\boxed{\mathbf{T}(u,v)}$ *is a one-to-one, differentiable transformation that maps the region* G *in the* uv*-plane to the region* R *in the* xy*-plane. Then*



$$\overset{\text{HARD}}{\underbrace{\iint_R f(x,y)\,dx\,dy}_{\text{Given}}} = \overset{\text{EASIER}}{\iint_G} \underbrace{f(\mathbf{T}(u,v))}_{\text{convert f into u,v}}\,|\det(D\underbrace{\mathbf{T}(u,v)}_{\text{multiply by ``derivative''}})|\,du\,dv.$$

$\boxed{\text{Jacobian}}$ ↓ "derivative",

It measures scaling of area from the transformation, e.g. if $|\det(D\vec{T}(u,v))| = 2$ then $\text{area}(R) = 2\cdot\text{area}(G)$

Example 100. Evaluate $\int_0^4 \int_{y/2}^{y/2+1} \frac{2x-y}{2}\,dx\,dy$ via the transformation $x = u+v$, $y = 2v$.

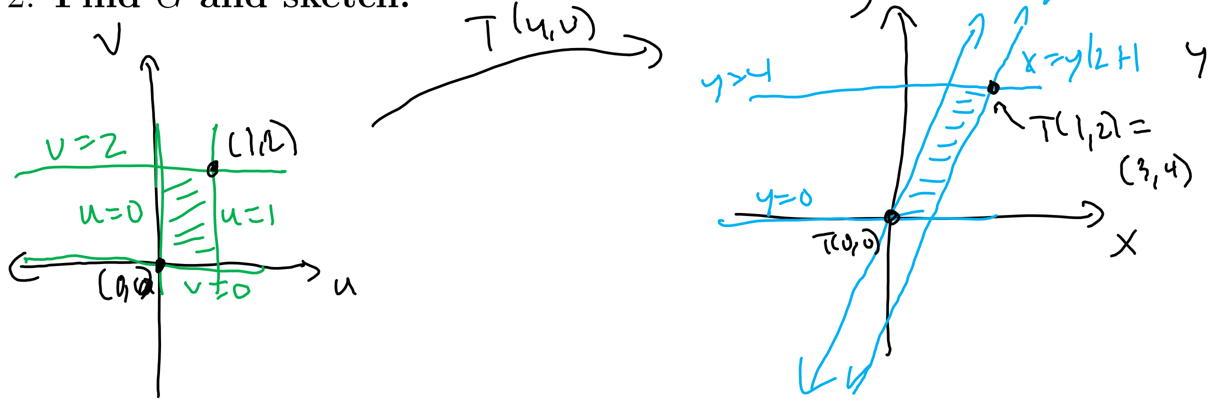
1. **Find T:**

$$\vec{T}(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \end{bmatrix} = \begin{bmatrix} u+v \\ 2v \end{bmatrix}$$

often $\vec{T}^{-1}$ is given: $u = \boxed{\frac{2x-y}{2}}$, $v = \boxed{\frac{y}{2}}$

↳ solve for x, y to get $\vec{T}$

2. **Find G and sketch:**



R

$y/2 \le x \le y/2 + 1$

$0 \le y \le 4$

$y=0 \to 2v=0 \to v=0$

$y=4 \to 2v=4 \to v=2$

$x=y/2 \to u+v = v \to u=0$

$x=y/2+1 \to u+v = v+1 \to u=1$

$\vec{T}(u,v) = \begin{bmatrix} u+v \\ 2v \end{bmatrix}$

3. **Find Jacobian:**

$$|\det(D\vec{T}(u,v))| = \left|\det\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\right| = |2-0| = 2$$

(So, Area(R) = 2 · Area(G))

4. **Convert and use theorem:**

$$\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x-y}{2}\,dx\,dy = \int_0^2 \int_0^1 \underbrace{u}_{} \cdot 2\,du\,dv$$

$$= 2$$

$$\frac{2(u+v)-2v}{2} = \frac{2u}{2} = u$$

Example 101. a) (Piazza poll) Find the Jacobian of the transformation

$$x = u + (1/2)v, \;\; y = v.$$

$$\left| \det\left(\begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix} \right) \right| = | 1 - 0 | = 1$$

b) (Piazza poll) Which transformation(s) seem suitable for the integral

linear bounds

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3 \boxed{(2x-y)} e^{(2x-y)^2} \, dx \, dy?$$

i) $u = x, v = y$ ← does nothing

ii) $u = \sqrt{x^2 + y^2}, v = \arctan(y/x)$ ← polar, not helpful

Good

iii) $u = 2x - y$, $v = y^3$

Better

iv) $u = y$, $v = 2x - y$

v) $u = 2x - y$, $v = y$

The same! ← $u^3 v e^{v^2}$

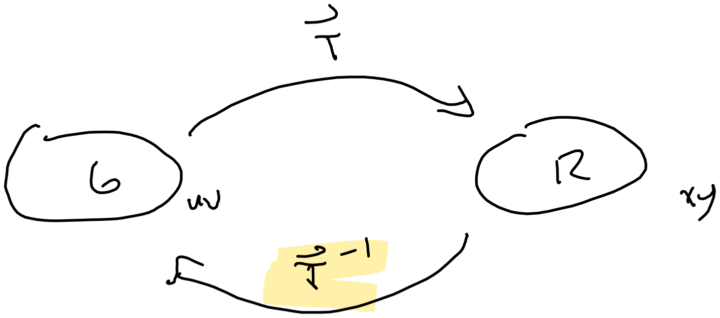
vi) $u = e^{(2x-y)^2}$, $v = y^3$ strands (2x-y)

c) Change variables in the integral above to one which is easier to compute using your work in a) and b).

Theorem 102 (Derivative of Inverse Coordinate Transformation)**.** *If* $\mathbf{T}(u, v)$ *is a one-to-one differentiable transformation that maps a region* G *in the* uv*-plane to a region* R *in the* xy*-plane, then we have*

$\vec{T}(u_0, v_0) = (x_0, y_0)$

$$|\det(D\mathbf{T}(u_0, v_0))| = \frac{1}{|\det(D\mathbf{T}^{-1}(x_0, y_0))|}$$



Example 103. Let's evaluate $\iint_R \frac{y(x+y)}{x^3}$ where R is the region in the xy-plane bounded by $y = x, y = 3x, y = 1 - x$, and $y = 2 - x$. Consider the coordinate transformation $u = x + y, v = y/x$.

$$\underbrace{u = x+y, \quad v = y/x} \hookrightarrow \vec{T}^{-1}(x,y) = \begin{bmatrix} x+y \\ y/x \end{bmatrix} \begin{matrix} u \\ v \end{matrix} \qquad \vec{T}(u,v) = \begin{bmatrix} ? \\ \vdots \\ ? \end{bmatrix}$$

1. **Find the rectangle G in the uv plane that is mapped to R**

$y=x$	$\underline{y/x}=1$	$v=1$
$y=3x$	$\underline{y/x}=3$	$v=3$
$y=1-x$	$\underline{x+y}=1$	$u=1$
$y=2-x$	$\underline{x+y}=2$	$u=2$

2. **Evaluate $f(\mathbf{T}(u,v))|\det(D\mathbf{T}(u,v))|$ in terms of u and v without directly solving for $\mathbf{T}$ using the theorem above**

$$f(x,y) = \frac{y(x+y)}{x^3}$$

$$f(\vec{T}(u,v))\,|\det(D\vec{T}(u,v))|$$

We know: $|\det(DT^{-1}(x,y))| = \left|\det\begin{pmatrix} 1 & 1 \\ -\frac{y}{x^2} & \frac{1}{x} \end{pmatrix}\right|$

$$= \left|\frac{1}{x} + \frac{y}{x^2}\right| = \frac{x+y}{x^2}$$

$$\frac{y(x+y)}{x^3} \cdot \frac{1}{\left(\frac{x+y}{x^2}\right)} = \frac{y(x+y)}{x^3} \cdot \frac{x^2}{(x+y)} = \frac{y}{x} = \boxed{V}$$

3. **Use the Substitution Theorem to compute the integral.**

$$\iint_R \frac{y(x+y)}{x^3}\,dy\,dx = \int_1^3 \int_1^2 v \; du\,dv$$

$\uparrow$

b/c $f(\vec T(u,v))\,|\det(D\vec T(u,v))| = v$

Why $dA = r\,dr\,d\theta$ in polar coords?

$$\vec T(r,\theta) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$$

$$D\vec T(r,\theta) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$\det(D\vec T(r,\theta)) = r\cos^2\theta + r\sin^2\theta$$
$$= r$$

Spherical coords

$$T(\rho,\varphi,\theta) = \begin{bmatrix} \rho\sin\varphi\cos\theta \\ \rho\sin\varphi\sin\theta \\ \rho\cos\varphi \end{bmatrix}$$

$$\det(D\vec T(\rho,\varphi,\theta)) = \rho^2\sin\varphi$$

Daily Announcements & Reminders:

- HW 15.8 due tonight
- Quiz 8 tomorrow on 15.7/15.8
- Exam 2 regrades open until Thursday morning
- Do warmup poll on Piazza

Goals for Today: Section 16.1, 16.2

- Define a line integral for a scalar function $f(x, y)$ or $f(x, y, z)$
- Compute line integrals using parameterizations

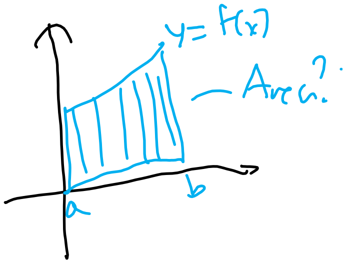
Unit 4: Vector Calculus

Goals:

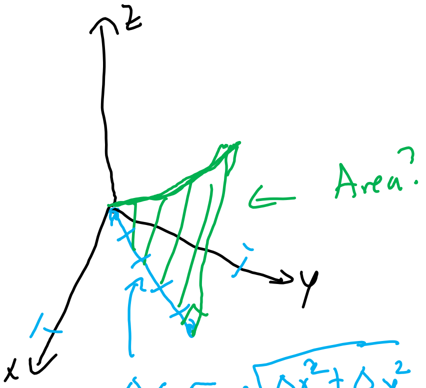
- Extend 1D/2D integrals to 1D/2D objects living in higher-dimensional space
- Extend the fundamental theorem of calculus in new ways

We will use tools from everything we have covered so far to do this: parameterizations, derivatives and gradients, and multiple integrals.

