MATH 2551 K - Dr. Hunter Lehmann

- Dr. Lehmann, Dr. H, Dr. Hunter, as you prefer

Daily Announcements & Reminders:

- Welcome! Meet your neighbors
- Practice Quiz 0 in studio tomorrow
- HW 12.2/12.3 due on Th at 11:59 pm
- Pearson MML is not required
  - Go to PLUS

Goals for Today: Sections 12.1, 12.3, 12.4

- Set classroom norms
- Syllabus highlights
- Describe the big-picture goals of the class
- Review $\mathbb{R}^3$ and the dot product
- Introduce the cross product and its properties

Class Values/Norms:

- Mistakes are a learning opportunity
- Mathematics is collaborative
- Make sure everyone is included ⟸ explain thoroughly
- Criticize ideas, not people
- Be respectful of everyone
- Ask questions.
- Creativity is valued

Big Idea: Extend differential & integral calculus.

Why Linear Algebra?
- not just 2D
- inputs/outputs as vectors

What are some key ideas from these two courses?

<- FTC ->

Differential Calculus

- Optimization of functions
- Computing derivatives
- Mean Value Theorem
- Derivative = slope = Inst. Rate of Change
- Limits <- L'Hospital
- IVT

Integral Calculus

- Integral <=> area under a curve
- Volumes of revolution
- Sums / convergence
- Integration techniques
- Polar coord.
- Parametric equations

Before: we studied **single-variable functions** $f : \mathbb{R} \to \mathbb{R}$ like $f(x) = 2x^2 - 6$.

domain codomain

Now: we will study **multi-variable functions** $f : \mathbb{R}^n \to \mathbb{R}^m$: each of these functions is a rule that assigns one output vector with m entries to each input vector with n entries.

$f : \mathbb{R}^2 \to \mathbb{R}^2$ translation by $\langle 1, 0 \rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$f(x,y) = \cancel{\begin{bmatrix} x \\ 0 \end{bmatrix}}$

$= \begin{bmatrix} x+1 \\ y \end{bmatrix}$

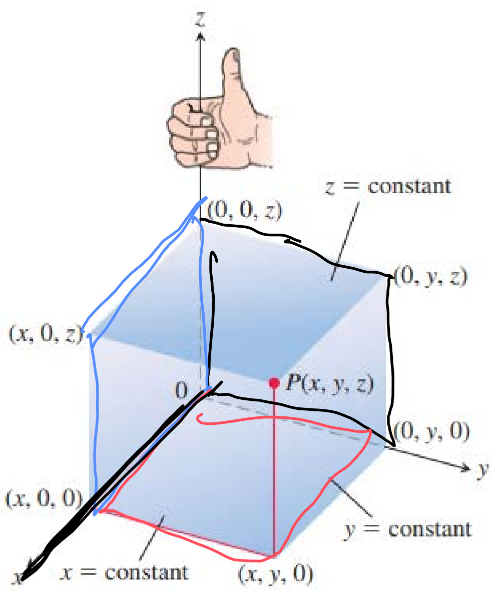
$g(x,y) = x^2 + y$

$g : \mathbb{R}^2 \to \mathbb{R}$

$h(t) = \langle \cos(t), \sin(t) \rangle$

$h : \mathbb{R} \to \mathbb{R}^2$

Section 12.1: Three-Dimensional Coordinate Systems



- Right-handed system
- $x=0$ (yz)-plane
  $y=0$ (xz)-plane
  $z=0$ (xy)-plane
- x-axis: $y=z=0$

Spheres

- $(x-1)^2+(y-2)^2+(z+1)^2=4$
  sphere of radius 2 centered at $(1,2,-1)$
  $(x-4)^2+(y+4)^2+(z-8)^2=9$
  ? sphere of radius 3 centered at $(4,-4,8)$

Question: What shape is the set of solutions $(x, y, z) \in \mathbb{R}^3$ to the equation $x^2+y^2 = 1$?

- radius 1 circle in xy-plane, centered at origin
- infinite circular cylinder, oriented in z-direction

Section 12.3/4: Dot & Cross Products

Definition 1. The **dot product** of two vectors $\mathbf{u} = \langle u_1, u_2, \ldots, u_n \rangle$ and $\mathbf{v} = \langle v_1, v_2, \ldots, v_n \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 \cdot v_1 + u_2 v_2 + \ldots + u_n v_n$$

This product tells us about orthogonality.

- $\vec{u} \cdot \vec{v}$ is a scalar
- related to projection
- $\vec{u} \cdot \vec{v} = 0 \iff \vec{u}, \vec{v}$ are orthogonal

In particular, two vectors are **orthogonal** if and only if their dot product is 0.

Example 2. Are $\mathbf{u} = \langle 1, 1 \rangle$ and $\mathbf{v} = \langle 2, -1 \rangle$ orthogonal?

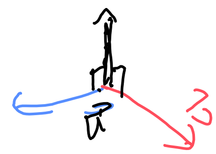
$\vec{u} \cdot \vec{v} = 1 \cdot 2 + 1(-1) = 1$

If $\vec{w}$ is $\perp$ to $\vec{u}$ (and $\vec{w} \neq \vec{0}$)

$\vec{w} = \langle -1, 1 \rangle$, then $\vec{u} \cdot \vec{w} = -1 + 1 = 0$

$\langle -\lambda, \lambda \rangle$

Goal: Given two vectors, in $\mathbb{R}^3$ produce a vector orthogonal to both of them in a "nice" way.



$\vec{u} \times \vec{v}$

1. Right-handed

2. Scalar multiplication & addition work

$a(\vec{u} \times \vec{v}) = (a\vec{u}) \times \vec{v} = \vec{u} \times (a\vec{v})$

$(\vec{u}+\vec{v}) \times \vec{w} = (\vec{u} \times \vec{w}) + (\vec{v} \times \vec{w})$

Definition 3. The **cross product** of two vectors $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ <u>in $\mathbb{R}^3$</u> is

$\vec{i} = \langle 1,0,0\rangle = \vec{e}_1$

$\vec{j} = \langle 0,1,0\rangle = \vec{e}_2$

$\vec{k} = \langle 0,0,1\rangle = \vec{e}_3$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

- $\vec{i} \times \vec{j} = \vec{k}$
  $\vec{j} \times \vec{k} = \vec{i}$
  $\vec{k} \times \vec{i} = \vec{j}$
- $\vec{j} \times \vec{i} = -\vec{k}$

- Anticommutative
  $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$

Example 4. Find $\langle 1, 2, 0\rangle \times \langle 3, -1, 0\rangle$.

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ 3 & -1 & 0 \end{vmatrix} = \vec{i}\begin{vmatrix} 2 & 0 \\ -1 & 0 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(-1-6)$$

$$= -7\vec{k}$$

Daily Announcements & Reminders:

- 12.2, 12.3 HW due tonight
- Useful formulas:

$$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$$

$$|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta$$

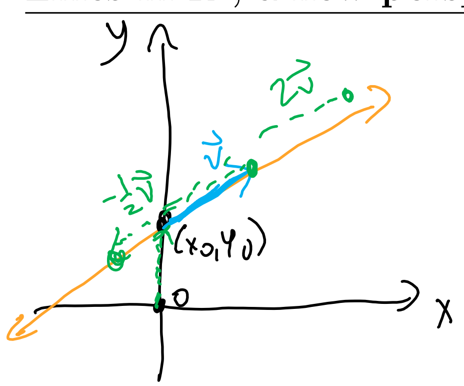
- Try the warmup

itempool.com/hlehmann3/live

Goals for Today: Sections 12.5

- Apply the cross product to solve problems
- Learn the equations that describe lines and planes in $\mathbb{R}^3$
- Solve problems involving the equations of lines and planes

Lines in $\mathbb{R}^2$, a new perspective:



- slope $m = \frac{\text{rise}}{\text{run}}$ = rate of y wrt x
- point (x_0, y_0)

$$y - y_0 = m(x - x_0)$$

Now:

- point: $(x_0, y_0) = P$
- direction vector: $\vec{v}$

line: $\boxed{\overrightarrow{OP} + t\vec{v} = \vec{r}(t)}$

- Any 2 points in space determine a unique line

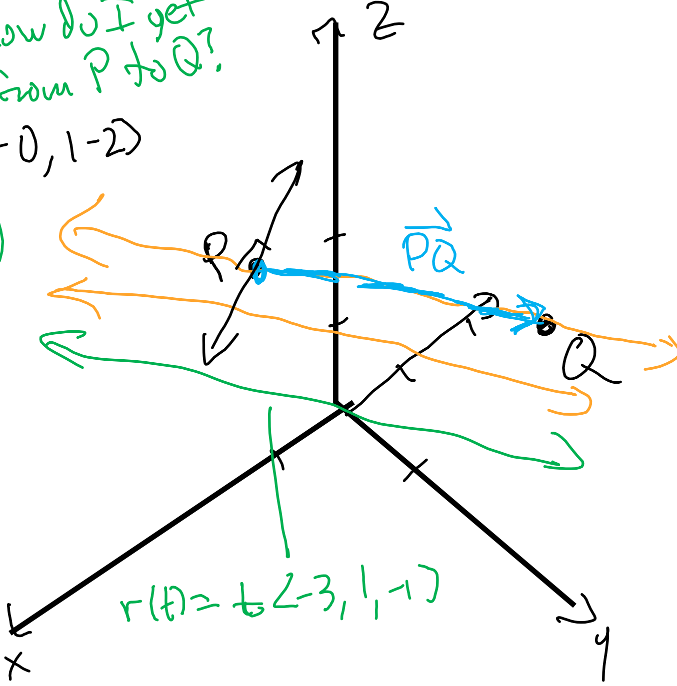
Example 5. Find a vector equation for the line that goes through the points $P = (1, 0, 2)$ and $Q = (-2, 1, 1)$.

how do I get from P to Q?

<u>Need</u>: <u>point</u>: $(1,0,2)$

<u>direction</u>: $\overrightarrow{PQ} = \langle -2-1, 1-0, 1-2 \rangle$

$= \langle -3, 1, -1 \rangle$



vector equation:

$\vec{r}(t) = \cancel{(1,0,2)} + t\langle -3, 1, -1 \rangle$

$= \langle -3t+1, t, -t+2 \rangle$

OR

$\vec{r}_2(t) = \langle -3t-2, t+1, -t+1 \rangle$

(use Q as point)

Q: How to find a line $\perp$ to this one?

A: Find a vector $\perp$ $\overrightarrow{PQ}$.

<u>Parametric Equations</u> for a Line:

$\vec{r}(t) = \langle -3t+1, t, -t+2 \rangle \iff \begin{cases} x(t) = -3t + 1 \\ y(t) = t \\ z(t) = -t + 2 \end{cases}$

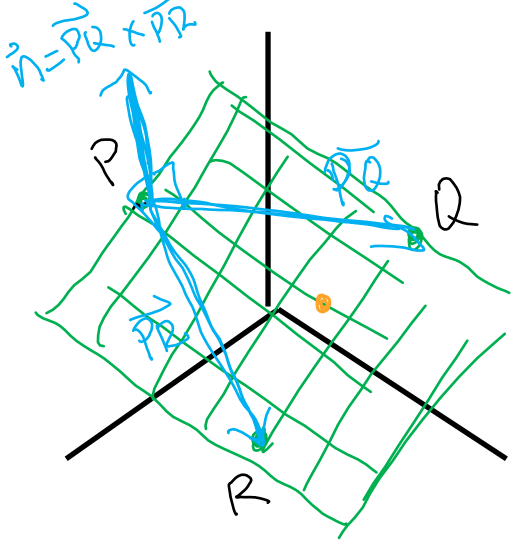
$\vec{r}(t) = \begin{bmatrix} -3t+1 \\ t \\ -t+2 \end{bmatrix}$

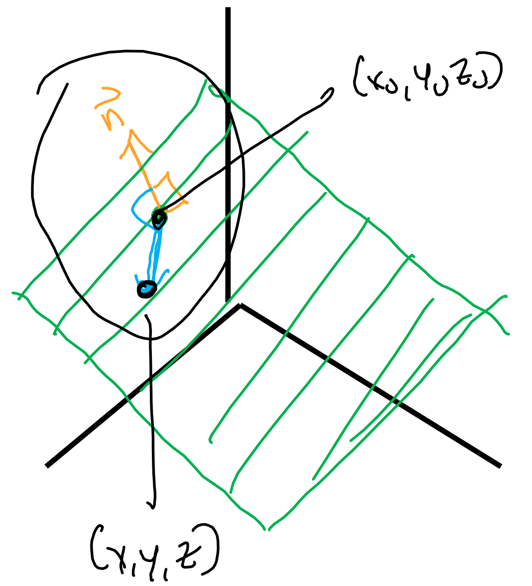
ex: Find direction vector for $\vec{r}(t) = \langle 1, 1, 1 \rangle + t\langle -2, 0, 4 \rangle$

$\vec{v} = \langle -2, 0, 4 \rangle$

Planes in $\mathbb{R}^3$

Conceptually: A plane is determined by either three points in $\mathbb{R}^3$ or by a single point and a direction $\mathbf{n}$, called the *normal vector*.