Example 104. Suppose we build a wall whose base is the straight line from $(0,0)$ to $(1,1)$ in the xy-plane and whose height at each point is given by $h(x,y) = 2x + y^2$ meters. What is the area of this wall?

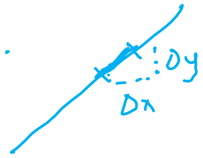


$A = \int_a^b f(x)\,dx$



$\Delta s_k \approx \sqrt{\Delta x_k^2 + \Delta y_k^2}$

$h = h(x_k, y_k)$



$$\text{area} \approx \sum_{k=1}^{n} h(x_k, y_k)\sqrt{\Delta x_k^2 + \Delta y_k^2}$$

$$\text{area} = \lim_{n\to\infty} \sum_{k=1}^{n} h(x_k, y_k)\sqrt{\Delta x_k^2 + \Delta y_k^2}$$

$$= \int_a^b h(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$$

1) Parameterize C: $\vec{r}(t) = (1-t)\langle 0,0\rangle + t\langle 1,1\rangle$, $0 \le t \le 1$

$= \langle t, t\rangle$

2) Substitute: $\int_0^1 \left(2(t) + (t)^2\right)\sqrt{1^2+1^2}\,dt = \int_0^1 \sqrt{2}(2t + t^2)\,dt$

$= 4\sqrt{2}/3$

Definition 105. The **line integral** of a scalar function $f(x,y)$ over a curve C in $\mathbb{R}^2$ is

$$\int_C f(x,y)\,\boxed{ds} = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k)\sqrt{\Delta x_k^2 + \Delta y_k^2}$$

$$= \int_a^b f(\vec{r}(t))\,\underline{|\vec{r}\,'(t)|\,dt}$$

where $\vec{r}(t)$ parameterizes C with $a \le t \le b$

$ds = |\vec{r}\,'(t)|\,dt$

What things can we compute with this?

- If $f = 1$: $\int_C 1\,ds$ = length of $C = \int_a^b |\vec{r}\,'(t)|\,dt$
- If $f = \delta$ is a density function: $\int_C \delta\,ds$ = total mass of object lying along C w/ density δ
- If f is a height: $\int_C h\,ds$ = area between $h(\vec{r}(t))$ & xy-planes

Strategy for computing line integrals:

1. Parameterize the curve C with some $\mathbf{r}(t)$ for $a \le t \le b$
2. Compute $ds = |\mathbf{r}'(t)|\ dt$
3. Substitute: $\int_C f(x,y,z)\ ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\ dt$
4. Integrate

Example 106. (Piazza poll) Compute $\int_C \underbrace{2x + y^2}\ ds$ along the curve C given by $\mathbf{r}(t) = \overset{x(t)}{\underline{10t}}\mathbf{i} + \overset{y(t)}{\underline{10t}}\mathbf{j}$ for $0 \le t \le \frac{1}{10}$.

$f(x,y) = 2x + y^2$

$f(\vec{r}(t)) = 2(10t) + (10t)^2$

1) Parameterize C. Given!

2) $|\vec{r}\,'(t)| = \sqrt{10^2+10^2} = \sqrt{200} = 10\sqrt{2}$

3/4) Substitute & Integrate:

$$\int_0^{1/10} \underbrace{(20t + 100t^2)}_{f(\vec{r}(t))} 10\sqrt{2}\ dt = \left(10t^2 + \frac{100}{3}t^3\right)10\sqrt{2}\Big|_0^{1/10}$$

$$= \left(\frac{1}{10} + \frac{1}{30}\right)10\sqrt{2}$$

$$= \sqrt{2} + \frac{\sqrt{2}}{3}$$

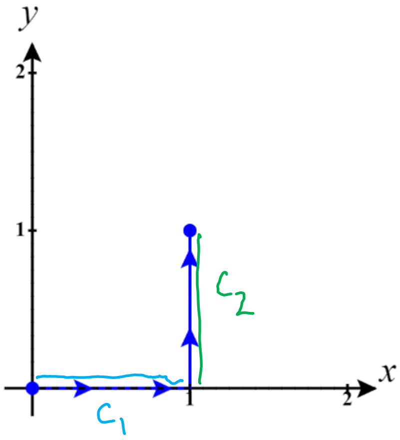
$$= \frac{4\sqrt{2}}{3}$$

- If $\vec{r}_1(t)$ & $\vec{r}_2(t)$ are parameterizations of the same curve C then $\int_{a_1}^{b_1} f(\vec{r}_1(t))\,|\vec{r}_1\,'(t)|\,dt = \int_{a_2}^{b_2} f(\vec{r}_2(t))\,|\vec{r}_2\,'(t)|\,dt$

- If $-C = C$ w/ opposite orientation

$$\int_{-C} f(x,y)\,ds = -\int_C f(x,y)\,ds$$

Example 107. Compute $\int_C 2x + y^2 \; ds$ along the curve C pictured below.



$$\int_C 2x+y^2\,ds = \int_{C_1} 2x+y^2\,ds + \int_{C_2} 2x+y^2\,ds$$

C_1: 1) $\vec{r}(t) = (1-t)\langle 0,0\rangle + t\langle 1,0\rangle$, $0\le t\le 1$

$= \langle t,0\rangle$, $0\le t\le 1$

2) $|\vec{r}\,'(t)| = |\langle 1,0\rangle| = 1$

C_2: 1) $\vec{r}_2(t) = (1-t)\langle 1,0\rangle + t\langle 1,1\rangle$, $0\le t\le 1$

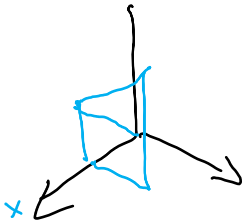
$= \langle 1,t\rangle$, $0\le t\le 1$

2) $|\vec{r}_2\,'(t)| = |\langle 0,1\rangle| = 1$

3) $\int_C 2x+y^2\,ds = \int_0^1 (2(t)+0^2)(1)\,dt + \int_0^1 (2(1)+t^2)(1)\,dt$

$= t^2+2t+\frac{t^3}{3}\Big|_0^1$

$= 1+2+\frac{1}{3} = \boxed{\frac{10}{3}}$



- most line integrals are path-dependent

Example 108. (Piazza Poll) Let C be a curve parameterized by $\mathbf{r}(t)$ from $a \leq t \leq b$. Select all of the true statements below.

a) $\mathbf{r}(t+4)$ for $a \leq t \leq b$ is also a parameterization of C with the same orientation

False

b) $\mathbf{r}(2t)$ for $a/2 \leq t \leq b/2$ is also a parameterization of C with the same orientation

True

c) $\mathbf{r}(-t)$ for $a \leq t \leq b$ is also a parameterization of C with the opposite orientation

False

d) $\mathbf{r}(-t)$ for $-b \leq t \leq -a$ is also a parameterization of C with the opposite orientation

True

e) $\mathbf{r}(b-t)$ for $0 \leq t \leq b-a$ is also a parameterization of C with the opposite orientation

True

Things to know how to parametrize

1) lines / line segment

2) circles / ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\vec{r}(t) = \langle a\cos(t), b\sin(t) \rangle$$

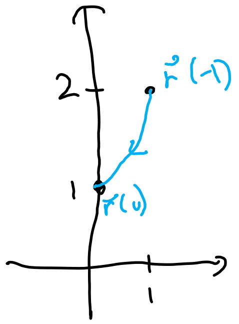
$$\alpha \leq t \leq \beta$$

3) segments $y = f(x)$, $x = g(y)$

$$\downarrow$$

$$\langle t, f(t) \rangle$$

Example 109. Find a parameterization of the curve C that consists of the portion of the curve $y = x^2 + 1$ from $(1, 2)$ to $(0, 1)$ and use it to write the integral $\int_C x^2 + y^2 \, ds$ as an integral with respect to your parameter.



Note: B/c of orientation $\vec{r}_1(\underset{-t}{t}) = \langle t, t^2+1 \rangle$ for $0 \le t \le 1$ $-1 \le t \le 0$

Does not work

1) So take: $\vec{r}(t) = \langle -t, t^2+1 \rangle$ for $-1 \le t \le 0$

$y = t^2 + 1 = (-t)^2 + 1$

2) $|\vec{r}\,'(t)| = |\langle -1, 2t \rangle| = \sqrt{4t^2+1}$

3) $\int_C x^2 + y^2 \, ds = \int_{-1}^{0} \left[(-t)^2 + (t^2+1)^2 \right] \sqrt{4t^2+1} \, dt$

OR: use $\vec{r}_1(t)$ and $-\int_C f \, ds = \int_{-C} f \, ds$.

Daily Announcements & Reminders:

- HW 16.1 due tonight
- HW 16.2 due next Tuesday
- Take advantage ~~of~~ weekly PLUS

CULC 125 { 5-6 pm Sundays
7-8 pm Tuesdays

5-6pm Monday } CULC 278
6-7pm Wednesday }

Goals for Today: Section 16.2

- Define and explore vector fields
- Define tangential and normal line integrals for vector fields
- Apply vector line integrals to problems involving work, flow, and flux
- Compute vector line integrals using parameterizations

Vector Fields:

Definition 110. A vector field is a function $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ which associates a vector to every point in its domain. $\vec{F}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$

Examples:

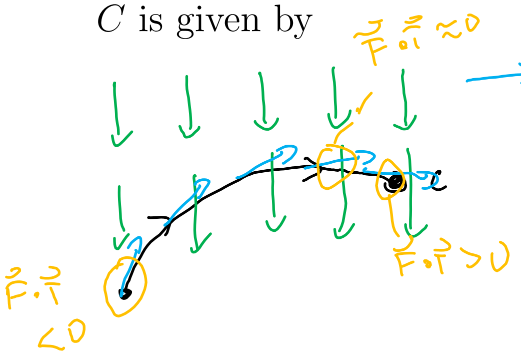
- Velocity field of a flowing fluid
- Slope field for a D.E.
- Electromagnetic field (force field)
- Tangent vectors on a curve or surface
- $\nabla f = \langle f_x, f_y, f_z \rangle$

Graphically: For each point (a, b, c) in the domain of $\mathbf{F}$, draw the vector $\mathbf{F}(a, b, c)$ with its base at (a, b, c).