Algebraically: A plane in $\mathbb{R}^3$ has a *linear* equation (back to Linear Algebra! imposing a single restriction on a 3D space leaves a 2D linear space, i.e. a plane)

$$ax + by + cz = d$$

(x_0, y_0, z_0) is on plane & so is (x, y, z)

$\langle x - x_0, y - y_0, z - z_0 \rangle$ lies in the plane (blue vector above)

$\vec{n} = \langle a, b, c \rangle$

$$\vec{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

$$d = ax_0 + by_0 + cz_0 = \vec{n} \cdot \vec{OP}$$

Example 6. Find ~~the~~ a normal vector and an equation for the plane that contains the points $P = (1, 2, -1), Q = (1, 0, -1)$, and $R = (0, 1, 3)$.

Find $\vec{n}$: $\vec{n} = \vec{PQ} \times \vec{PR} = $ $\begin{array}{cc} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -2 & 0 \\ -1 & -1 & 4 \end{vmatrix} & \begin{array}{l} \\ \leftarrow \vec{PQ} \\ \leftarrow \vec{PR} \end{array} \end{array}$

$$= \vec{i} \begin{vmatrix} -2 & 0 \\ -1 & 4 \end{vmatrix} - \vec{j} \begin{vmatrix} 0 & 0 \\ -1 & 4 \end{vmatrix} + \vec{k} \begin{vmatrix} 0 & -2 \\ -1 & -1 \end{vmatrix}$$

$$= \langle -8 - 0, -(0 - 0), 0 - 2 \rangle$$

$$= \langle -8, 0, -2 \rangle \uparrow \vec{n}$$

eqn: $-8(x - 1) + 0(y - 2) + (-2)(z - (-1)) = 0$

$\vec{OP} = \langle 1, 2, -1 \rangle$; $P = (1, 2, -1)$

$$-8x - 2z + 6 = 0$$

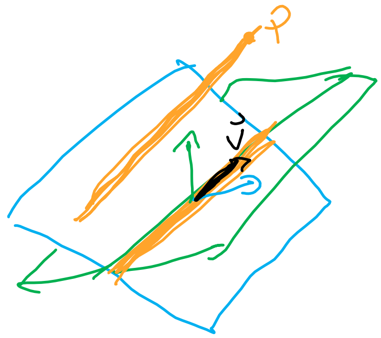
$$4x + z = 3$$

Q: Is $(2, 1, 1)$ on ? ?

Check $4(2) + 0(1) + 1 \stackrel{?}{=} 3$

$9 \neq 3$, so $(2, 1, 1)$ is not on this plane

Example 7. Consider the planes $y - z = -2$ and $x - y = 0$. Show that the planes intersect and find an equation for the line passing through the point $P = (-8, 0, 2)$ which is parallel to the line of intersection of the planes.



1) Intersect?

$x = z - 2$

$y = z - 2$ $y = z - 2$

$y = x$ so e.g. $(0, 0, 2)$

2) direction vector for line: $\vec{v} = \vec{n}_1 \times \vec{n}_2$

Finish on Tuesday.

12.6: Quadric Surfaces.

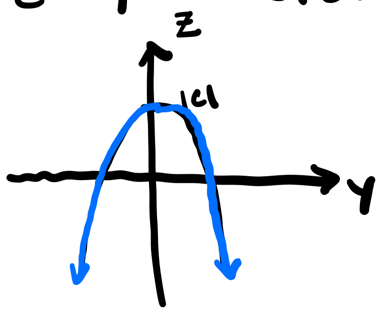
- Points in $\mathbb{R}^3$ which solve a quadratic equation.
  - * Sphere: $x^2+y^2+z^2=r^2$
  - * Cylinder: $x^2+y^2=r^2$ $\quad -1 \le z \le 1$
- How do we analyze these?
  - * Cross-sections

<u>ex</u>: $z = x^2 - y^2$

Cross-section:

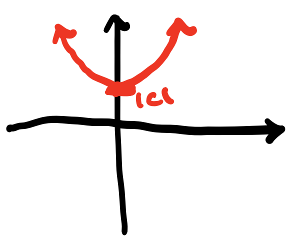
1. Slice by $x=c$

   $z = c^2 - y^2$ (Parabola in yz-plane)



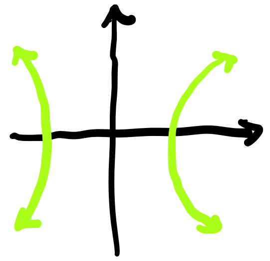
2. Slice by $y=c$

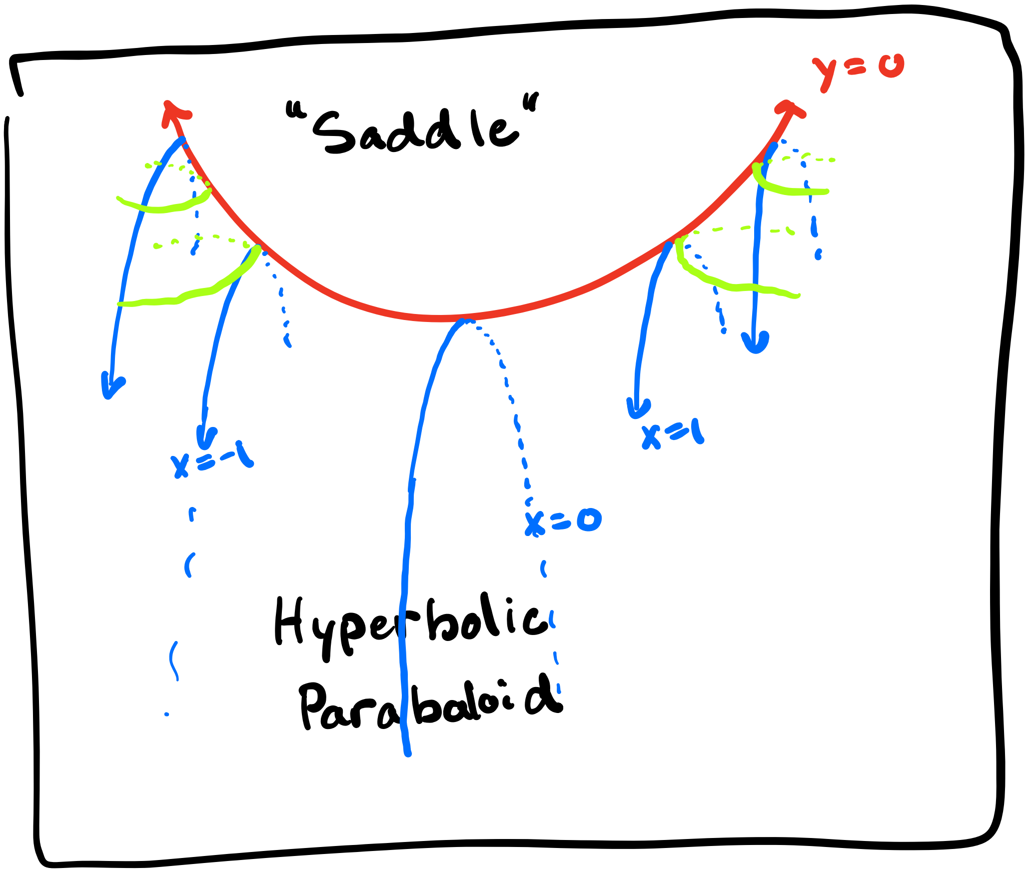
   $z = x^2 - c^2$ (Parabola in xz-plane)



3. Slice by $z=c$

   $c = x^2 - y^2$ (Hyperbola)





ex. $x = z^2 + y^2$

$x = c \geq 0$: circle in yz-plane of radius $\sqrt{c}$.

$y = c$: parabola in zx-plane

$z = c$: parabola in yx-plane

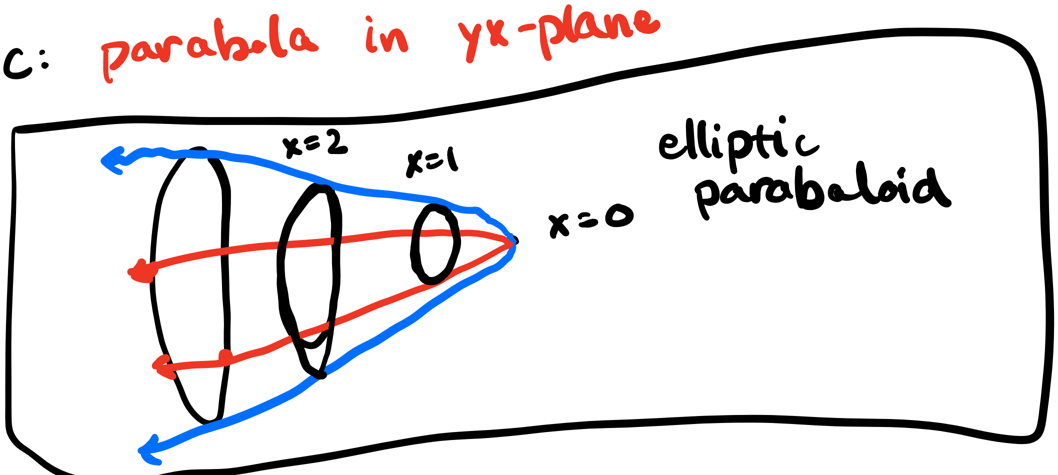
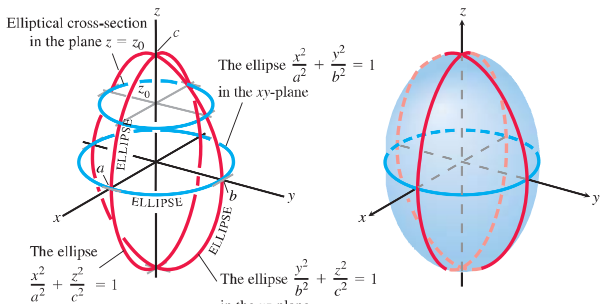
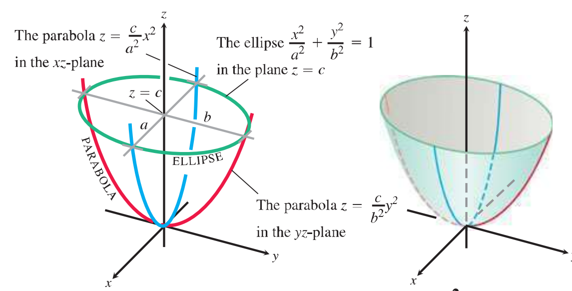


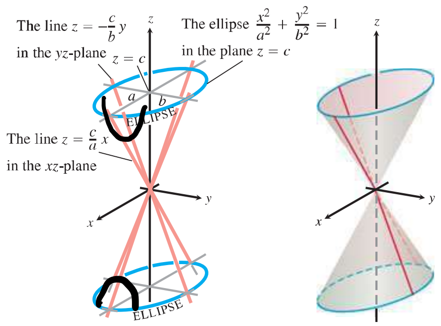
TABLE 12.1 Graphs of Quadric Surfaces



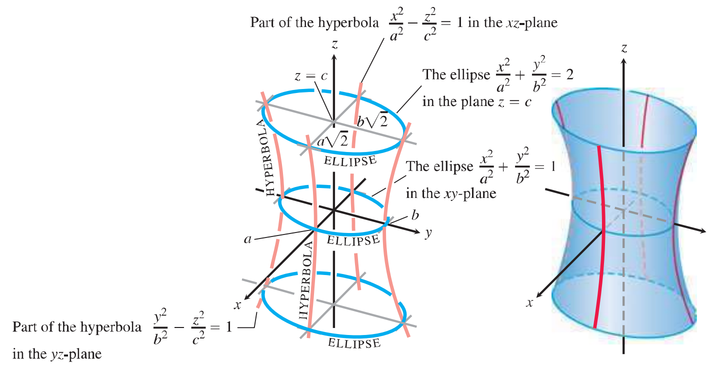
ELLIPSOID $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$