Tools: CalcPlot3d
Field Play

Idea: In many physical processes, we care about the total sum of the strength of that part of a field that lies either in the direction of a curve or perpendicular to that curve.

1. The work done by a field **F** on an object moving along a curve C is given by



$\rightarrow \vec{T}$ unit tangent

component of $\vec{F}$ in dir. of C is $\boxed{\vec{F}\cdot\vec{T}}$

$$\text{Work done} = \int_C \vec{F}\cdot\vec{T}\, ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}\, |\vec{r}\,'(t)|\, dt$$

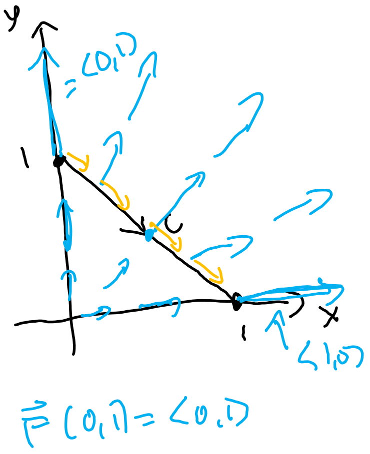
$\vec{r}$ parametrizes C

$$= \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\, dt \quad ⊗$$

$$= \int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz$$

Example 111. Work Done by a Field. Suppose we have a force field $\mathbf{F}(x,y) = \langle x,y\rangle$ N. Find the work done by **F** on a moving object from $(0,1)$ to $(1,0)$ in a straight line, where x,y are measured in meters.

$\langle 1,0\rangle - \langle 0,1\rangle = \langle 1,-1\rangle$



1) Parameterize C: $\vec{r}(t) = \langle 1,-1\rangle t + \langle 0,1\rangle \quad 0 \le t \le 1$

2) Find $\vec{r}\,'(t)$: $\vec{r}\,'(t) = \langle 1,-1\rangle$

3) Substitute:

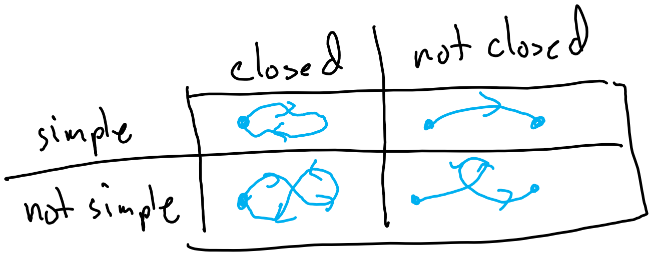
$\vec{F}(\vec{r}(t)) = \langle t, 1-t\rangle$

$$\text{work done} = \int_C \vec{F}\cdot\vec{T}\, ds$$
$$= \int_0^1 \langle t, 1-t\rangle \cdot \langle 1,-1\rangle\, dt$$
$$= \int_0^1 t - (1-t)\, dt$$
$$= \int_0^1 2t - 1\, dt = t^2 - t\Big|_0^1 = 0$$

should not be vector

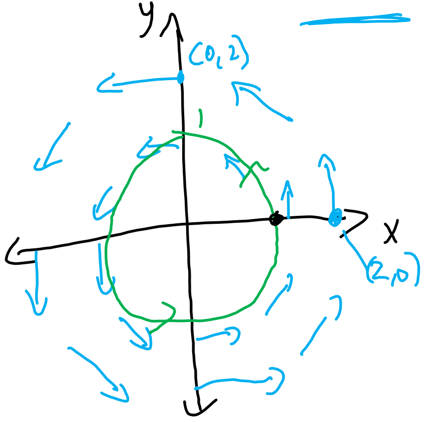
1. The flow along a curve C of a velocity field $\mathbf{F}$ for a fluid in motion is given by $\int_C \vec{F}\cdot\vec{T}\,ds$

When C is closed, this is called circulation. C is called simple if it does not intersect itself.



$\vec{F}(x,y,z)$ depends only position

Example 112. Flow of a Velocity Field. Find the circulation of the velocity field $\mathbf{F}(x, y) = \langle -y, x\rangle$ cm/s around the unit circle, parameterized counterclockwise.



$\vec{F}(2,0) = \langle -0, 2\rangle$
$\vec{F}(0,2) = \langle -2, 0\rangle$

Goal: $\int_C \vec{F}\cdot\vec{T}\,ds$

1) Parameterize C: $\vec{r}(t) = \langle \cos(t), \sin(t)\rangle$, $0 \le t \le 2\pi$

2) Compute $\vec{r}\,'(t)$: $\vec{r}\,'(t) = \langle -\sin(t), \cos(t)\rangle$

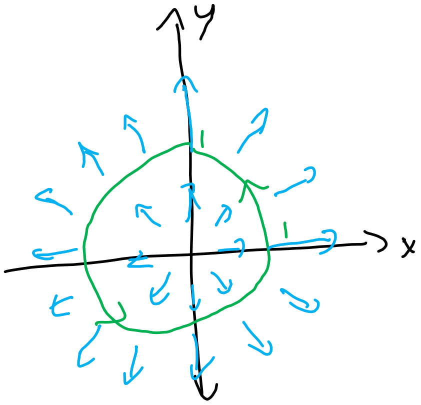
3) Substitute

$$\text{circulation} = \int_C \vec{F}\cdot\vec{T}\,ds = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$
$$= \int_0^{2\pi} \langle -\sin(t), \cos(t)\rangle \cdot \langle -\sin(t), \cos(t)\rangle\,dt$$
$$= \int_0^{2\pi} \sin^2(t) + \cos^2(t)\,dt$$
$$= 2\pi \text{ cm}^2/\text{s}$$

Q: What is the clockwise circulation of the field around the unit circle?

-2π

Example 113. (Piazza Poll) What is the circulation of $\mathbf{F}(x, y) = \langle x, y\rangle$ around the unit circle, parameterized counterclockwise?



$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_0^{2\pi} \overbrace{\langle \cos(t), \sin(t)\rangle \cdot \langle -\sin(t), \cos(t)\rangle}^{\text{orthogonal}}\,dt$$

$$= \int_0^{2\pi} 0\,dt$$

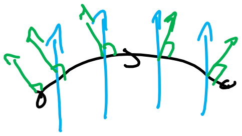
$$= 0$$

Strategy for computing tangential component line integrals

e.g. work, flow, circulation integrals

1. Find a parameterization $\mathbf{r}(t)$, $a \le t \le b$ for the curve C.
2. Compute $\mathbf{r}'(t)$.
3. Substitute: $\underline{\int_C \mathbf{F}\cdot\mathbf{T}\,ds} = \underline{\int_C \mathbf{F}\cdot d\mathbf{r}} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt$
4. Integrate

Idea: flux across a plane curve of a 2D-vector field measures the flow of the field across that curve (instead of along it).



We compute this with the integral

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds.$$

↑ unit normal

The sign of the flux integral tells us whether the net flow of the field across the curve is in the direction of $\vec{n}$ or in the opposite direction.

$\int_C \vec{F} \cdot \vec{n}\, ds$

$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{\langle y'(t), -x'(t)\rangle}{|r'(t)|} |\vec{r}'(t)|\, dt$

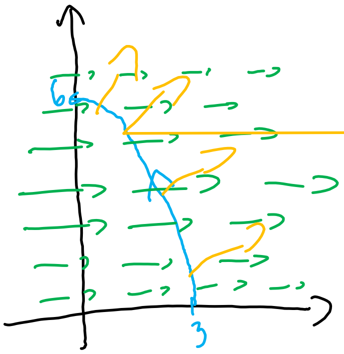
$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \langle y'(t), -x'(t)\rangle\, dt$

We can choose **n** to be either of

$\vec{n} = \frac{\langle y'(t), -x'(t)\rangle}{|\vec{r}'(t)|}$ – usual choice or $\frac{\langle -y'(t), x'(t)\rangle}{|\vec{r}'(t)|}$

($\vec{r}(t) = \langle x(t), y(t)\rangle$ parameterizes C)

Example 114. Flux of a Velocity Field. Compute the flux of the velocity field $\mathbf{v} = \langle 3 + 2y - y^2/3, 0\rangle$ cm/s across the quarter of the ellipse $\frac{x^2}{9} + \frac{y^2}{36} = 1$ in the first quadrant.



1) Parametrize: $\vec{r}(t) = \langle 3\cos(t), 6\sin(t)\rangle$, $0 \leq t \leq \frac{\pi}{2}$

2) Get normal: $\vec{r}'(t) = \langle -3\sin(t), 6\cos(t)\rangle$

$\vec{n} = \langle y', -x'\rangle$

$= \langle 6\cos(t), 3\sin(t)\rangle$

3) Substitute:

$\text{flux} = \int_0^{\pi/2} \langle 3 + 2(6\sin(t)) - \frac{(6\sin(t))^2}{3}, 0\rangle \cdot \langle 6\cos(t), 3\sin(t)\rangle\, dt$

$u = \sin(t)$

$= 30$

Strategy for computing normal component line integrals

e.g. flux integrals

1. Find a parameterization $\mathbf{r}(t)$, $a \leq t \leq b$ for the curve C.
2. Compute $x'(t)$ and $y'(t)$ and determine which normal to work with.
3. Substitute: $\int_C \mathbf{F} \cdot \mathbf{n}\ ds = \pm \int_a^b P(\mathbf{r}(t))y'(t) - Q(\mathbf{r}(t))x'(t)\ dt$ (sign based on choice of normal)
4. Integrate

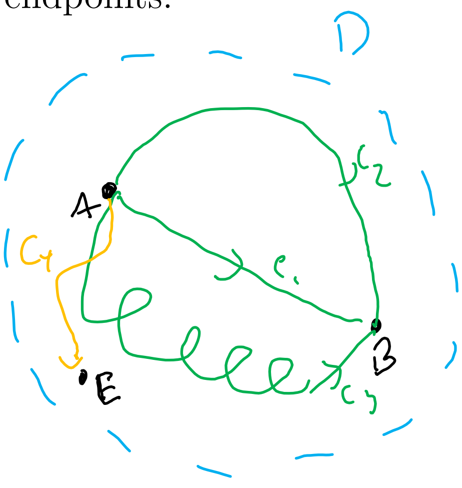
Daily Announcements & Reminders:

- HW 16.2 due tonight, 16.3 Due Th.
- Quiz 9 tomorrow on 16.1, 16.2
- Do warmup Piazza poll

Goals for Today: Section 16.3

- Define conservative vector fields and recognize examples from physics
- Learn how to check if a field is conservative
- Compute potential functions
- Apply the Fundamental Theorem of Line Integrals to compute line integrals of conservative vector fields

Definition 115. A vector field **F** is **path independent** on an open region D if $\int_C \vec{F}\cdot\vec{T}\,ds$ is the same for all paths C in the region that have the same endpoints.



$$\int_{C_1} \vec{F}\cdot\vec{T}\,ds = \int_{C_2} \vec{F}\cdot\vec{T}\,ds = \int_{C_3} \vec{F}\cdot\vec{T}\,ds$$

$$\neq \int_{C_4} \vec{F}\cdot\vec{T}\,ds$$

- gravitational fields are path ind.
- electrostatic fields are " "
- spring forces are " "

When **F** is path independent, we can use the simplest path from point A to point B to compute a line integral, and will often denote the line integral with points as bounds, e.g.

$$\Bigg[\int_{(0,1,2)}^{(3,1,1)} \mathbf{F}\cdot\mathbf{T}\,ds \qquad \text{or} \qquad \int_{(a,b)}^{(c,d)} \mathbf{F}\cdot d\mathbf{r}.$$

Example 116. If C is any closed path and **F** is path independent on a region containing C, then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = 0$$

Question: Given **F**, how do we tell if it is path independent on a particular region?

??

For example, is $\mathbf{F}(x,y) = \langle x, y\rangle$ a path independent vector field on its domain?

We don't know?

Example 117. (Piazza Poll) Last time, we saw that if C is the unit circle about the origin, oriented counterclockwise, then $\int_C \langle -y, x\rangle \cdot d\mathbf{r} = 2\pi$. From this, we can conclude:

$\vec{F} = \langle -y, x\rangle$ is not path ind. because C is a closed curve and $\int_C \vec{F}\cdot\vec{T}\,ds = 2\pi \neq 0$

$\vec{F}(x,y) = \langle f_x, f_y \rangle$

A different idea: Suppose **F** is a gradient vector field, i.e. **F** = ∇f for some function of multiple variables f. f is called a potential for **F**. In this case we also say that **F** is **conservative.**

$\boxed{f(x,y) = \frac{x^2}{2} + \frac{y^2}{2} + C}$

e.g. Is $\vec{F} = \langle x, y \rangle$ conservative?

Need f s.t. $\nabla f = \vec{F}$ i.e $\langle f_x, f_y \rangle = \langle x, y \rangle$

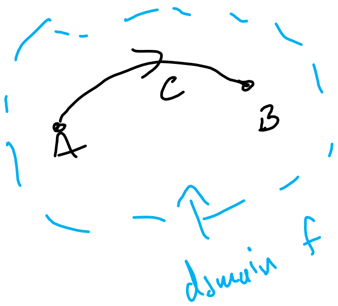
$$\begin{cases} f_x = x \\ f_y = y \end{cases} \longrightarrow f = \int f_x \, dx = \int x \, dx = \frac{x^2}{2} + C(y) = f(x,y)$$

$$y = f_y = \frac{\partial}{\partial y}\left(\frac{x^2}{2} + C(y)\right)$$

$$y = C'(y)$$

$$\int y \, dy = C(y) = \frac{1}{2}y^2 + C$$

Theorem 118 (Fundamental Theorem of Line Integrals)**.** *If C is a smooth curve from the point A to the point B in the domain of a function f with continuous gradient on C, then*

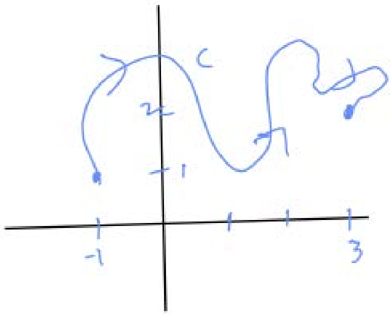


$$\int_C \nabla f \cdot \mathbf{T} \, ds = f(B) - f(A)$$

FTC: $\int_a^b f'(x) dx = f(b) - f(a)$

Example 119. Compute $\int_C \langle x, y \rangle \cdot d\mathbf{r}$ for the curve C shown below from $(-1, 1)$ to $(3, 2)$.

$$\nabla\left(\frac{1}{2}x^2 + \frac{1}{2}y^2\right) = \langle x, y \rangle$$



By FToLI:

$$\int_C \langle x, y \rangle \cdot d\vec{r} = \frac{1}{2}x^2 + \frac{1}{2}y^2 \Big|_{(-1,1)}^{(3,2)}$$

$$= \frac{1}{2}\left((9+4) - (1+1)\right)$$

$$= 11/2$$

It follows that **every conservative field is path independent.**

In fact, by carefully constructing a potential function, we can show the converse is also true: every path independent field is conservative

This leads to a better way to test for path-independence and a way to apply the FToLI.

Curl Test for Conservative Fields: Let $\mathbf{F} = P\mathbf{i}+Q\mathbf{j}+R\mathbf{k}$ be a vector field defined on a **simply-connected** region. If $\operatorname{curl}\mathbf{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle = \langle 0,0,0\rangle$, then $\mathbf{F}$ is conservative.

If $\vec{F}=\nabla f$, $\vec{F}=\langle f_x, f_y, f_z\rangle$

$= \langle$

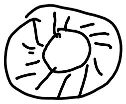
$\vec{F}=\langle P, Q, 0\rangle$

- If $\mathbf{F}$ is a 2-d vector field, $\operatorname{curl}\mathbf{F} = \langle 0, 0, Q_x-P_y\rangle$
- This is also called the **mixed-partials test**, because

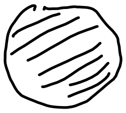
$R_y=Q_z,\ P_z=R_x,\ Q_x=P_y$

simply-connected: no holes

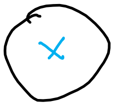
e.g. $\mathbb{R}^2$ or $\mathbb{R}^3$ is simply connected



is not simply connected



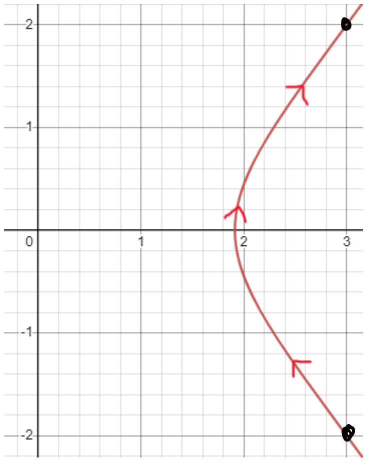
is simply connected



is not simply connected

$\int_C P\, dx + Q\, dy$

Example 120. Evaluate $\int_C (10x^4 - 2xy^3)\ dx - 3x^2y^2\ dy$ where C is the part of the curve $x^5 - 5x^2y^2 - 7x^2 = 0$ from $(3, -2)$ to $(3, 2)$.



Check if $\vec{F}$ is conservative

$$\vec{F} = \langle \underset{P}{10x^4 - 2xy^3}, \underset{Q}{-3x^2y^2} \rangle$$

note domain of $\vec{F}$ is $\mathbb{R}^2$, simply connected

curl Test: Is $\langle 0, 0, Q_x - P_y \rangle = \langle 0, 0, 0 \rangle$?

$Q_x = -6xy^2$

$P_y = -6xy^2$

so $\langle 0, 0, Q_x - P_y \rangle = \langle 0, 0, -6xy^2 + 6xy^2 \rangle = \langle 0, 0, 0 \rangle$

so $\vec{F}$ is conservative.

Find potential $f(x,y)$: $\nabla f = \vec{F}$

$\int f_x\, dx = \int 10x^4 - 2xy^3\, dx = \underline{2x^5} - \underline{x^2y^3} + C(y)$

$\int f_y\, dy = \int -3x^2y^2\, dy = -x^2y^3 + K(x)$

$$\boxed{f(x,y) = 2x^5 - x^2y^3 + C}$$

- Keep every term w/ only x or y and one copy of each term w/ both x & y that appears in both equations

Use FTOLI:

- given integral $= f(3,2) - f(3,-2)$

$= \cancel{2(3)^5} - 3^2(2)^3 - \cancel{2(3)^5} + (3)^2(-2)^3$

$= \boxed{-144}$

Daily Announcements & Reminders:

- HW 16.3 due tonight
- Exam 3 in 2 weeks
- Final in 3 weeks
- Review spherical/cylindrical
- Do warm up poll

Goals for Today: Section 16.4

- Define the divergence and curl of a vector field
- Interpret divergence and curl geometrically
- Apply Green's Theorem to compute line integrals over the boundary of a simply-connected region

Useful notation: $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$

So if $f(x, y, z)$ is a function of three variables, $\nabla f = \left\langle \frac{\partial}{\partial x}(f), \frac{\partial}{\partial y}(f), \frac{\partial}{\partial z}(f) \right\rangle$

If $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ is a vector field:

- $\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(P) + \frac{\partial}{\partial y}(Q) + \frac{\partial}{\partial z}(R)$ • $\mathbb{R}^n$ for any n
- $\nabla \times \mathbf{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left\langle R_y - Q_z, -(R_x - P_z), Q_x - P_y \right\rangle$

  • $\mathbb{R}^3$ only

e.g. $\vec{F} = \langle xy, 2y^2, x+z \rangle$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(2y^2) + \frac{\partial}{\partial z}(x+z)$$

$$= \underline{y} + \underline{4y+1} = \underline{5y+1}$$

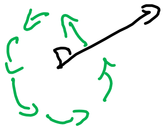
$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\underline{\partial x}} & \frac{\partial}{\underline{\partial y}} & \frac{\partial}{\underline{\partial z}} \\ xy & 2y^2 & x+z \end{vmatrix} = \Big\langle \frac{\partial}{\partial y}(x+z) - \frac{\partial}{\partial z}(2y^2), -\Big(\frac{\partial}{\partial x}(x+z) - \frac{\partial}{\partial z}(xy)\Big), \frac{\partial}{\partial x}(2y^2) - \frac{\partial}{\partial y}(xy) \Big\rangle$$

$$= \langle 0, -1, -x \rangle$$

How do we measure the change of a vector field?