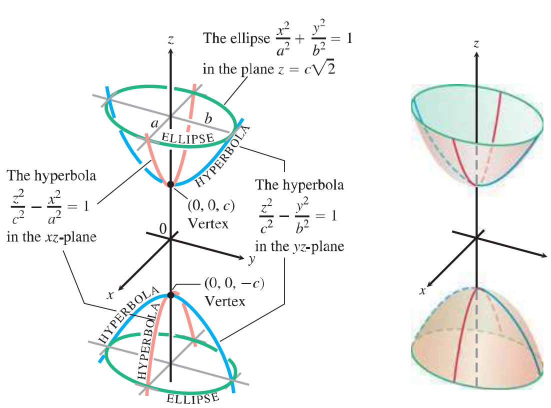
ELLIPTICAL PARABOLOID $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$



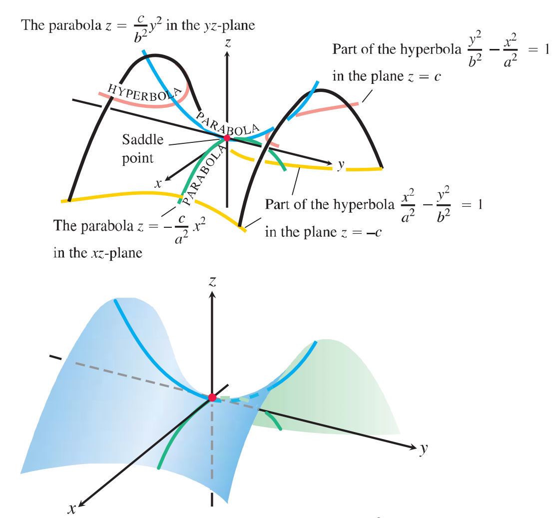
ELLIPTICAL CONE $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$



HYPERBOLOID OF ONE SHEET $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$



HYPERBOLOID OF TWO SHEETS $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



HYPERBOLIC PARABOLOID $\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}, c > 0$

13.1 Vector-valued functions

ex. $\vec{r}(t) = \langle t, t^2+1 \rangle$ $t \geq 0$

$= \langle x(t), y(t) \rangle$

$x(t) = t$

$y(t) = t^2+1 = [x(t)]^2+1$

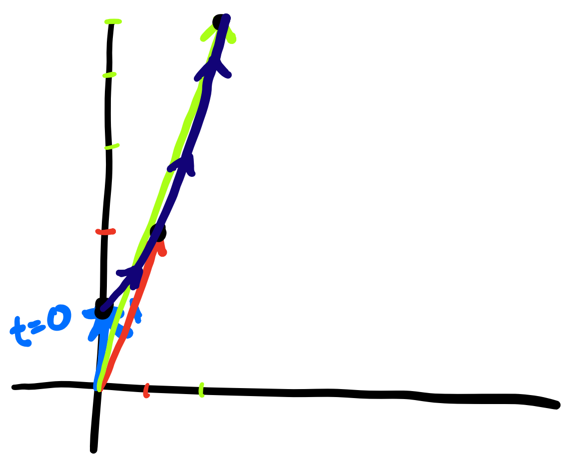
a) $\vec{r}(0) = \langle 0, 1 \rangle$

$\vec{r}(1) = \langle 1, 2 \rangle$

$\vec{r}(2) = \langle 2, 5 \rangle$

b) Graph?

2 plots : plot each coordinate separately.



c) Is there a function whose graph is the same

$y = f(x)$

yes! $y = x^2+1$.

Some examples:

Vector equation for a line:

$$\vec{r}(t) = \vec{p} + t \cdot \vec{v} \quad , \quad \vec{p} = \langle x_0, y_0, z_0 \rangle$$

$$= \langle x_0 + t v_1, \quad \vec{v} = \langle v_1, v_2, v_3 \rangle$$

$$y_0 + t v_2,$$

$$z_0 + t v_3 \rangle.$$

Question: Given a function $y = f(x)$, can we write the graph as a vector-valued function $\vec{r}(t)$?

Question: is there $x(t)$ and $y(t)$ which causes $\vec{r}(t)$ and $y = f(x)$ to have the same graph?

$$\vec{r}(t) = \langle x(t), y(t) \rangle$$

- $y(t) = f(t)$
- $x(t) = t$

$$\boxed{\vec{r}(t) = \langle t, f(t) \rangle}$$

How about $x = g(y)$?

$$\boxed{\vec{r}(t) = \langle g(t), t \rangle}$$

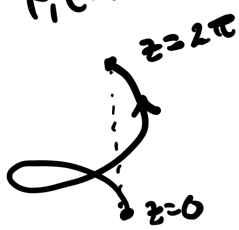
---

ex. $\vec{r}_1(t) = \langle \cos t, \sin t, t \rangle$

$\vec{r}_2(t) = \langle \cos 2t, \sin 2t, 2t \rangle$

Graph(s)

$\vec{r}_1(t)$, $0 \le t \le 2\pi$



$\vec{r}_2(t)$, $0 \le t \le 2\pi$

ex. Find a vector-valued function with graph: a circle in xy-plane of radius r.

polar form for a circle of radius r:

$$\begin{cases} x(\theta) = r\cos\theta \\ y(\theta) = r\sin\theta \end{cases}$$

$$\boxed{\begin{aligned} \vec{r}(t) &= \langle x(t), y(t) \rangle \\ &= \langle r\cos t, r\sin t \rangle \end{aligned}}$$

$$x^2 + y^2 = r^2$$

Daily Announcements & Reminders:

- HW for 12.5 due tonight
- No studio Monday
- Try the warmup problem

Goals for Today: Sections 13.1-13.2

- Compute limits, derivatives, and tangent lines for vector-valued functions
- Compute integrals of vector-valued functions and solve initial value problems

Calculus of vector-valued functions

Unifying theme: Do what you already know, componentwise.

This works with limits:

Example 12. Compute $\lim_{t\to e}\langle t^2, 2, \ln(t)\rangle$.

$$\vec{r}(t) = \begin{bmatrix} t^2 \\ 2 \\ \ln(t) \end{bmatrix}$$

$$\lim_{t\to e} \langle t^2, 2, \ln(t)\rangle$$

$$= \langle \lim_{t\to e} t^2, \lim_{t\to e} 2, \lim_{t\to e} \ln(t)\rangle$$

$$= \langle e^2, 2, 1\rangle$$

And with continuity:

Example 13. Determine where the function $\mathbf{r}(t) = t\mathbf{i} - \dfrac{1}{t^2-4}\mathbf{j} + \sin(t)\mathbf{k}$ is continuous.

$(t-2)(t+2)$ if $z(t) = \dfrac{1}{t-3}$

$\Rightarrow (-\infty,-2)\cup(-2,2)\cup(2,3)\cup(3,\infty)$

Continuous at $t=t_0$

- 1) $\lim_{t\to t_0} \vec{r}(t)$ needs to exist
- 2) $\lim_{t\to t_0} \vec{r}(t) = \vec{r}(t_0)$

Domain: x: $(-\infty,\infty)$

y: $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$ | All t except $t=-2,2$

z: $(-\infty,\infty)$

So overall:

- x, z-components are cts on all $\mathbb{R}$
- $\lim_{t\to a} \dfrac{-1}{t^2-4} = \dfrac{-1}{a^2-4} \Rightarrow y(t)$ is cts on its entire domain

  if $a \neq -2, 2$

Conclusion: $\vec{r}(t)$ is cts on $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$

Ex: $\vec{r}(t) = \langle t, f(t)\rangle$

$f(t) = \begin{cases} -1, & x<0 \\ 0, & x=0 \\ 1, & x>1 \end{cases}$

Domain: $(-\infty,\infty$

Cts: $(-\infty,0)\cup(0,\infty)$

CalcPlot3D

And with derivatives:

Example 14. If $\mathbf{r}(t) = \langle 2t - \frac{1}{2}t^2 + 1, t - 1\rangle$, find $\mathbf{r}'(t)$. $= \frac{d}{dt}(\vec{r}(t))$

$$\vec{r}'(t) = \left\langle \frac{d}{dt}\left(2t - \frac{1}{2}t^2 + 1\right), \frac{d}{dt}(t-1) \right\rangle$$

$$= \langle 2 - t, 1 \rangle$$

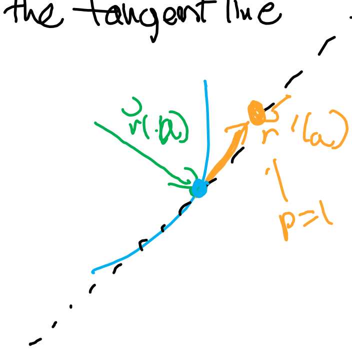
Interpretation: If $\mathbf{r}(t)$ gives the position of an object at time t, then

- $\mathbf{r}'(t)$ gives velocity
- $|\mathbf{r}'(t)|$ gives speed
- $\mathbf{r}''(t)$ gives accelration

Let's see this graphically

Example 15. Find an equation of the tangent line to $\mathbf{r}(t) = \langle 2t - \frac{1}{2}t^2 + 1, t - 1\rangle$ at time $t = 2$.

• If $\vec{r}(t)$ is smooth ($\vec{r}\,'(t) \neq \vec{0}$) then the tangent line at $t=a$ to the graph of $\vec{r}(t)$ is



$$\ell(\square) = \square \cdot \vec{r}\,'(a) + \vec{r}(a)$$

e.g. $a=2$, so $\vec{r}(2) = \langle 3, 1\rangle$

$\vec{r}\,'(2) = \langle 0, 1\rangle$

so $\ell(\square) = \square\langle 0,1\rangle + \langle 3,1\rangle$

OR

$\ell(t) = t\langle 0,1\rangle + \langle 3,1\rangle$

~~Let's revisit the~~ Itempool question ~~from the end of last lecture:~~

Example 16. Find parametric equations for the tangent line to the curve $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + t\mathbf{k}$ at the point $(1, 0, 2\pi)$.

$a = 2\pi$

b/c $\vec{r}(2\pi) = \langle 1, 0, 2\pi\rangle$

$\cos(t) = 1$

$\sin(t) = 0$

$t = 2\pi$

same t

$\vec{r}\,'(t) = \langle -\sin(t), \cos(t), 1\rangle$

$\vec{r}\,'(2\pi) = \langle 0, 1, 1\rangle$

$\ell(t) = \langle 0, 1, 1\rangle t + \langle 1, 0, 2\pi\rangle$

different t from $\vec{r}(t), \vec{r}\,'(t)$

Continuing with integrals:

Example 17. Find $\int_0^1 \langle t, e^{2t}, \sec^2(t)\rangle \; dt$.

At this point we can solve initial-value problems like those we did in single-variable calculus:

Example 18. Wallace is testing a rocket to fly to the moon, but he forgot to include instruments to record his position during the flight. He knows that his velocity during the flight was given by

$$\mathbf{v}(t) = \langle -200\sin(2t), 200\cos(t), 400 - \frac{400}{1+t}\rangle \; m/s.$$

If he also knows that he started at the point $\mathbf{r}(0) = \langle 0, 0, 0\rangle$, use calculus to reconstruct his flight path.

Visualizing (quadric surfaces)

- Cross-sections (reduce 3d problem → 2d problem)
- Use a computer (CalcPlot3d, GeoGebra, Mathematica)

Reference:

$$49x^2 + 4y^2 + z^2 = 17$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad ?$$

Continuing with integrals:

Example 17. Find $\int_0^1 \langle t, e^{2t}, \sec^2(t)\rangle\ dt$.

$$= \left\langle \int_0^1 t\,dt,\ \int_0^1 e^{2t}\,dt,\ \int_0^1 \sec^2(t)\,dt \right\rangle$$

$$= \left\langle \tfrac{1}{2}t^2\Big|_0^1,\ \tfrac{1}{2}e^{2t}\Big|_0^1,\ \tan(t)\Big|_0^1 \right\rangle$$

$$= \left\langle \tfrac{1}{2},\ \tfrac{1}{2}e^2 - \tfrac{1}{2}\cancel{e}^{\,1},\ \tan(1) - \cancel{\tan(0)}^{\,0} \right\rangle$$

If $\langle t, e^{2t}, \sec^2(t)\rangle$ is a velocity function: $\vec{?}$ is the displacement vector from position at $t=0$ to $t=1$.

At this point we can solve initial-value problems like those we did in single-variable calculus:

Example 18. Wallace is testing a rocket to fly to the moon, but he forgot to include instruments to record his position during the flight. He knows that his velocity during the flight was given by

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If he also knows that he started at the point $\mathbf{r}(0) = \langle 0,0,0\rangle$, use calculus to reconstruct his flight path.