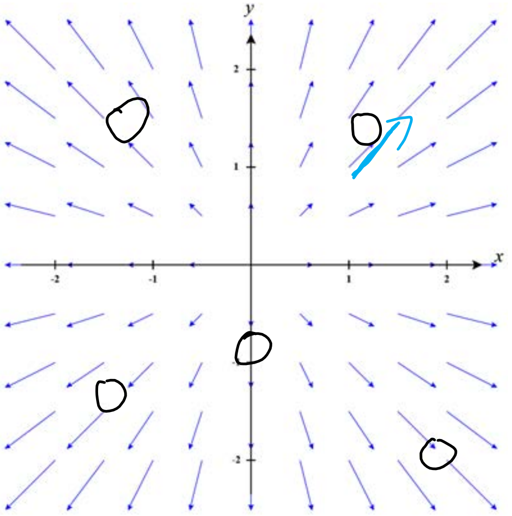
1. **Curl** (in $\mathbb{R}^3$)
    - Tells us circulation density
    - Measures instantaneous / local circulation
    - Is a vector
    - Direction gives RHR oriented axis of rotation
    - Magnitude gives rate of rotation
    - curl **F** = $\nabla \times \vec{F}$

      define scalar curl: $(\nabla \times \vec{F}) \cdot \hat{k} = Q_x - P_y$
    - If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$: we use $\nabla \times \mathbf{F} = \nabla \times \langle P, Q, 0 \rangle = \langle 0, 0, Q_x - P_y \rangle$
    - If $\nabla \times \vec{F} = \vec{0}$, then $\vec{F}$ is conservative: $\vec{F} = \nabla f$ and $\vec{F}$ is called <u>irrotational</u>.



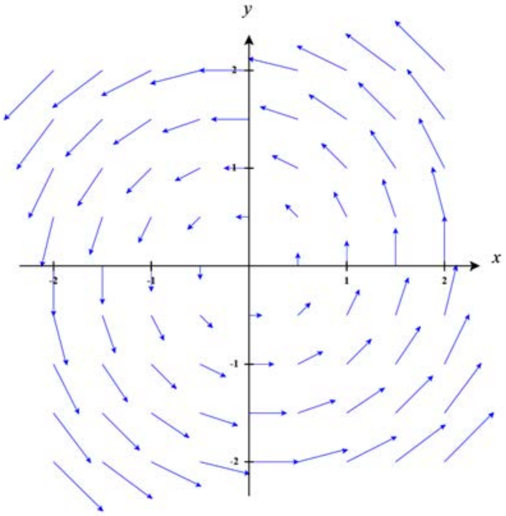
2. **Divergence** (in any $\mathbb{R}^n$)
    - Tells us flux density
    - Measures expansion / compression
    - Is a scalar
    - div **F** = $\nabla \cdot \vec{F} = P_x + Q_y + R_z$ (in $\mathbb{R}^3$)
    - If $\nabla \cdot \vec{F} = 0$, $\vec{F}$ is <u>incompressible</u> and $\vec{F} = \nabla \times \vec{G}$ for some $\vec{G}$.

Example 121. Let $\mathbf{F}(x,y) = \langle x, y\rangle$. Based on the visualization of this vector field below, what can we say about the sign (+,-,0) of the divergence and scalar curl of this vector field? Verify by computing the divergence and curl.



- For each ○, strength of $\vec{F}$ going into ○ is less than going out, so $\nabla \cdot \vec{F} > 0$
- For each ○, push CCW on right is exactly cancelled by push on left CW
- $\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) = 2$
- $(\nabla \times \vec{F}) \cdot \vec{k} = \frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x) = 0$

Example 122. (Piazza Poll) Let $\mathbf{F}(x,y) = \langle -y, x\rangle$. Based on the visualization of this vector field below, what can we say about the sign (+,-,0) of the divergence and curl of this vector field? Verify by computing the divergence and curl.



Question: How is this useful?

Answer: We can relate rates of change of a vector inside a region to the behavior of the vector field on the boundary of the region.

Theorem 123 (Green's Theorem)**.** *Suppose C is a piecewise smooth, simple, closed curve enclosing on its left a region R in the plane. If $\mathbf{F} = \langle P, Q\rangle$ has continuous partial derivatives around R, then*

a) Circulation form:

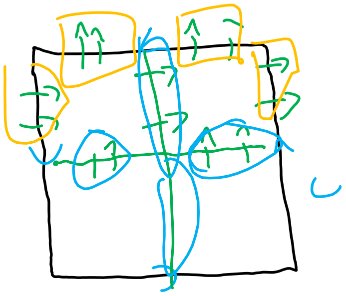
$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_C P\,dx + Q\,dy = \iint_R \overset{\text{curl }\vec{F}}{(\nabla\times\mathbf{F})\cdot\mathbf{k}}\,dA = \iint_R Q_x - P_y\,dA$$

b) Flux form:

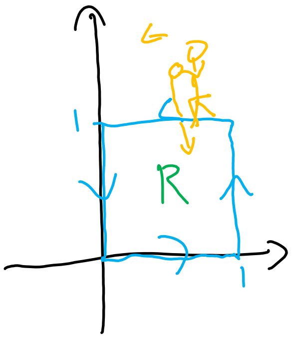
$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_C P\,dy - Q\,dx = \iint_R \overset{\text{div }\vec{F}}{(\nabla\cdot\mathbf{F})}\,dA = \iint_R P_x + Q_y\,dA$$

a) Total circulation of $\vec{F}$ around the boundary of a region is the double integral of the local circulation on the interior

b) Total flux of $\vec{F}$ across the boundary of a region is the double integral of the local 'flux' on the interior .



Example 124. Evaluate the line integral $\int_C \mathbf{F} \cdot \mathbf{T}\ ds$ for the vector field $\mathbf{F} = \langle -y^2, xy \rangle$ where C is the boundary of the square bounded by $x = 0, x = 1, y = 0$, and $y = 1$. oriented CCW

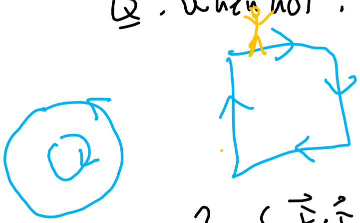


- Direct calculation needs 4 parameterizations
- FToLI: not conservative
- Green's Thm:

$$\langle 0, 0, Q_x - P_y \rangle \cdot \langle 0, 0, 1 \rangle = Q_x - P_y$$

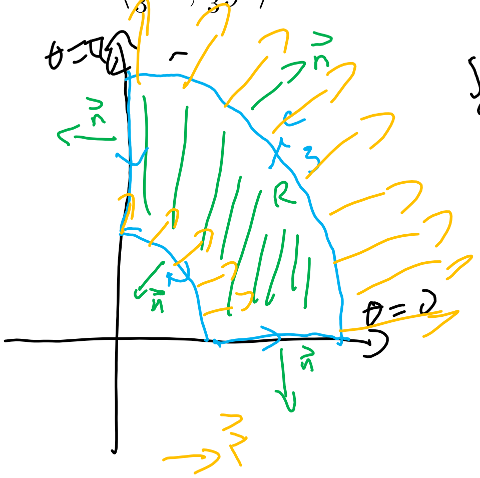
$$\int_C \vec{F} \cdot \vec{T}\, ds = \iint_R (\text{curl } \vec{F}) \cdot \vec{k}\ dA$$
$$= \iint_R 3y\ dA$$
$$= \int_0^1 \int_0^1 3y\ dy\, dx$$
$$= \int_0^1 \frac{3}{2} y^2 \Big|_0^1 dx$$
$$= \boxed{\frac{3}{2}}$$

Q: when not?



$$? \int_{-C} \vec{F} \cdot \vec{T}\, ds = -\int_C \vec{F} \cdot \vec{T}\, ds$$

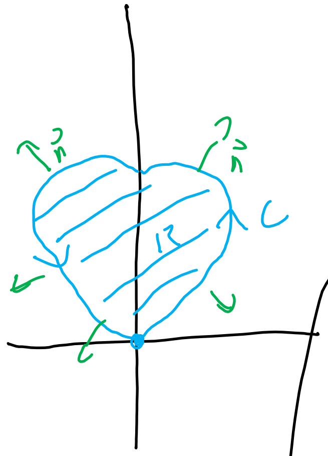
Example 125. Compute the flux out of the region R which is the portion of the annulus between the circles of radius 1 and 3 in the first ~~octant~~ quadrant for the vector field $\mathbf{F} = \langle \frac{1}{3}x^3, \frac{1}{3}y^3 \rangle$.



$$\nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) \cdot \left\langle \frac{1}{3}x^3, \frac{1}{3}y^3 \right\rangle$$
$$= \frac{\partial}{\partial x}\left(\frac{1}{3}x^3\right) + \frac{\partial}{\partial y}\left(\frac{1}{3}y^3\right)$$

$$\int_C F \cdot \vec{n}\, ds = \iint_R \nabla \cdot \vec{F}\ dA$$
$$= \iint_R x^2 + y^2\ dA$$
$$= \int_0^{\pi/2} \int_1^3 r^2 \cdot r\ dr\, d\theta$$
$$= \frac{1}{4} r^4 \Big|_1^3 \cdot \frac{\pi}{2}$$
$$= \frac{20\pi}{2} = \boxed{10\pi}$$

Example 126. Let R be the region bounded by the curve $\mathbf{r}(t) = \langle \sin(2t), \sin(t) \rangle$ for $0 \leq t \leq \pi$. Find the area of R, using Green's Theorem applied to the vector field $\mathbf{F} = \frac{1}{2}\langle x, y \rangle$.



$$\text{area}(R) = \iint_R 1 \, dA$$

} want this :

$$\nabla \cdot \langle \tfrac{1}{2}x, \tfrac{1}{2}y \rangle = \tfrac{1}{2} + \tfrac{1}{2} = 1$$

$$= \iint_R \nabla \cdot \vec{F} \, dA$$

Green's Thm

$$= \int_C \vec{F} \cdot \vec{n} \, ds$$

1) Parameterize: $\vec{r}(t) = \langle \sin(2t), \sin(t) \rangle, \quad 0 \leq t \leq \pi$

2) Get normal: $\vec{r}\,'(t) = \langle 2\cos(2t), \cos(t) \rangle$

$$\vec{n} = \langle y', -x' \rangle = \langle \cos(t), -2\cos(2t) \rangle$$

3) compute:

$$= \int_0^\pi \vec{F}(\vec{r}(t)) \cdot \vec{n}(t) \, dt$$

$$= \int_0^\pi \langle \tfrac{1}{2}\sin(2t), \tfrac{1}{2}\sin(t) \rangle \cdot \langle \cos(t), -2\cos(2t) \rangle \, dt$$

$\cos(2t) = 2\cos^2 t - 1$

$$= \int_0^\pi \tfrac{1}{2}\sin(2t)\cos(t) - \sin(t)\cos(2t) \, dt$$

$$= \int_0^\pi \underline{\sin(t)}\,\underline{\cos^2(t)} - \underline{\sin(t)}\,(\underline{2\cos^2(t) - 1}) \, dt$$

$u = \cos(t) \quad du = -\sin(t)\,dt$

$$= \frac{4}{3}$$

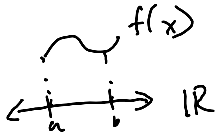
Note: This is the idea behind the operation of the measuring instrument known as a planimeter.