Know $\vec{v}(t) = \vec{r}\,'(t)$ $\implies$ $\vec{r}(t) = \int \vec{v}(t)\,dt + \vec{C}$

$$= \left\langle \int -200\sin(2t)\,dt,\ \int 200\cos(t)\,dt,\ \int 400 - \frac{400}{1+t}\,dt \right\rangle + \vec{C}$$

$$= \langle 100\cos(2t),\ 200\sin(t),\ 400t - 400\ln|1+t| \rangle + \vec{C}\ m$$

OR

$$\langle 100\cos(2t) + C_1,\ 200\sin(t) + C_2,\ 400t - 400\ln|1+t| + C_3 \rangle\ m$$

$\vec{C} = \langle C_1, C_2, C_3\rangle$

Get: $C_1 = ?$
$C_2 = ?$
$C_3 = ?$

$$\langle 0,0,0\rangle = \vec{r}(0) = \langle 100, 0, 0\rangle + \vec{C}$$

$$\vec{C} = \langle -100, 0, 0\rangle\ m$$

$$\vec{r}(t) = \langle 100\cos(2t) - 100,\ 200\sin(t),\ 400t - 400\ln|1+t| \rangle \text{ meters}$$

Daily Announcements & Reminders:

- HW 12.6 due tonight
- Quiz 2 tomorrow: 12.6, 13.1, 13.2
  - do not need to memorize quadrics (yet)
- Do warm-up problem

Goals for Today: Sections 13.3

- Compute arc lengths of curves using parameterizations
- Introduce the idea of an arc-length parameterization
- Compute arc-length parameterizations of curves

13.3 Arc length of curves

We have discussed motion in space using by equations like $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$.

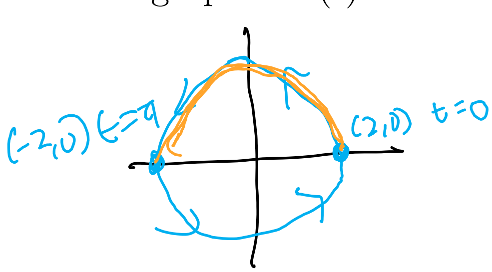
Our next goal is to be able to measure distance traveled or arc length.

Motivating problem: Suppose the position of a fly at time t is

$$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t) \rangle,$$

where $0 \le t \le 2\pi$.

a) Sketch the graph of $\mathbf{r}(t)$. What shape is this?



Circle of radius 2

b) How far does the fly travel between $t = 0$ and $t = \pi$?

How long is the orange arc? ~~4π~~ $\boxed{2\pi}$ (half circumference)

c) What is the speed $|\mathbf{v}(t)|$ of the fly at time t?

$$|\langle -2\sin(t), 2\cos(t)\rangle| = \sqrt{4\sin^2(t) + 4\cos^2(t)}$$
$$= \sqrt{4(\underbrace{\sin^2(t) + \cos^2(t)}_{1})}$$
$$= 2$$

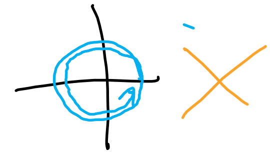
d) Compute the integral $\displaystyle\int_0^{\pi} |\mathbf{v}(t)|\, dt$. What do you notice?

distance = speed · time

$$\int_0^{\pi} 2\, dt = 2t\Big|_0^{\pi} = 2\pi$$

$$\int_a^b |\vec{v}(t)|\, dt \neq \left| \int_a^b \vec{v}(t)\, dt \right|$$

Definition 19. We say that the **arc length** of a smooth curve $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ from $t=a$ to $t=b$ that is traced out exactly once is



$$L = \int_a^b |\vec{r}\,'(t)|\,dt$$

Example 20. Set up an integral for the arc length of the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ from the point $(1, 1, 1)$ to the point $(2, 4, 8)$.

$$L = \int_a^b |\vec{r}\,'(t)|\,dt$$

$$= \int_1^2 \sqrt{1+4t^2+9t^4}\,dt$$

$\vec{r}\,'(t) = \langle 1, 2t, 3t^2\rangle$

$|\vec{r}\,'(t)| = \sqrt{1+4t^2+9t^4}$

$t=a$: $t=1$

$\langle 1,1,1\rangle = \langle t, t^2, t^3\rangle$

$t=b$: $t=2$

$\langle 2,4,8\rangle = \langle t, t^2, t^3\rangle$

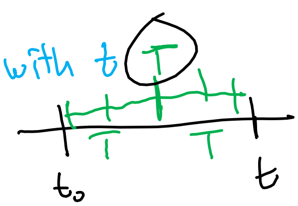
Sometimes, we care about the distance traveled from a fixed starting time t_0 to an arbitrary time t, which is given by the **arc length function**.

variable

nothing to do with t

$$s(t) = \int_{t_0}^t |\vec{r}\,'(T)|\,dT$$

fixed number



We can use this function to produce parameterizations of curves where the parameter s measures distance along the curve: the points where $s = 0$ and $s = 1$ would be exactly 1 unit of distance apart.

- This is like marking position on a highway by mile markers instead of time.
- Finding such a parameterization is always possible, but usually hard.

Example 21. Find an arc length parameterization of the circle of radius 4 about the origin in $\mathbb{R}^2$, $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t)\rangle, 0 \le t \le 2\pi$.

1) Compute $s(t)$.

$$\vec{r}\,'(t) = \langle -4\sin(t), 4\cos(t)\rangle$$

$$|\vec{r}\,'(t)| = \sqrt{16\sin^2(t) + 16\cos^2(t)} = \sqrt{16(\underbrace{\sin^2(t)+\cos^2(t)}_{1})} = 4$$

$$s(t) = \int_0^t |\vec{r}\,'(\tau)|\, d\tau$$

often tricky

$$= \int_0^t 4\, d\tau = 4\tau \Big|_0^t = 4t$$

$s(t) = 4t \iff$ starting at $t_0 = 0$, traveling on curve for t units of time results in traveling $4t$ units of distance

2) Invert to get t as a function s

$$t = \frac{s}{4} = f(s)$$

Usually very hard

3) Write $\vec{r}_2(s) := \vec{r}(f(s))$

$$= \langle 4\cos\left(\tfrac{s}{4}\right), 4\sin\left(\tfrac{s}{4}\right)\rangle$$

$0 \le s \le 8\pi$

$s(0) = 0$

$s(2\pi) = 8\pi$

$0 \le \frac{s}{4} \le 2\pi$

Check: Is $|\vec{r}_2\,'(s)| = 1$?

$$\frac{d}{ds}\left(\vec{r}_2(s)\right) = \langle -4\sin\left(\tfrac{s}{4}\right)\cdot\tfrac{1}{4}, 4\cos\left(\tfrac{s}{4}\right)\cdot\tfrac{1}{4}\rangle$$

$$= \langle -\sin\left(\tfrac{s}{4}\right), \cos\left(\tfrac{s}{4}\right)\rangle$$

$$|\vec{r}_2\,'(s)| = \sqrt{\sin^2\left(\tfrac{s}{4}\right) + \cos^2\left(\tfrac{s}{4}\right)} = \sqrt{1} = 1$$

Example 21. Find an arc length parameterization of the circle of radius 4 about the origin in $\mathbb{R}^2$, $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t)\rangle, 0 \le t \le 2\pi$.

1) Compute $s(t)$.

$$s(t) = \int_0^t |\vec{r}\,'(\tau)|\,d\tau$$

$\vec{r}\,'(t) = \langle -4\sin(t), 4\cos(t)\rangle$

$|\vec{r}\,'(t)| = \sqrt{16\sin^2(t) + 16\cos^2(t)}$

$= \sqrt{16(\cancel{\sin^2(t) + \cos^2(t)})}$ 1

$= 4$

often tricky

$$= \int_0^t 4\,d\tau$$

$$= 4\tau\Big|_0^t = 4t$$

$s(t) = 4t \iff$ starting at $t_0 = 0$, traveling on curve for t units of time results in traveling $4t$ units of distance

2) Invert to get t as a function s

Usually very hard

$t = \frac{s}{4} = f(s)$

3) Write $\vec{r}_2(s) := \vec{r}(f(s))$

$$= \left\langle 4\cos\left(\frac{s}{4}\right), 4\sin\left(\frac{s}{4}\right)\right\rangle$$

$0 \le s \le 8\pi$

$s(0) = 0$

$s(2\pi) = 8\pi$

$0 \le \frac{s}{4} \le 2\pi$

Check: Is $|\vec{r}_2\,'(s)| = 1$?

$\frac{d}{ds}\left(\vec{r}_2(s)\right) = \left\langle -4\sin\left(\frac{s}{4}\right)\cdot\frac{1}{4}, 4\cos\left(\frac{s}{4}\right)\cdot\frac{1}{4}\right\rangle$

$= \left\langle -\sin\left(\frac{s}{4}\right), \cos\left(\frac{s}{4}\right)\right\rangle$

$|\vec{r}_2\,'(s)| = \sqrt{\sin^2\left(\frac{s}{4}\right) + \cos^2\left(\frac{s}{4}\right)} = \sqrt{1} = 1$

Daily Announcements & Reminders:

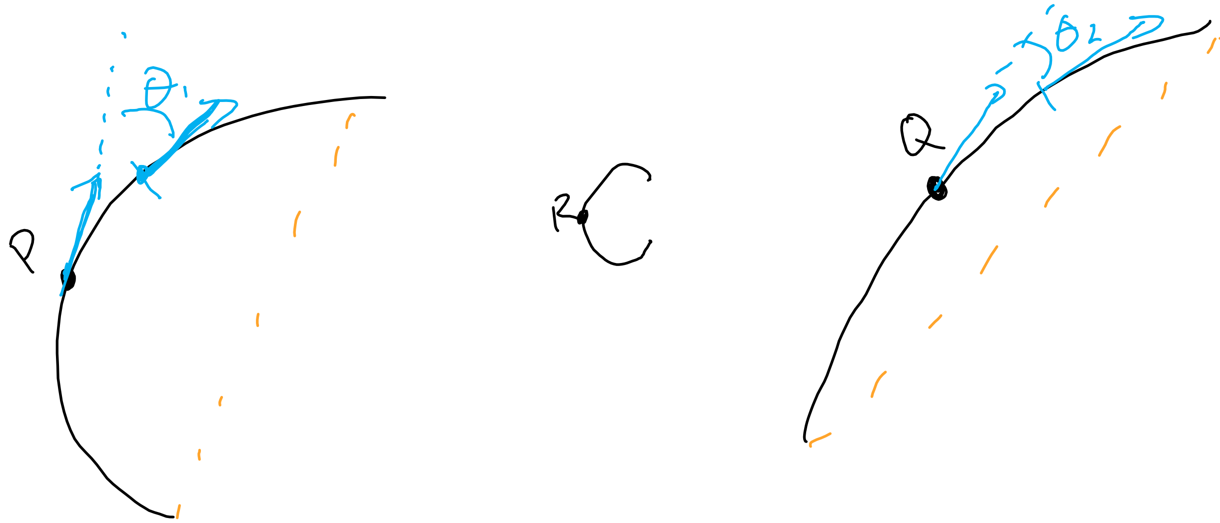
- 13.1, 13.2 HW due tonight
- Do warmup problem
- Quiz 1 grades released after class

Goals for Today: Sections 13.4, 14.1

- Define, interpret, and compute the curvature of a curve
- Compute the unit tangent and principal unit normal vectors of a curve
- Give examples of functions of multiple variables
- Find the domain of functions of two or three variables

13.4 - Curvature, Tangents, Normals

The next idea we are going to explore is the <u>curvature</u> of a curve in space along with two vectors that orient the curve.



First, we need the **unit tangent vector**, denoted **T**:

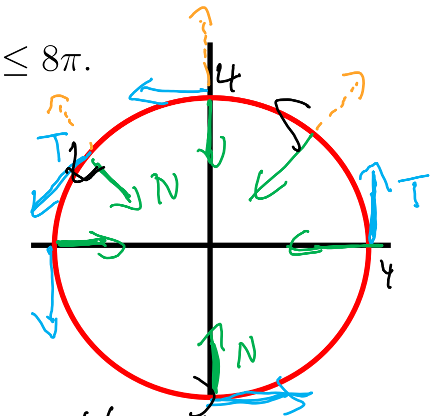
- In terms of an arc-length parameter s: $\vec{r}\,'(s)$
- In terms of any parameter t: $\cancel{r'(t)}$ / $|\vec{r}\,'(t)|$

This lets us define the **curvature**, $\kappa(s) =$ $|d\vec{T}/ds|$

Example 22. ~~Last class~~ Earlier we found an arc length parameterization of the circle of radius 4 centered at $(0, 0)$ in $\mathbb{R}^2$:

$$\mathbf{r}(s) = \left\langle 4\cos\left(\frac{s}{4}\right), 4\sin\left(\frac{s}{4}\right)\right\rangle, \qquad 0 \leq s \leq 8\pi.$$

Use this to find $\mathbf{T}(s)$ and $\kappa(s)$.



$\vec{T}(s) = \vec{r}\,'(s)$
$= \langle -\sin(\frac{s}{4}), \cos(\frac{s}{4}) \rangle$

$\kappa(s) = |d\vec{T}/ds| = |\langle -\frac{1}{4}\cos(\frac{s}{4}), -\frac{1}{4}\sin(\frac{s}{4}) \rangle|$
$= \frac{1}{4}$

rate of change of direction of motion is $\frac{1}{4}$ unit for every 1 unit of distance travelled

all circles have constant curvature $= \frac{1}{\text{radius}}$

- helices have constant curvature

$\vec{N} = \langle -\cos(\frac{s}{4}), -\sin(\frac{s}{4}) \rangle$

Question: In which direction is **T** changing?

$|\vec{N}| = 1$

This is the direction of the **principal unit normal**, $\mathbf{N}(s) =$ $d\vec{T}/ds \,/\, |d\vec{T}/ds|$

$\vec{N} \cdot \vec{T} \overset{?}{=} 0$

We said that it is often hard to find arc length parameterizations, so what do we do if we have a generic parameterization $\mathbf{r}(t)$?

- $\mathbf{T}(t) =$ $\vec{r}\,'(t) / |\vec{r}\,'(t)|$
- $\mathbf{N}(t) =$ $\vec{T}\,'(t) / |\vec{T}\,'(t)|$
- $\kappa(t) =$ $|\vec{T}\,'(t)| / |\vec{r}\,'(t)|$ or $|\vec{r}\,'(t) \times \vec{r}\,''(t)| / |\vec{r}\,'(t)|^3$

Example 23. Find $\mathbf{T}, \mathbf{N}, \kappa$ for the helix $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), t-1\rangle$.

$$\vec{T}(t) = \langle -2\sin(t), 2\cos(t), 1\rangle / \sqrt{4\sin^2 t + 4\cos^2(t) + 1}$$

$$\sqrt{4\sin^2 t + 4\cos^2(t) + 1} = \sqrt{5}$$

$$= \frac{1}{\sqrt{5}}\langle -2\sin(t), 2\cos(t), 1\rangle$$

$$\vec{N}(t) = \frac{1}{\sqrt{5}}\langle -2\cos(t), -2\sin(t), 0\rangle \Big/ \frac{1}{\sqrt{5}}\sqrt{4\cos^2(t) + 4\sin^2(t) + 0}$$

$$\sqrt{4} = 2$$

$$= \langle -\cos(t), -\sin(t), 0\rangle$$

Check: $\vec{T} \cdot \vec{N} = \frac{1}{\sqrt{5}}\left(2\sin(t)\cos(t) - 2\sin(t)\cos(t) + 0\right) = 0$

$$\kappa(t) = \frac{2/\sqrt{5}}{\sqrt{5}} = \boxed{\frac{2}{5}}$$

Daily Announcements & Reminders:

- 13.3 & 13.4 HW due tonight
- Quiz 3 tomorrow: TF from basic 14.1, FR from 13.3/13.4
- Exam 1 next Tuesday - covers through today's material
  - formula sheet will be provided
  - sample exams on Canvas → Modules (content doesn't exactly match; no 14.1 on old exams)

Goals for Today: Sections 14.1

- Give examples of functions of multiple variables
- Find the domain of functions of two or three variables
- Graph functions of two variables
- Introduce and sketch traces and contours of functions of two variables
- Find level surfaces of functions of three variables

14.1 Functions of Multiple Variables

Definition 24. A function of two variables is a rule that assigns to each pair of real numbers (x, y) in a set D a value denoted by $f(x, y)$.

$$f : D \to \mathbb{R}, \text{ where } D \subseteq \mathbb{R}^2$$

D is a subset of $\mathbb{R}^2$

$D \neq$ an interval

$z = f(x,y)$ → name

domain: pairs (x,y) that we can plug into f

codomain: outputs

- some quadric surfaces, all planes are examples

$z = \boxed{3x^2 + 2y^2 = f(x,y)}$

Example 25. Three examples are

$$f(x, y) = x^2 + y^2, \qquad g(x, y) = \ln(x + y), \qquad h(x, y) = \frac{1}{\sqrt{x + y}}.$$

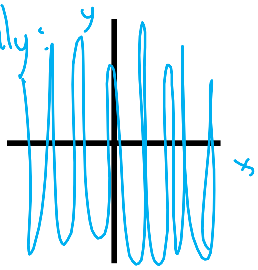
Example 26. Find the domains of f, g, and h.

$f(x,y) = x^2 + y^2$

Domain: $\mathbb{R}^2$

$\{(x,y) \mid x, y \in \mathbb{R}\}$

Graphically:

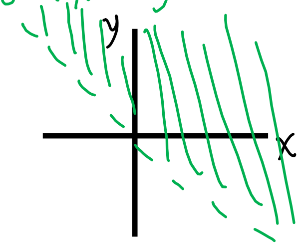


$g(x,y) = \ln(x+y)$

Domain: $\boxed{x+y>0}$

$\{(x,y) \mid x+y>0\}$

OR $\{(x,y) \mid y > -x\}$



$h(x,y) = \dfrac{1}{\sqrt{x+y}}$

Domain: $x+y \geq 0$

$\sqrt{x+y} \neq 0$

$\Leftrightarrow \{(x,y) \mid x+y>0\}$

same as g

Definition 27. If f is a function of two variables with domain D, then the graph of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

Here are the graphs of the three functions above.

Example 28. Suppose a small hill has height $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$ m at each point (x, y). How could we draw a picture that represents the hill in 2D?

- Use cross-sections

$0 = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$

$x^2+y^2=16$

$1 = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$

$\frac{1}{4}x^2 + \frac{1}{4}y^2 = 3$

$x^2+y^2=12$

$5 = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$

$x^2+y^2=-4$

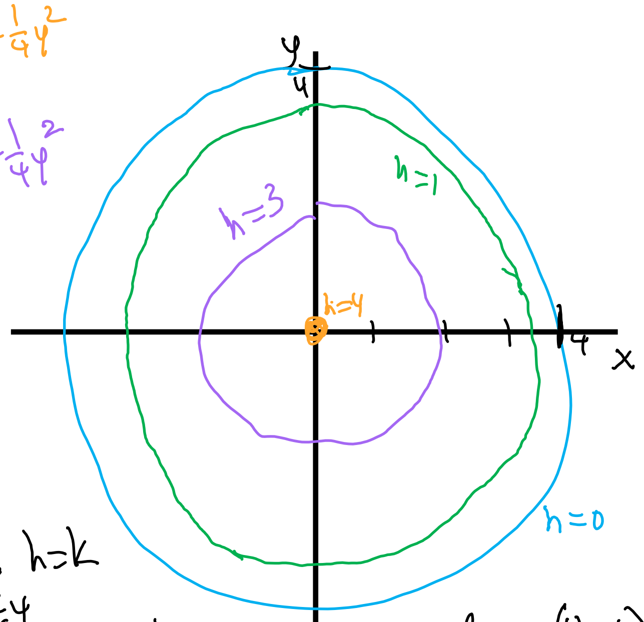
no points where $h=5$

$4 = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$

$x^2+y^2=0$

$3 = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$

$x^2+y^2=4$



At height: $h=k$

$0 \leq k \leq 4$

$k = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2 \Rightarrow x^2+y^2 = (4-k)4$

In 3D, it looks like this.

Definition 29. The contours (also called level curves) of a function f of two variables are the curves with equations $f(x,y)=k$, where k is a constant (in the range of f). A plot of contours for various values of z is a contour map (or contour plot).

Some common examples of these are:

- topographical
- thermal imaging
- weather maps
- ocean depth maps
- population (density)

Example 30. Create a contour diagram of $f(x, y) = x^2 - y^2$ (added post-lecture)

$k=0$: $0 = x^2 - y^2$ (two lines)

$x^2 = y^2$

$y = \pm x$

$k=1$: $1 = x^2 - y^2$ (hyperbola, opening L/R)

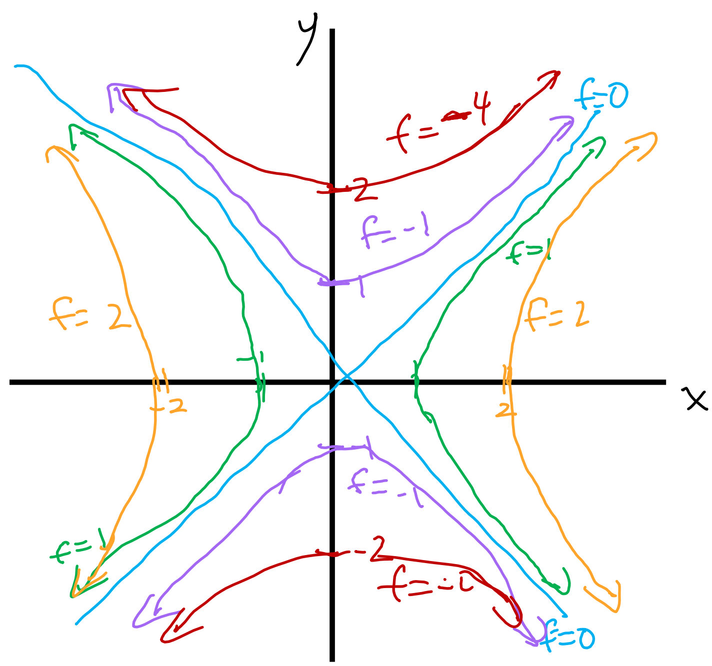
$k=4$: $4 = x^2 - y^2$ (further shifted L/R hyperbola)

$k=-1$: $-1 = x^2 - y^2$ (hyperbola, opening U/D)

$1 = y^2 - x^2$

$k=-4$: $-4 = x^2 - y^2$ (further shifted L/R hyperbola)

$4 = y^2 - x^2$



Example 31. Create a contour diagram of $g(x, y) = y \sin(x)$

Definition 32. The traces of a surface are the curves of intersection of the surface with planes parallel to the xz-plane & yz-plane.

Example 33. Use the traces and contours of $z = f(x, y) = 4 - 2x - y^2$ to sketch the portion of its graph in the $\underbrace{\text{first octant}}_{x \geq 0,\ y \geq 0,\ z \geq 0}$.

contours

$k = 4 - 2x - y^2$

$2x = 4 - k - y^2$

$x = (2 - \frac{1}{2}k) - \frac{1}{2}y^2$

parabolas opening left

x-traces:

$z = (4 - 2k) - y^2$

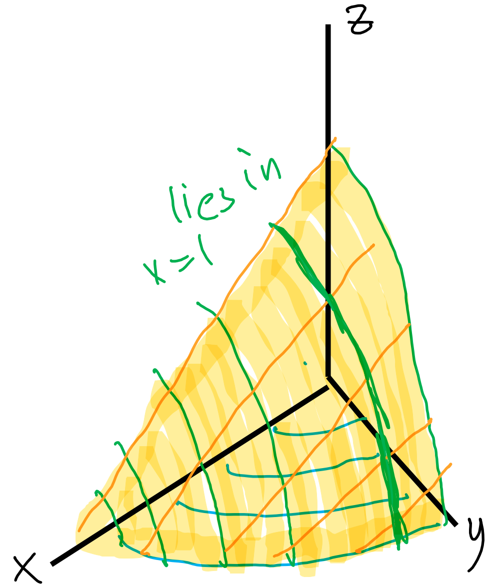
parabola opening downward in $x = k$ plane

y-traces:

$z = 4 - 2x - k^2$

$z = (4 - k^2) - 2x$

line in $y = k$ plane



Let's check our work: https://tinyurl.com/math2551-f23-2var-graph

Definition 34. A function of 3 variables is a rule that assigns to each triplet of real numbers (x, y, z) in a set D a value denoted by $f(x, y, z)$.

$$f : D \to \mathbb{R}, \text{ where } D \subseteq \mathbb{R}^3$$

We can still think about the domain and range of these functions. Instead of level curves, we get level surfaces.

Example 35. Describe the domain of the function $f(x, y, z) = \dfrac{1}{4 - x^2 - y^2 - z^2}$.

Domain: $4 - x^2 - y^2 - z^2 \neq 0$

$x^2 + y^2 + z^2 \neq 4$

All points in $\mathbb{R}^3$ except the sphere $x^2 + y^2 + z^2 = 4$

Example 36. Describe the level surfaces of the function $g(x, y, z) = 2x^2 + y^2 + z^2$.