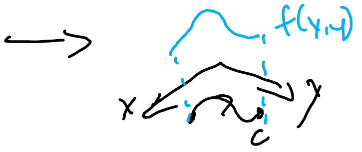
Daily Announcements & Reminders:

- HW 16.4 due tonight
- Quiz 10 tomorrow on 16.3, 16.4
- Exam 3 info tomorrow on Canvas
- Do warmup poll

Goals for Today:

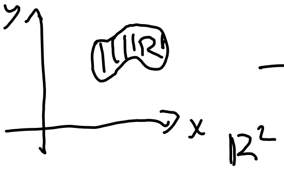
Sections 16.5/16.6

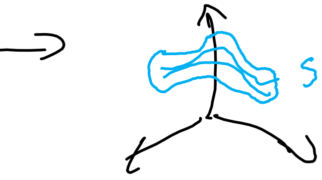




- Describe surfaces in $\mathbb{R}^3$ parametrically
- Define and compute surface integrals
- Use surface integrals to compute meaningful quantities: surface areas, masses, flux, etc.

Now:





Different ways to think about curves and surfaces:

$\iint_R f(x,y)\,dA$

	Curves	Surfaces
Explicit:	$y = f(x)$ $y = \sqrt{4-x^2}$	$z = f(x,y)$ $z = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$
Implicit:	$F(x,y) = 0$ $x^2+y^2=4$	$F(x,y,z) = 0$ $x^2+y^2+z^2=4$
Parametric Form:	$\mathbf{r}(t) = \langle x(t), y(t) \rangle$ $\vec{r}(t) = \langle 2\cos(t), 2\sin(t) \rangle$ $0 \le t \le 2\pi$	$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$ or $\vec{r}(s,t) = \langle x(s,t), y(s,t), z(s,t) \rangle$

Example 127. Give parameteric representations for the surfaces below.

Goal: $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ with domain R in uv-plane

a) $x = y^2 + \frac{1}{2}z^2 - 2$ • elliptic paraboloid

• $\vec{r}(u,v) = \langle u^2 + \frac{1}{2}v^2 - 2, u, v\rangle$ $(u,v) \in \mathbb{R}^2$

y = u

z = v

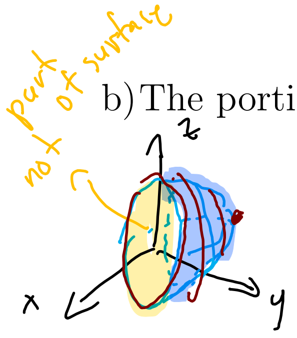
$y = u^2$

$z = v^2$ → $\vec{r}_2(u,v) = \langle u^4 + \frac{1}{2}v^4 - 2, u^2, v^2\rangle$ $(u,v) \in \mathbb{R}^2$

↑ not a parametrization of whole paraboloid

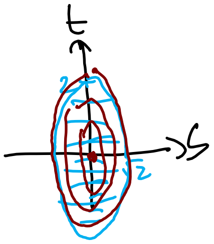
$\vec{r}_3(u,v) = \langle u^{10} + \frac{1}{2}v^{10} - 2, u^5, v^5\rangle$ $(u,v) \in \mathbb{R}^2$

b) The portion of the surface $x = y^2 + \frac{1}{2}z^2 - 2$ which lies $\underbrace{\text{behind the } yz\text{-plane}}_{x < 0}$.



$\vec{r}(s,t) = \langle s^2 + \frac{1}{2}t^2 - 2, s, t\rangle$

$s^2 + \frac{1}{2}t^2 - 2 < 0$ → $\frac{s^2}{2} + \frac{t^2}{4} < 1$



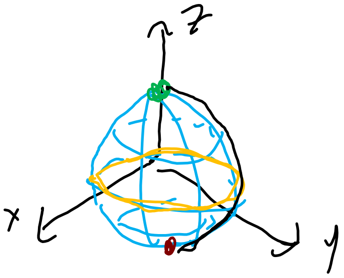
c) $x^2 + y^2 + z^2 = 9$

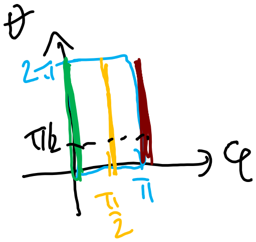
Think in spherical coords: $\rho^2 = 9$ so $\rho = 3$

$\vec{r}(\varphi,\theta) = \langle 3\sin\varphi\cos\theta, 3\sin\varphi\sin\theta, 3\cos\varphi\rangle$

$0 \le \varphi \le \pi$

$0 \le \theta \le 2\pi$



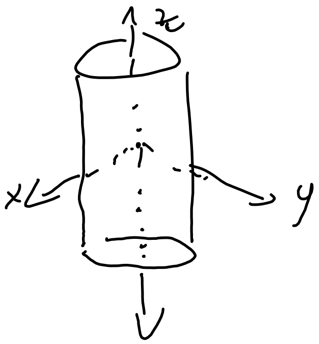


d) $x^2 + y^2 = 25$ → r = 5

$\vec{r}(\theta, z) = \langle 5\cos\theta, 5\sin\theta, z\rangle$

$0 \le \theta \le 2\pi$

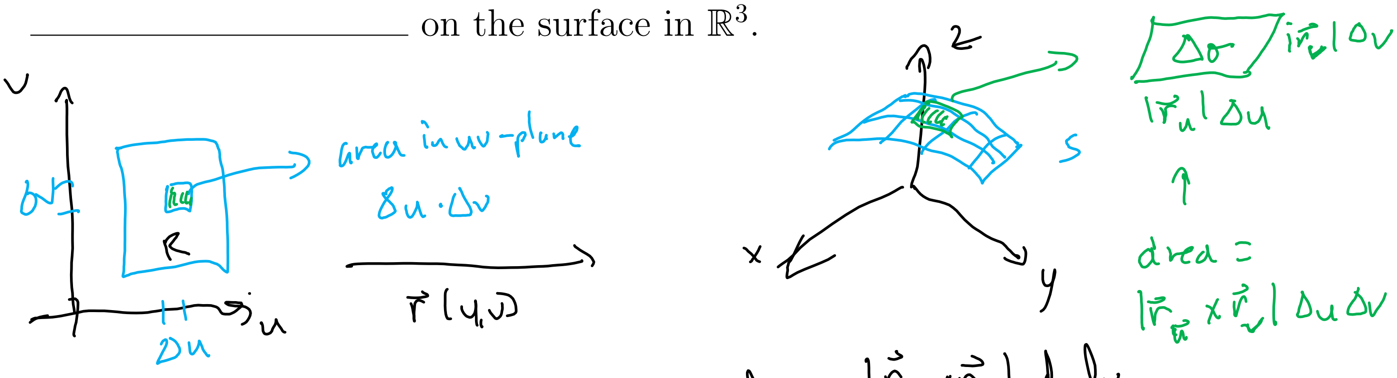
$z \in \mathbb{R}$



What can we do with this? $\vec{r}(u,v)$ is a parameterization of S

If our parameterization is **smooth** ($\mathbf{r}_u, \mathbf{r}_v$ not parallel in the domain), then:

- $\mathbf{r}_u \times \mathbf{r}_v$ is a normal vector to surface
- A rectangle of size $\Delta u \times \Delta v$ in the uv-domain is mapped to a ~~rectangle~~ parallelogram of size ____________ on the surface in $\mathbb{R}^3$.



$$d\sigma = |\vec{r}_u \times \vec{r}_v|\, du\, dv$$

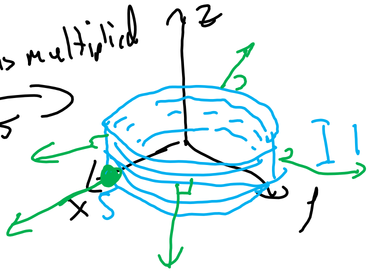
↑ "infinitesimal surface area"

- Thus, Area(S) = $\displaystyle\iint_S 1\, d\sigma = \iint_R |\vec{r}_u \times \vec{r}_v|\, dA$

Example 128. (Piazza Poll) Find the area of the portion of the cylinder $x^2+y^2 = 25$ between $z = 0$ and $z = 1$.

$$\vec{r}(\theta, z) = \langle 5\cos\theta, 5\sin\theta, z\rangle \quad \{0 \le \theta \le 2\pi,\ 0 \le z \le 1\}$$

area is multiplied by 5



$$\vec{r}_\theta = \langle -5\sin\theta, 5\cos\theta, 0\rangle$$

$$\vec{r}_z = \langle 0, 0, 1\rangle$$

✓ $\vec{r}_\theta \times \vec{r}_z = \langle 5\cos\theta, 5\sin\theta, 0\rangle$]