$k = 2x^2 + y^2 + z^2$ · Surface in $\mathbb{R}^3$

Level surfaces are ellipsoids

Daily Announcements & Reminders:

- HW 14.2 due tonight
- Exam 1 graded by 9/29

Goals for Today: Sections 14.3, 14.4

- Understand derivatives as transformations
- Define the total derivative
- Find linearizations
- Learn the Chain Rule for derivatives of functions of multiple variables
- Be able to compute implicit partial derivatives

Example 43. Find $f_x(1,2)$ and $f_y(1,2)$ of the functions below.

a) $f(x,y) = \sqrt{5x-y}$

$$f_x = \frac{\partial}{\partial x}\left(\sqrt{5x-y}\right) = \frac{1}{2}(5x-y)^{-1/2}\cdot\frac{\partial}{\partial x}(5x-y) = \frac{5}{2\sqrt{5x-y}}$$

$$f_x(1,2) = \frac{5}{2\sqrt{3}}$$

$$f_y = \frac{\partial}{\partial y}\left(\sqrt{5x-y}\right) = \frac{1}{2}(5x-y)^{-1/2}\cdot\frac{\partial}{\partial y}(5x-y) = \frac{-1}{2\sqrt{5x-y}}$$

$$f_y(1,2) = -\frac{1}{2\sqrt{3}}$$

b) $f(x,y) = \tan(xy)$

$$\frac{\partial f}{\partial x} = f_x = \sec^2(xy)\cdot y \qquad f_x(1,2) = 2\sec^2(2)$$

$$f_y = \sec^2(xy)\cdot x \qquad f_y(1,2) = \sec^2(2)$$

Question: How would you define the second partial derivatives?

• Take partial Derivatives of the first partial derivatives?

$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$

$f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$

• inside to outside

$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$

$f_{yy} = (f_y)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$

pure partials

Example 44. Find f_{xx}, f_{xy}, f_{yx}, and f_{yy} of the functions below.

mixed partials

a) $f(x, y) = \sqrt{5x - y}$

$f_x = \frac{5}{2}(5x-y)^{-1/2}$ $\quad f_y = -\frac{1}{2}(5x-y)^{-1/2}$

$f_{xx} = \frac{\partial}{\partial x}(f_x)$
$= \frac{\partial}{\partial x}\left(\frac{5}{2}(5x-y)^{-1/2}\right)$
$= -\frac{5}{4}(5x-y)^{-3/2} \cdot 5$

$f_{yx} = \frac{\partial}{\partial x}\left(-\frac{1}{2}(5x-y)^{-1/2}\right)$
$= \frac{1}{4}(5x-y)^{-3/2} \cdot 5$

$f_{xy} = \frac{\partial}{\partial y}\left(\frac{5}{2}(5x-y)^{-1/2}\right)$
$= -\frac{5}{4}(5x-y)^{-3/2} \cdot (-1)$

$f_{yy} = \frac{\partial}{\partial y}\left(-\frac{1}{2}(5x-y)^{-1/2}\right)$
$= \frac{1}{4}(5x-y)^{-3/2} \cdot (-1)$

b) $f(x, y) = \tan(xy)$

$f_x = y\sec^2(xy)$ $\quad f_y = x\sec^2(xy)$

$f_{xx} = \frac{\partial}{\partial x}\left(y\sec^2(xy)\right) = y \cdot 2\sec(xy) \cdot \sec(xy)\tan(xy) \cdot y$
$= 2y^2\sec^2(xy)\tan(xy)$

$f_{xy} = \frac{\partial}{\partial y}\left(y\sec^2(xy)\right) = 1 \cdot \sec^2(xy) + y2\sec(xy)\sec(xy)\tan(xy) \cdot x$

$f_{yx} = \frac{\partial}{\partial x}\left(x\sec^2(xy)\right) = 1 \cdot \sec^2(xy) + x\,2\sec(xy)\sec(xy)\tan(xy)\,y$

$f_{yy} = \frac{\partial}{\partial y}\left(x\sec^2(xy)\right) = 2x^2\sec^2(xy)\tan(xy)$

What do you notice about f_{xy} and f_{yx} in the previous examples?

Theorem 45 (Clairaut's Theorem). *Suppose f is defined on a disk D that contains the point (a,b). If the functions $f, f_x, f_y, f_{xy}, f_{yx}$ are all continuous on D, then*

$$f_{xy} = f_{yx}$$

(and similarly for 3rd order: $f_{xxy} = f_{xyx} = f_{yxx}$ if all 1st, 2nd, & mixed 3rd partials are cts)

$\neq$ to not $f_{yyx} = f_{yxy} = f_{xyy}$

Example 46. Last time, we computed partial derivatives for functions of two variables. What about functions of three variables? How many partial derivatives should $f(x,y,z) = 2xyz - z^2y$ have? Compute them.

$$f_x = 2yz - 0$$

$$f_y = 2xz - z^2$$

$$f_z = 2xy - 2zy$$

Example 47. How many rates of change should the function $f(s,t) = \begin{array}{c|c|c} f_1(s,t) & \multirow{3}{*}{} & x(s,t) \\ f_2(s,t) & & y(s,t) \\ f_3(s,t) & & z(s,t) \end{array}$ $\begin{bmatrix} s^2+t \\ 2s-t \\ st \end{bmatrix}$ have? Compute them.

6 rates of change: each of 3 outputs has 2 partial derivatives

$$\frac{\partial f_1}{\partial s} = \frac{\partial x}{\partial s} = 2s \qquad \frac{\partial f_1}{\partial t} = \frac{\partial x}{\partial t} = 1$$

$$\frac{\partial f_2}{\partial s} = \frac{\partial y}{\partial s} = 2 \qquad \frac{\partial f_2}{\partial t} = \frac{\partial y}{\partial t} = -1$$

$$\frac{\partial f_3}{\partial s} = t = \frac{\partial z}{\partial s} \qquad \frac{\partial f_3}{\partial t} = \frac{\partial z}{\partial t} = s$$

$$Df(s,t) = \begin{bmatrix} 2s & 1 \\ 2 & -1 \\ t & s \end{bmatrix}$$

In the previous example, we computed $\underline{6 = 2 \cdot 3}$ partial derivatives. How might we **organize** this information?

n m

For any function $f : \mathbb{R}^n \to \mathbb{R}^m$ having the form $f(x_1, \ldots, x_n) = \begin{bmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_m(x_1, \ldots, x_n) \end{bmatrix}$,

we have $\underline{n}$ inputs, $\underline{m}$ output, and $\underline{n \cdot m}$ partial derivatives, which we can use to form the **total derivative**.

This is a <u>linear</u> map from $\mathbb{R}^n \to \mathbb{R}^m$, denoted Df, and we can represent it with an <u>matrix</u>, with one column per input and one row per output.

It has the formula $Df_{ij} = \frac{\partial}{\partial x_j}(f_i)$ (see ex above)

in row i, take derivatives of ith component of output

in column j, take derivatives wrt the jth variable

Example 48. Find the total derivatives of each function:

a) $f(x) = x^2 + 1$ $\quad Df(x) = [2x]$

b) $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$ $\quad D\vec{r}(t) = \begin{bmatrix} -\sin(t) \\ \cos(t) \\ 1 \end{bmatrix} = \vec{r}\,'(t)$

c) $f(x, y) = \sqrt{5x - y}$ $\quad Df(x,y) = \begin{bmatrix} \frac{5}{2\sqrt{5x-y}} & \frac{-1}{2\sqrt{5x-y}} \end{bmatrix}$

d) $f(x, y, z) = 2xyz - z^2y$ $\quad Df(x,y,z) = [f_x \quad f_y \quad f_z]$

e) $\mathbf{f}(s, t) = \langle s^2 + t, 2s - t, st \rangle$ $\quad D\vec{f}(s,t) = \begin{bmatrix} 2s & 1 \\ 2 & -1 \\ t & s \end{bmatrix}$

What does it mean? In differential calculus, you learned that one interpretation of the derivative is as a slope. Another interpretation is that the derivative measures how a function transforms a neighborhood around a given point.

Check it out for yourself: (credit to samuel.gagnon.nepton, who was inspired by 3Blue1Brown.)

In particular, the (total) derivative of **any** function $f : \mathbb{R}^n \to \mathbb{R}^m$, evaluated at $\mathbf{a} = (a_1, \ldots, a_n)$, is the linear function that best approximates $f(\mathbf{x}) - f(\mathbf{a})$ at $\mathbf{a}$.

This leads to the familiar linear approximation formula for functions of one variable: $f(x) \approx f(a) + f'(a)(x - a)$.

$\vec{f}(\vec{x}) = f(\vec{a}) + Df(\vec{a})(\vec{x} - \vec{a})$

Definition 49. The **linearization** or **linear approximation** of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$ at the point $\mathbf{a} = (a_1, \ldots, a_n)$ is

$$L(\mathbf{x}) = f(\vec{a}) + Df(\vec{a})(\vec{x} - \vec{a})$$

Example 50. Find the linearization of the function $f(x, y) = \sqrt{5x - y}$ at the point $(1, 1)$. Use it to approximate $f(1.1, 1.1)$.

$\vec{a} = (1,1)$

$$L(x,y) = f(1,1) + Df(1,1)\begin{bmatrix} x-1 \\ y-1 \end{bmatrix}$$

$$= \sqrt{5-1} + \left[\frac{5}{2\sqrt{5-1}} \quad \frac{-1}{2\sqrt{5-1}}\right]\begin{bmatrix} x-1 \\ y-1 \end{bmatrix}$$

$$= 2 + \left[\frac{5}{4} \quad -\frac{1}{4}\right]\begin{bmatrix} x-1 \\ y-1 \end{bmatrix}$$

$$= 2 + \frac{5}{4}(x-1) - \frac{1}{4}(y-1)$$

$f(1.1, 1.1) \approx L(1.1, 1.1) = 2 + 1.25(1.1-1) - 0.25(1.1-1)$

$= 2 + 0.125 - 0.025$

$= 2.1$

Question: What do you notice about the equation of the linearization?

$f(1.1, 1.1) = 2.0976\ldots$

Example 53. Suppose we are walking on our hill with height $h(x,y) = 4 - \dfrac{1}{4}x^2 - \dfrac{1}{4}y^2$ along the curve $\mathbf{r}(t) = \langle t+1, 2-t^2 \rangle$ in the plane. How fast is our height changing at time $t = 1$ if the positions are measured in meters and time is measured in minutes?

Daily Announcements & Reminders:

- 14.3 HW due tonight
- Quiz 4 tomorrow: partial derivatives & Chain Rule (today, basic Q)
- Practice problems added to Ch. 14 practice for total derivatives
- Do the warm up

Goals for Today: Sections 14.5, 14.6

- Learn the Chain Rule for derivatives of functions of multiple variables
- Be able to compute implicit partial derivatives
- Learn to compute the rate of change of a multivariable function in any direction
- Investigate the connection between the gradient vector and level curves/surfaces
- Discuss tangent planes to surfaces, how to find them, and when they exist

Example 51. The **differential** of a function $f(\mathbf{x})$ is

$$df = f_{x_1}\, dx_1 + f_{x_2}\, dx_2 + \ldots f_{x_n}\, dx_n. = Df \cdot \begin{bmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_n \end{bmatrix}$$

Find the differential df for $f(x, y, z) = x^2 + y^2 + z^2$ and use it to estimate the change in f between $(1, 1, 1)$ and $(1.1, 1, 0.9)$.

Idea: $df \approx \Delta f \approx f_{x_1}\Delta x_1 + \ldots + f_{x_n}\Delta x_n$

$df = 2x\,dx + 2y\,dy + 2z\,dz$

At $(1,1,1)$: $df = 2\,dx + 2\,dy + 2\,dz$

so $\Delta f \approx 2\Delta x + 2\Delta y + 2\Delta z$

$= 2(1.1-1) + 2(1-1) + 2(.9-1)$

$= 0.2 + 0 - 0.2$

$= 0$

Only change in f! no info about value of f

$f(1.1, 1, 0.9) - f(1,1,1)$

$= 3.02 - 3$

$= 0.02$

Checking error

The Chain Rule

Example 52. If $f(t) = \ln(t^2)$, then $\dfrac{df}{dt} = \dfrac{1}{t^2} \cdot 2t$

$$\frac{d}{dt}\left(f(g(t))\right) = f'(g(t)) \cdot g'(t)$$

p inputs

Similarly, the **Chain Rule** for functions of multiple variables says that if $f : \mathbb{R}^p \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ are both differentiable functions then

m outputs n inputs p outputs

$$D(\boxed{f(g(\mathbf{x}))}) = Df(g(\mathbf{x}))Dg(\mathbf{x}).$$

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "g"] \arrow[rr, bend right, "f \circ g"'] & \mathbb{R}^p \arrow[r, "f"] & \mathbb{R}^n \end{tikzcd}$$

e.g.

$f(x,y) =$ height as a function of position

$g(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ position as a function of time

$f(g(t)) =$ height as a function of time

Example 53. Suppose we are walking on our hill with height $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$ along the curve $\mathbf{r}(t) = \langle t + 1, 2 - t^2\rangle$ in the plane. How fast is our height changing at time $t = 1$ if the positions are measured in meters and time is measured in minutes?

Goal: $D(h(\vec{r}(t)))\big|_{t=1} = (Dh)(\vec{r}(1)) \cdot (D\vec{r})(1)$

$\vec{r}(1) = \langle 2, 1\rangle$

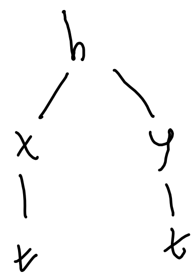
$Dh(x,y) = \left[-\frac{1}{2}x \quad -\frac{1}{2}y\right]$

$D\vec{r}(t) = \begin{bmatrix} 1 \\ -2t \end{bmatrix}$

$= Dh(\underset{x}{2}, \underset{y}{1}) \cdot D\vec{r}(\overset{t}{1})$

$= \left[-1 \quad -\frac{1}{2}\right] \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$= -1 + \frac{1}{2}(2) = 0$ m/min



An alternate perspective to organize the Chain Rule: tree diagram

Example 54. Suppose that $W(s,t) = F(u(s,t), v(s,t))$, where F, u, v are differentiable functions and we know the following information.

$\mathbb{R}^2 \to \mathbb{R}^2 \to \mathbb{R}$

$(s,t) \to (u,v) \to W$

$(1,0) \mapsto (2,3) \mapsto ?$

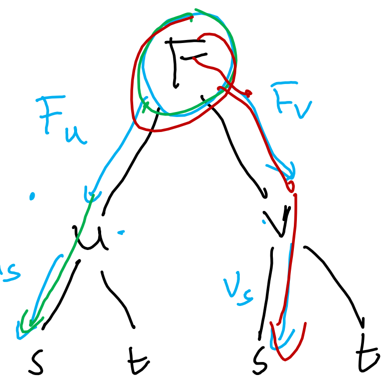
$u(1,0) = 2$	$v(1,0) = 3$
$u_s(1,0) = -2$	$v_s(1,0) = 5$
$u_t(1,0) = 6$	$v_t(1,0) = 4$
$F_u(2,3) = -1$	$F_v(2,3) = 10$

Find $W_s(1,0)$ and $W_t(1,0)$.

$s=1$ $t=0$

$u(1,0)=2$

$v(1,0)=3$



$W_s = F_u \cdot u_s + F_v \cdot v_s$

$W_s(1,0) = F_u(2,3)\cdot u_s(1,0) + F_v(2,3)\cdot v_s(1,0)$

$= (-1)(-2) + (10)(5)$

$= 52$

$DW(1,0) = [W_s(1,0) \quad W_t(1,0)] = [F_u(u(1,0),v(1,0)) \quad F_v(u(1,0),v(1,0))] \begin{bmatrix} u_s(1,0) & u_t(1,0) \\ v_s(1,0) & v_t(1,0) \end{bmatrix}$

$= \begin{bmatrix} -1 & 10 \end{bmatrix} \begin{bmatrix} -2 & 6 \\ 5 & 4 \end{bmatrix}$

$= [52 \quad 34]$

Application to Implicit Differentiation: If $F(x,y,z) = c$ is used to *implicitly* define z as a function of x and y, then the chain rule says:

$\frac{\partial z}{\partial x} = -\frac{2zx}{x^2 + \frac{1}{z}}$

$2x^2 + y^2 + \ln z = 3$

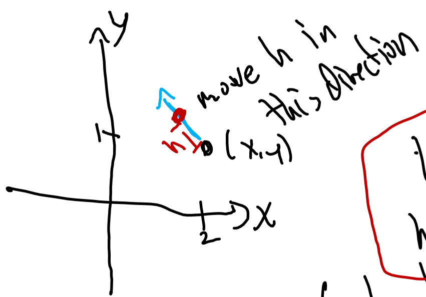
$$\boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}}$$

Example 55. Recall that if $z = f(x,y)$, then f_x represents the rate of change of z in the x-direction and f_y represents the rate of change of z in the y-direction. What about other directions?

Let's go back to our hill example again, $f(x,y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$. How could we figure out the rate of change of our height from the point $(2,1)$ if we move in the direction $\langle -1, 1\rangle$?

i) Make direction into unit vector

$\vec{u} = \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$



$$\boxed{\lim_{h\to 0} \frac{f\left(x+h\left(-\frac{1}{\sqrt{2}}\right), y+h\left(\frac{1}{\sqrt{2}}\right)\right) - f(x,y)}{h}}$$

$$\underbrace{(x,y)}_{\vec{p}} + h\underbrace{\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}_{\vec{u}}$$

Definition 56. The <u>directional derivative</u> of $f : \mathbb{R}^n \to \mathbb{R}$ at the point $\mathbf{p}$ in the direction of a unit vector $\mathbf{u}$ is

$$D_{\mathbf{u}} f(\mathbf{p}) = \lim_{h\to 0} \frac{f(\vec{p}+h\vec{u}) - f(\vec{p})}{h}$$

if this limit exists.

E.g. for our hill example from ~~the last page~~ we have:

$$\lim_{h\to 0} \frac{4 - \frac{1}{4}\left(2 - \frac{h}{\sqrt{2}}\right)^2 - \frac{1}{4}\left(1+\frac{h}{\sqrt{2}}\right)^2 - \frac{7}{2}}{h}$$

Note that $D_{\mathbf{i}} f = f_x$ $\qquad D_{\mathbf{j}} f = f_y$ $\qquad D_{\mathbf{k}} f = f_z$

Definition 57. If $f : \mathbb{R}^n \to \mathbb{R}$, then the gradient of f at $\mathbf{p} \in \mathbb{R}^n$ is the vector function ∇f (or grad f) defined by

hill height

$$\nabla f(\mathbf{p}) = (f_{x_1}(\vec{p}), \ldots, f_{x_n}(\vec{p})) = (Df(\vec{p}))^T$$

e.g. $\nabla h(x,y) = \begin{bmatrix} -\frac{1}{2}x \\ -\frac{1}{2}y \end{bmatrix}$

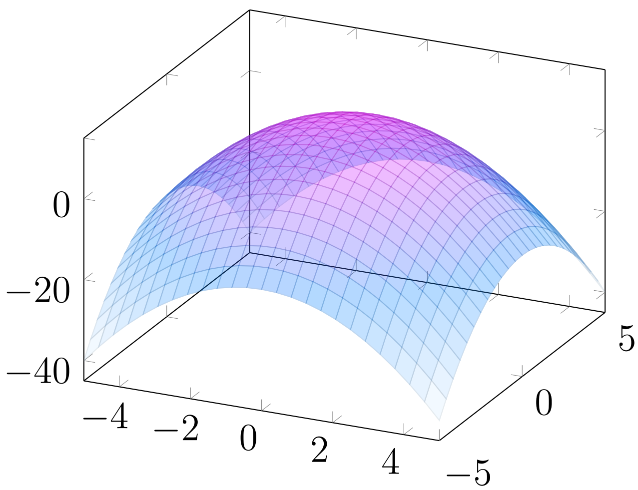
Note: If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at a point $\mathbf{p}$, then f has a directional derivative at $\mathbf{p}$ in the direction of any unit vector $\mathbf{u}$ and

$$D_{\mathbf{u}} f(\mathbf{p}) = Df(\vec{p})\,\vec{u} = \nabla f(\vec{p}) \cdot \vec{u}$$

dot product

Tangent planes to level surfaces

Suppose S is a surface with equation $F(x, y, z) = k$. How can we find an equation of the tangent plane of S at $P(x_0, y_0, z_0)$?



$x^2 + y^2 + z = 10, P = (-1, 3, 0)$

This is a good place to actually define differentiable! We say $f : \mathbb{R}^n \to \mathbb{R}$ is **differentiable** at $\mathbf{a}$ if its linearization is a good approximation of f near $\mathbf{a}$. (Technical definition in textbook).

In particular, if f is a function $f(x, y)$ of two variables, it is differentiable at (a, b) if it has a unique tangent plane at (a, b).

Example 60. Find the equation of the tangent plane at the point $(-2, 1, -1)$ to the surface given by

$$z = 4 - x^2 - y$$

Special case: if we have $z = f(x, y)$ and a point $(a, b, f(a, b))$, the equation of the tangent plane is

This should look familiar: it's ______________________________

Daily Announcements & Reminders:

- HW 14.4 due tonight, 14.5 pushed to T
- Exam grades should be out tomorrow

Goals for Today: Section 14.5, 14.6, 14.7

- Investigate the connection between the gradient vector and level curves/surfaces
- Discuss tangent planes to surfaces, how to find them, and when they exist
- Define local & global extreme values for functions of two variables
- Learn how to find local extreme values for functions of two variables

Last time: If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at a point $\mathbf{p}$, then f has a directional derivative at $\mathbf{p}$ in the direction of any unit vector $\mathbf{u}$ and

$$D_{\mathbf{u}}f(\mathbf{p}) = Df(\mathbf{p})\mathbf{u} = \nabla f(\mathbf{p}) \cdot \mathbf{u}$$

- $D_{\vec{u}}f(\vec{p})$ is the rate of change of f at $\vec{p}$ if we move in the direction $\vec{u}$
- If $\vec{u}$ is tangent to a contour of f at $\vec{p}$ then $D_{\vec{u}}f(\vec{p}) = 0$

Example 58. Find the gradient vector and the directional derivative of each function at the given point $\mathbf{p}$ in the direction of the given vector $\mathbf{u}$.

$Df(-1,1) = \begin{bmatrix} -1 & 1 \end{bmatrix}$

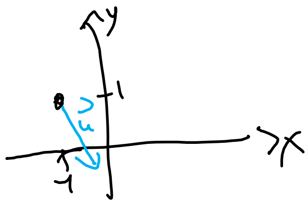
$\nabla f(-1,1) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

a) $f(x,y) = \ln(x^2+y^2), \mathbf{p} = (-1,1), \mathbf{u} = \left\langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \right\rangle$

$\nabla f = \langle f_x, f_y \rangle = \left\langle \frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \right\rangle$

$\nabla f(-1,1) = \left\langle \frac{-2}{2}, \frac{2}{2} \right\rangle = \langle -1, 1 \rangle$

$D_{\vec{u}} f(-1,1) = \nabla f(-1,1) \cdot \vec{u} = \langle -1, 1 \rangle \cdot \left\langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \right\rangle = -\frac{1}{\sqrt{5}} - \frac{2}{\sqrt{5}}$



rate of change of f $\rightarrow \; = \boxed{-\frac{3}{\sqrt{5}}}$

b) $g(x,y,z) = x^2 + 4xy^2 + z^2$, $\mathbf{p} = (1,2,1)$, $\mathbf{u}$ the unit vector in the direction of $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

1) Get unit vector
2) Gradient
3) Dot product

$\nabla g = \langle 2x + 4y^2, 8xy, 2z \rangle$

$\nabla g(1,2,1) = \langle 2+16, 16, 2 \rangle = \langle 18, 16, 2 \rangle$

$D_{\vec{u}} f(1,2,1) = \nabla f(1,2,1) \cdot \vec{u}$

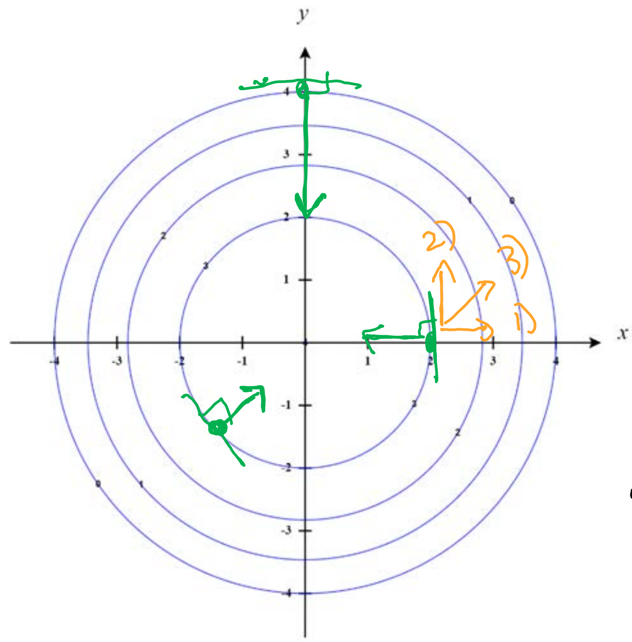
$\vec{u} = \frac{\langle 1,2,-1 \rangle}{|\langle 1,2,-1 \rangle|} = \frac{1}{\sqrt{6}} \langle 1,2,-1 \rangle$

$= \langle 18, 16, 2 \rangle \cdot \left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}} \right\rangle$

$= \frac{1}{\sqrt{6}} (18 + 32 - 2) = \boxed{\frac{48}{\sqrt{6}}}$

Example 59. If $h(x,y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$, the contour map is given below. Find and draw ∇h on the diagram at the points $(2,0), (0,4)$, and $(-\sqrt{2}, -\sqrt{2})$. At the point $(2,0)$, compute $D_{\mathbf{u}}h$ for the vectors $\mathbf{u}_1 = \mathbf{i}, \mathbf{u}_2 = \mathbf{j}, \mathbf{u}_3 = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$.