$\vec{r}(0,0) = \langle 5,0,0\rangle$

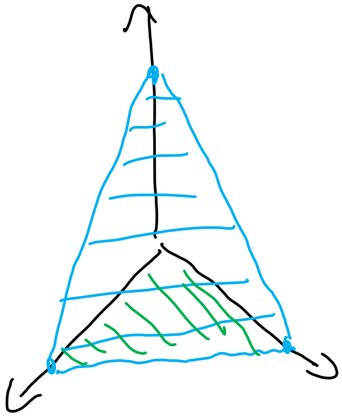
$\vec{r}_\theta \times \vec{r}_z(0,0) = \langle 5,0,0\rangle$

$$|\vec{r}_\theta \times \vec{r}_z| = \sqrt{25\cos^2\theta + 25\sin^2\theta} = 5$$

$$\text{Area}(S) = \iint_S 1\, d\sigma = \underbrace{\int_0^{2\pi}\int_0^1 5\, dz\, d\theta}_{} = 10\pi$$

Example 129. Suppose the density of a thin plate S in the shape of the portion of the plane $\underbrace{x + y + z = 1}_{\langle 1,1,1\rangle}$ in the first octant is $\delta(x, y, z) = 6xy$. Find the mass of the plate.

kg/m^2



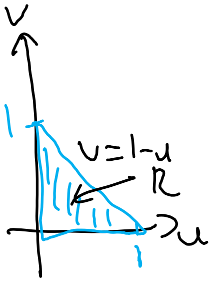
$$\text{mass} = \iint_S \delta(x,y,z)\,d\sigma$$

1) Parameterize S : $z = 1 - x - y$

$$\vec{r}(u,v) = \langle u, v, 1-u-v \rangle$$

$0 \le u \le 1$

$0 \le v \le 1-u$



2) Find $|\vec{r}_u \times \vec{r}_v|$:

$$\vec{r}_u = \langle 1, 0, -1 \rangle$$

$$\vec{r}_v = \langle 0, 1, -1 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle 0-(-1), -(-1-0), 1-0 \rangle = \langle 1, 1, 1 \rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$$

3) Substitute

$$\iint_S \delta(x,y,z)\,d\sigma = \iint_R \delta(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\,dA$$

$$= \int_0^1 \int_0^{1-u} 6uv \cdot \sqrt{3}\,dv\,du$$

$$= \sqrt{3}/4 \text{ kg}$$

Daily Announcements & Reminders:

- HW 16.5 due tonight
- Exam 3 info after class on Canvas (4/18)
- Final Exam is in 2 weeks on 4/25
  ↳ 3 parts, take 0 to 3
  - each part ~ 50 min to complete
  - max of midterm & average of midterm & final part
    is your grade

Goals for Today: Section 16.6/16.7

- Compute flux surface integrals
- Interpret the physical significance of flux surface integrals
- Introduce and apply Stokes' Theorem for surface integrals

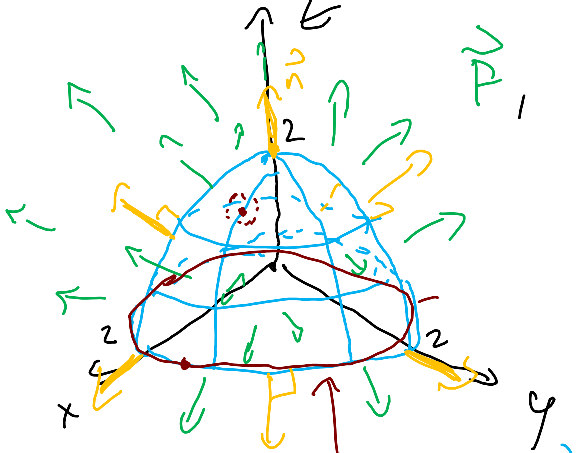
Goal: If $\mathbf{F}$ is a vector field in $\mathbb{R}^3$, find the total flux of $\mathbf{F}$ through a surface S.

Note: If the flux is positive, that means the net movement of the field through S is in the direction of the normal vector to S

If $\mathbf{r}(u,v)$ is a smooth parameterization of S with domain R, we have

$$\text{flux of } \mathbf{F} \text{ through } S = \iint_S \underbrace{(\mathbf{F}\cdot\overset{\text{unit normal}}{\mathbf{n}})}_{\substack{\text{normal component}\\ \text{of } \vec{F} \text{ to } S}}\, d\sigma = \iint_R \underbrace{\mathbf{F}(\mathbf{r}(u,v))\cdot(\overset{\text{not a magnitude!}}{\mathbf{r}_u\times\mathbf{r}_v})\, dA}_{}.$$

Example 130. Find the flux of $\mathbf{F} = \langle x, y, z\rangle$ through the upper hemisphere of $x^2 + y^2 + z^2 = 4$, oriented away from the origin.



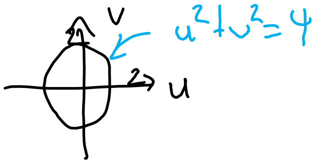
$\vec{C}$ is the circle $x^2+y^2=4$ in xy-plane

$\vec{F}$, $\vec{n}$

1) Parameterize S

a) $z = \sqrt{4-x^2-y^2}$ $-2 \le u \le 2$

$\vec{r}(u,v) = \langle u, v, \sqrt{4-u^2-v^2}\rangle$ $-\sqrt{4-u^2} \le v \le \sqrt{4-u^2}$

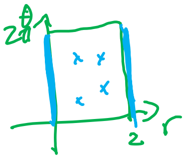


b) $z = \sqrt{4-r^2}$ ⇐

$\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, \sqrt{4-r^2}\rangle$

$0 \le r \le 2$

$0 \le \theta \le 2\pi$



c) $\rho = 2$

$\vec{r}(\varphi,\theta) = \langle 2\sin\varphi\cos\theta, 2\sin\varphi\sin\theta, 2\cos\varphi\rangle$

$0 \le \varphi \le \pi/2$

$0 \le \theta \le 2\pi$

2) Compute $\vec{r}_r \times \vec{r}_\theta$:

$$\vec{r}_r = \left\langle \cos\theta, \sin\theta, \frac{-r}{\sqrt{4-r^2}}\right\rangle$$

$$\vec{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle$$

$$\vec{r}_r \times \vec{r}_\theta = \left\langle 0 + \frac{r^2\cos\theta}{\sqrt{4-r^2}}, -\left(0 - \frac{r^2\sin\theta}{\sqrt{4-r^2}}\right), r\cos^2\theta + r\sin^2\theta \right\rangle$$

$$= \left\langle \frac{+r^2\cos\theta}{\sqrt{4-r^2}}, \frac{r^2\sin\theta}{\sqrt{4-r^2}}, r \right\rangle$$

- Boundaries are iffy, but OK on inside

3) Plug in:

$$\text{flux} = \iint_S \vec{F}\cdot\hat{n}\, d\sigma = \int_0^{2\pi}\int_0^2 \vec{F}(\vec{r}(r,\theta))\cdot(\vec{r}_r \times \vec{r}_\theta)\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 \langle r\cos\theta, r\sin\theta, \sqrt{4-r^2}\rangle \cdot \left\langle \frac{r^2\cos\theta}{\sqrt{4-r^2}}, \frac{r^2\sin\theta}{\sqrt{4-r^2}}, r\right\rangle dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 \frac{r^3}{\sqrt{4-r^2}}\cos^2\theta + \frac{r^3}{\sqrt{4-r^2}}\sin^2\theta + r\sqrt{4-r^2}\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 \frac{r^3}{\sqrt{4-r^2}} + r\sqrt{4-r^2}\, dr\, d\theta \,,$$

$$\frac{r^3}{\sqrt{4-r^2}} + r\sqrt{4-r^2} \quad dr$$

$$\left(\frac{r^2}{\sqrt{4-r^2}} + \sqrt{4-r^2}\right) \boxed{r\,dr}$$

$$\left(\frac{4-u}{\sqrt{u}} + \sqrt{u}\right)\left(-\frac{1}{2}du\right)$$

$u = 4 - r^2 \rightarrow$ $r=0 \rightarrow u=4$

$\downarrow$ $\boxed{du = -2r\,dr}$ $r=2 \rightarrow u=0$

$r^2 = 4-u$

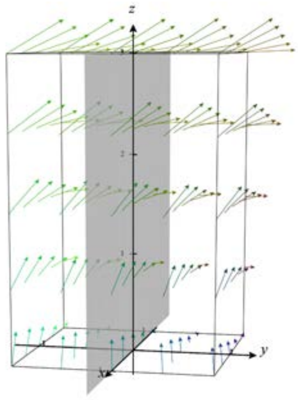
$$= 2\pi \int_4^0 \left(\frac{4-u}{\sqrt{u}} + \sqrt{u}\right) -\frac{1}{2}du$$

$$= \pi \int_0^4 4u^{-1/2} - \cancel{u^{1/2}} + \cancel{u^{1/2}} \quad du$$

$$= \pi \cdot 8u^{1/2}\Big|_0^4 = \boxed{16\pi}$$

Example 131. (Piazza Poll) Suppose S is a smooth surface in $\mathbb{R}^3$ and $\mathbf{F}$ is a vector field in $\mathbb{R}^3$. **True or False:** If $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma > 0$, then the angle between $\mathbf{F}$ and $\mathbf{n}$ is acute at all points on S.

Example 132. (Piazza Poll) Based on the plot of the vector field $\mathbf{F}$ and the surface S below, oriented in the positive y-direction, is the flux integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$ positive, negative, or zero?

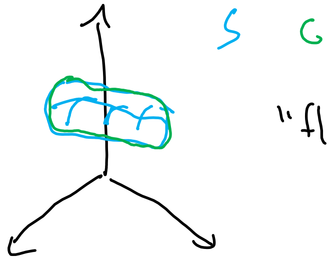


Recall: If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field, we defined its:

1. *divergence:* $\nabla \cdot \mathbf{F} = P_x + Q_y + R_z$
2. *curl:* $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$

Example 133. (Piazza Poll) Suppose $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field in $\mathbb{R}^3$ with continuous partial derivatives. Compute the divergence of the curl of $\mathbf{F}$, i.e. $\nabla \cdot (\nabla \times \mathbf{F})$.

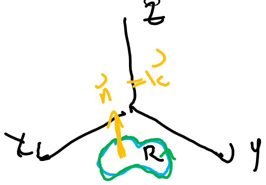
Theorem 134 (Stokes' Theorem)**.** *Let S be a smooth oriented surface and C be its compatibly oriented boundary. Let $\mathbf{F}$ be a vector field with continuous partial derivatives. Then*



$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \int_C \mathbf{F} \cdot \mathbf{T}\, ds.$$

"flux of curl of $\vec{F}$ through S" = "circulation of $\vec{F}$ around the boundary of S"

- If S is a region R in the xy-plane, then we get:



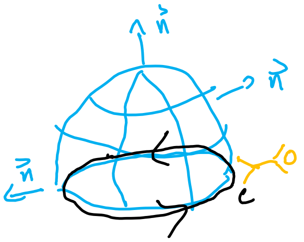
$$\iint_R (\nabla \times \vec{F}) \cdot \hat{k}\, dA = \int_C \vec{F} \cdot \vec{T}\, ds$$ Green's Thm!

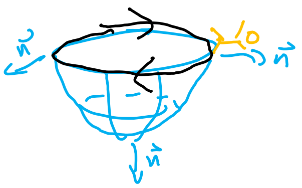
has no boundary

- An **oriented surface** is one where the normal vectors are consistent on all of S
    - Möbius strip is non-orientable

- S and C are oriented compatibly if:

walking along C in its orientation with your head in direction of $\vec{n}$ to S results in S being on your left





Daily Announcements & Reminders:

- Do warm up poll
- Exam 3 in class Thursday
- Studypalooza 4/24 11-1 in Clough 144
- Final Exam 4/25, 2:40-5:30pm, Howey L3 (usual room)
  - survey for part 1/part 2 today
  - info this week
- HW: 16.6 Due tonight, 16.7/8 due Th, Practice Sets due T

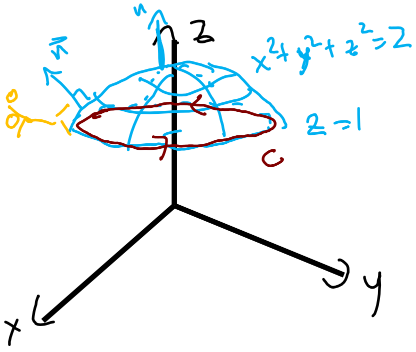
Goals for Today: Section 16.7/16.8

- Apply Stokes' Theorem to flux integral problems.
- Use Stokes' Theorem to simplify flux integrals
- Introduce and apply the Divergence Theorem to flux integral problems

Theorem 135 (Stokes' Theorem)**.** *Let S be a smooth oriented surface and C be its compatibly oriented boundary. Let $\mathbf{F}$ be a vector field with continuous partial derivatives. Then*

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma = \int_C \mathbf{F} \cdot \mathbf{T} \, ds.$$

Example 136 (DD). Let $\mathbf{F} = \langle -y, x+(z-1)x^{x\sin(x)}, x^2+y^2\rangle$. Find $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,d\sigma$ over the surface S which is the part of the sphere $x^2+y^2+z^2 = 2$ above $z = 1$, oriented away from the origin.



Option 1) Parameterize S, compute $\nabla\times\vec{F}$, compute $\vec{r}_u\times\vec{r}_v$, Substitute:

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x+(z-1)x^{x\sin(x)} & x^2+y^2 \end{vmatrix}$$

$$= \langle 2y - x^{x\sin(x)}, \quad \ldots, \quad \underbrace{\frac{\partial}{\partial x}\left(x+(z-1)x^{x\sin(x)}\right)\cdot 1}\ \rangle$$

↑ very hard

Option 2: Use Stokes' Thm

$$\iint_S (\nabla\times\vec{F})\cdot\vec{n}\,d\sigma = \int_C \vec{F}\cdot\vec{T}\,ds$$

- Orient C: CCW
- Parameterize C: $x^2+y^2+\underset{z=1}{\underset{\uparrow}{1^2}} = 2 \rightarrow \begin{matrix} x^2+y^2=1 \\ z=1 \end{matrix}$

$\rightarrow \vec{r}(t) = \langle \cos(t), \sin(t), 1\rangle \qquad 0 \le t \le 2\pi$

$\vec{r}\,'(t) = \langle -\sin(t), \cos(t), 0\rangle$

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_0^{2\pi} \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

$$= \int_0^{2\pi} \langle -\sin(t), \cos(t) + 0, \overset{1}{\cos^2(t)+\sin^2(t)}\rangle\cdot\langle -\sin(t), \cos(t), 0\rangle\,dt$$

$$= \int_0^{2\pi} \sin^2(t) + \cos^2(t)\,dt$$

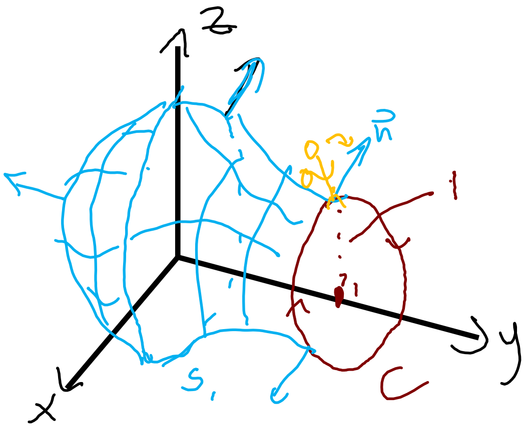
$$= 2\pi$$

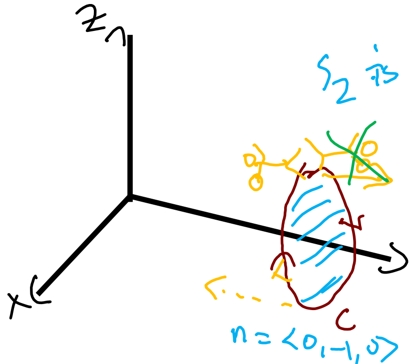
Question: What can we say if two different surfaces S_1 and S_2 have the same oriented boundary C?

$$\iint_{S_1} (\nabla \times \vec{F})\cdot\vec{n}\, d\sigma = \int_C \vec{F}\cdot\vec{T}\, ds = \iint_{S_2} (\nabla\times\vec{F})\cdot\vec{n}\, d\sigma$$

y=1 in S_2

Example 137. Suppose curl $\mathbf{F} = \langle y^{y^y}\sin(z^2), (y - \cancel{1})e^{x^{x^x}} + 2, -ze^{x^{x^x}}\rangle$. Compute the net flux of the curl of $\mathbf{F}$ over the surface pictured below, which is oriented outward and whose boundary curve is a unit circle centered on the y-axis in the plane $y = 1$.





S_2 is disk of radius 1 in y=1

Goal: Choose S_2 so that) is nice

don't know $\vec{F}$

$$\int_C \vec{F}\cdot\vec{T}\, ds = \iint_{S_2} (\nabla\times\vec{F})\cdot n\, d\sigma$$

$$= \iint_{S_2} (\nabla\times\vec{F})\cdot\langle 0,-1,0\rangle\, d\sigma$$

$$= \iint_{S_2} 0 + 2(-1) + 0\, d\sigma$$

y-coord of $\vec{F}$ in plane y=1

$$= -2\boxed{\iint_{S_2} d\sigma}$$

$$= -2(\pi)$$

area of S_2

$$\iint_{S_1} (\nabla\times\vec{F})\cdot\vec{n}\, d\sigma =$$

S_1 impossible to parameterize

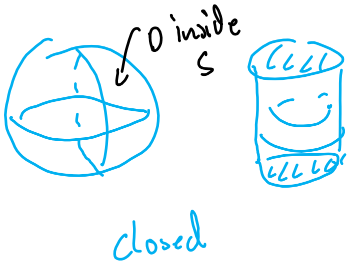
$\nabla\times\vec{F}$ looks hard

$$\vec{r}(u,v) = \langle \sim, 1, \sim\rangle$$

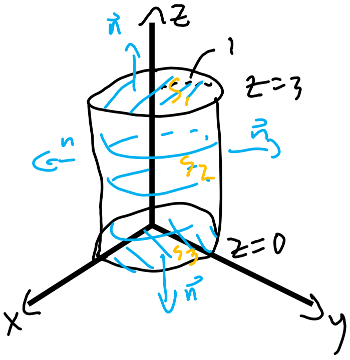
Theorem 138 (Divergence Theorem)**.** *Let S be a closed surface oriented outward, D be the volume inside S, and $\mathbf{F}$ be a vector field with continuous partial derivatives. Then*

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iiint_D \nabla\cdot\mathbf{F}\,dV.$$

"net flux of $\vec F$ out of $\underset{\uparrow\ \text{closed}}{S}$" = "sum of local flux of F inside surface"



Example 139. Let $\mathbf{F} = \langle \overset{P}{y^{1234}e^{\sin(yz)}}, \overset{Q}{y - x^{z^x}}, \overset{R}{z^2}$ [redacted] $- z\rangle$ and S be the surface consisting of the portion of cylinder of radius 1 centered on the z-axis between $z = 0$ and $z = 3$, together with top and bottom disks, oriented outward. Find the flux of $\mathbf{F}$ through S.



$$\iint_S \vec F\cdot\vec n\,d\sigma = \iint_{S_1}\vec F\cdot\vec n\,d\sigma + \iint_{S_2}\vec F\cdot\vec n\,d\sigma + \iint_{S_3}\vec F\cdot\vec n\,d\sigma$$

- tedious & hard

$$= \iiint_D \overset{\to\ P_x+Q_y+R_z}{\vec\nabla\cdot\vec F}\,dV$$

$$= \iiint_D 0 + 1 + (2z-1)\,dV$$

$$= \iiint_D 2z\,dV$$

$$= \int_0^{2\pi}\int_0^1\int_0^3 2z\cdot r\,dz\,dr\,d\theta$$

$$= 9\pi$$

Stokes Theorem:

$$\int_{\partial D} f = \int_D \partial f$$

Div Thm: D is a closed 3d region

∂D is a closed surface

f is a vector field

∂f is $\nabla \cdot \vec{F}$

$$f(b)-f(a) = \int_a^b f'(x)\,dx$$

FTC: D is $[a, b]$

∂D is $\{b, \underline{a}\}$

f is $f(x)$

$\partial f = f'(x)$