$\nabla h = \langle -\frac{1}{2}x, -\frac{1}{2}y \rangle$



Gradients:

$(2,0)$	$\langle -1, 0 \rangle$
$(0,4)$	$\langle 0, -2 \rangle$
$(-\sqrt{2}, -\sqrt{2})$	$\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$

- At $\vec{p}$, $\nabla f(\vec{p})$ is $\perp$ to contour
- $\nabla f(\vec{p})$ points in direction of greatest increase
- To maximize $D_{\vec{u}} f(\vec{p})$:

$$= \nabla f(\vec{p}) \cdot \vec{u}$$

$$|D_{\vec{u}} f(\vec{p})| = |\nabla f(\vec{p})| \, \cancelto{1}{|\vec{u}|} \, \cancelto{1}{\cos\theta}$$

(if $\theta = 0$)

↑ maximum rate of change of f at $\vec{p}$ is $|\nabla f(\vec{p})|$

$D_{\vec{u}} h(2,0)$: $\langle -1, 0 \rangle = \nabla h$

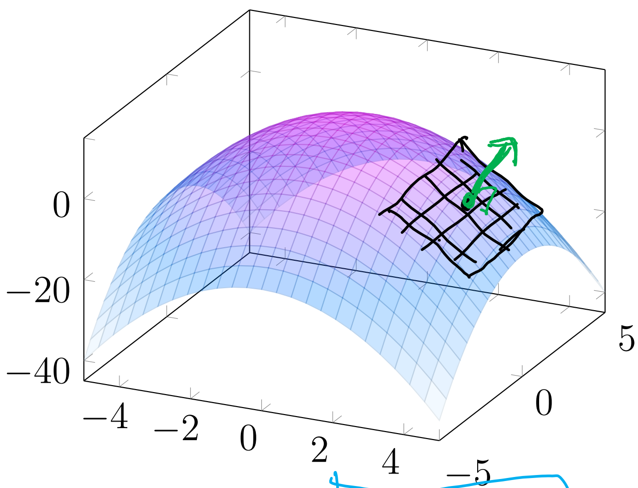
1)	$\langle 1, 0 \rangle$	$(1)(-1) + (0)(0) = -1$
2)	$\langle 0, 1 \rangle$	$(-1)(0) + (0)(1) = 0$
3)	$\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$	$-\frac{1}{\sqrt{2}} + 0 = -\frac{1}{\sqrt{2}}$
	$\langle -1, 0 \rangle$	$(-1)^2 + 0 = 1$

Note that the gradient vector is orthogonal to level curves.

Similarly, for $f(x,y,z)$, $\nabla f(a,b,c)$ is orthogonal to level surfaces

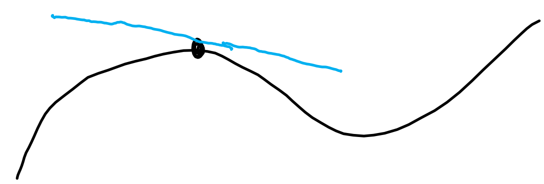
Tangent planes to level surfaces

Suppose S is a surface with equation $F(x,y,z)=k$. How can we find an equation of the tangent plane of S at $P(x_0,y_0,z_0)$?



$x^2+y^2+z=10$, $\boxed{P=(-1,3,0)}$

tangent line:



tangent plane:
- need point: use given point
- need $\vec{n}$: use ∇f at given point

1) Identify f which our surface is a level surface of:

$f = x^2+y^2+z$

$x^2+y^2+z-10=0$

$f = x^2+y^2+z-10$

2) ∇f : $\langle 2x, 2y, 1\rangle$

3) Plug in: $\nabla f(-1,3,0) = \langle -2, 6, 1\rangle$

tangent plane $\boxed{-2(x+1)+6(y-3)+(z-0)=0}$

This is a good place to actually define differentiable! We say $f:\mathbb{R}^n \to \mathbb{R}$ is **differentiable** at $\mathbf{a}$ if its linearization is a good approximation of f near $\mathbf{a}$. (Technical definition in textbook).

In particular, if f is a function $f(x,y)$ of two variables, it is differentiable at (a,b) if it has a unique tangent plane at (a,b).

Example 60. Find the equation of the tangent plane at the point $\boxed{(2, 1, -1)}$ to the surface given by

$$z = 4 - x^2 - y$$

$0 = 4 - x^2 - y - z$

$x^2 + y + z = 4$

1) Identify f:

$f = x^2 + y + z$

2) ∇f: $\langle 2x, 1, 1 \rangle$

3) Plug in: $\nabla f(2,1,-1) = \langle 4, 1, 1 \rangle$

plane: $\boxed{4(x-2) + (y-1) + (z+1) = 0}$

Special case: if we have $z = f(x, y)$ and a point $(a, b, f(a, b))$, the equation of the tangent plane is

$$0 = \overbrace{f(x,y) - z}^{F(x,y,z)}$$

then $\nabla F = \langle f_x, f_y, -1 \rangle$

so tangent plane is

$f_x(a,b)(x-a) + f_y(a,b)(y-b) - 1(z - f(a,b)) = 0$

$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

This should look familiar: it's the linearization

In $\mathbb{R}^3$, another interesting thing can happen. Let's look at $z = x^2 - y^2$ (a hyperbolic paraboloid!) near $(0, 0)$.

This is called a ________________________

Notice that in all of these examples, we have a horizontal tangent plane at the point in question, i.e.

Definition 62. If $f(x, y)$ is a function of two variables, a point (a, b) in the domain of f with $Df(a, b) =$________________ or where $Df(a, b)$ ________________________ is called a ________________________ of f.

Example 63. Find the critical points of the function $f(x, y) = x^3 + y^3 - 3xy$.

Example 65. Classify the critical points of $f(x, y) = x^3 + y^3 - 3xy$ from Example 63.

Daily Announcements & Reminders:

- 14.5 - 14.6 HW due tonight
- Quiz 5 tomorrow on 14.5/14.6
- Exam 1 : ~82% median, 80% mean
  - regrade requests open until F
- Jury duty Th, Dr. Powell will cover lecture

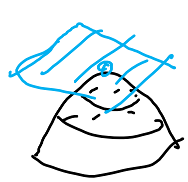
Goals for Today: Section 14.7

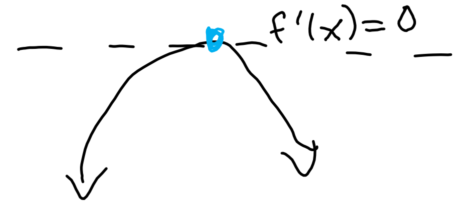
- Learn how to find local extreme values for functions of two variables
- Learn how to classify critical points for functions of two variables
- Find global extreme values of continuous functions of two variables on closed & bounded domains

Recall: If $f(x, y)$ is a function of two variables, we said $\nabla f(a, b)$ points in the direction of greatest change of f.

Back to the hill $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$! What should we expect to get if we compute $\nabla h(0, 0)$? Why? What does the tangent plane to $z = h(x, y)$ at $(0, 0, 4)$ look like?

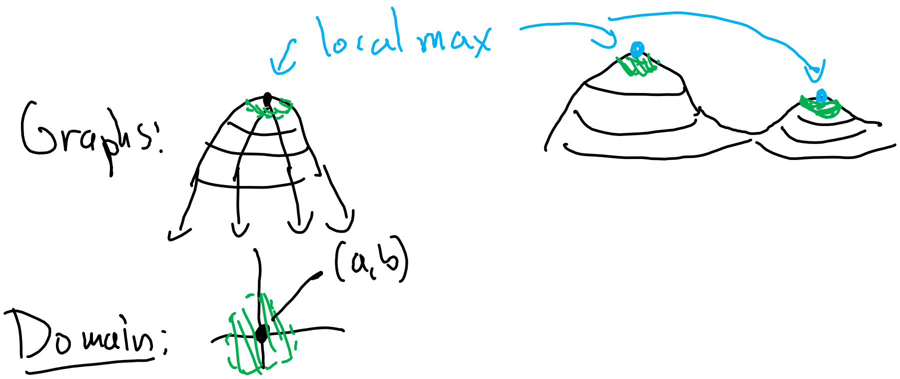
- $\nabla h(0,0) = \langle 0,0 \rangle$ b/c $h(0,0) = 4$ is the largest output of h, so there is no direction to move to get bigger
- tangent plane to our graph at $(0,0,4)$ is parallel to xy-plane





Definition 61. Let $f(x, y)$ be defined on a region containing the point (a, b). We say

- $f(a, b)$ is a local minimum value of f if $f(a, b) \leq f(x, y)$ for all domain points (x, y) in a disk centered at (a, b)
- $f(a, b)$ is a local maximum value of f if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in a disk centered at (a, b)



In $\mathbb{R}^3$, another interesting thing can happen. Let's look at $z = x^2 - y^2$ (a hyperbolic paraboloid!) near $(0, 0)$.

This is called a saddle point

Notice that in all of these examples, we have a horizontal tangent plane at the point in question, i.e. $\nabla f = \vec{0}$ OR $Df = [0 \ 0]$

↑ total derivative of f

Definition 62. If $f(x, y)$ is a function of two variables, a point (a, b) in the domain of f with $Df(a, b) =$ $[0 . 0]$ or where $Df(a, b)$ does not exist is called a critical point of f.

Example 63. Find the critical points of the function $f(x,y) = x^3 + y^3 - 3xy$.

1) Compute Df:

$$Df = \begin{bmatrix} 3x^2 - 3y & 3y^2 - 3x \end{bmatrix}$$

2) Set equal to $[0 \ \ 0]$ & solve

$$\begin{bmatrix} 3x^2 - 3y & 3y^2 - 3x \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

BOTH

① $3x^2 - 3y = 0$

② $3y^2 - 3x = 0$

$3y = 3x^2 \implies y = x^2$ ③ [Plug into ②

$3(x^2)^2 - 3x = 0$

$3x^4 - 3x = 0$

$3x(x^3 - 1) = 0$

$3x = 0$ $\qquad$ $x^3 - 1 = 0$

$\boxed{x = 0, \ y = 0}$ $\qquad$ $\boxed{x = 1, \ y = 1}$ [Plug back into ③

$(0,0)$ & $(1,1)$ are the critical pts of f.

To classify critical points, we turn to the **second derivative test** and the **Hessian matrix**.

The **Hessian matrix** of $f(x, y)$ at (a, b) is

$$Hf(a,b) = \begin{bmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{bmatrix}$$

Theorem 64 (2nd derivative test)**.** *Suppose (a, b) is a critical point of $f(x, y)$ and $Hf(a, b)$ exists. Then we have:*

or $f_{yy}(a,b)$

- *If* $\det(Hf(a, b)) > 0$ *and* $f_{xx}(a, b) > 0$, *$f(a, b)$ is a local minimum*
- *If* $\det(Hf(a, b)) > 0$ *and* $f_{xx}(a, b) < 0$, *$f(a, b)$ is a local maximum*
- *If* $\det(Hf(a, b)) < 0$, *f has a saddle point at (a, b)*
- *If* $\det(Hf(a, b)) = 0$, *the test is inconclusive.*

the function is changing in the same way in all directions if $\det(Hf(a,b)) > 0$

f is not changing the same way in all directions

Example 65. Classify the critical points of $f(x, y) = x^3 + y^3 - 3xy$ from Example 63.

crit pts: $(0,0)$, $(1,1)$ $\quad f_x = 3x^2 - 3y \quad f_y = 3y^2 - 3x$

Find Hf:

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6x & -3 \\ -3 & 6y \end{bmatrix}$$

At $(0,0)$: $Hf(0,0) = \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix}$

$$\det(Hf(0,0)) = 0 - (-3)(-3) = -9 < 0$$

By the 2^{nd} derivative test f has a saddle point at $(0,0)$

At $(1,1)$: $Hf(1,1) = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}$

$$\det(Hf(1,1)) = (6)(6) - (-3)(-3) = 27 > 0$$

$$f_{xx} > 0$$

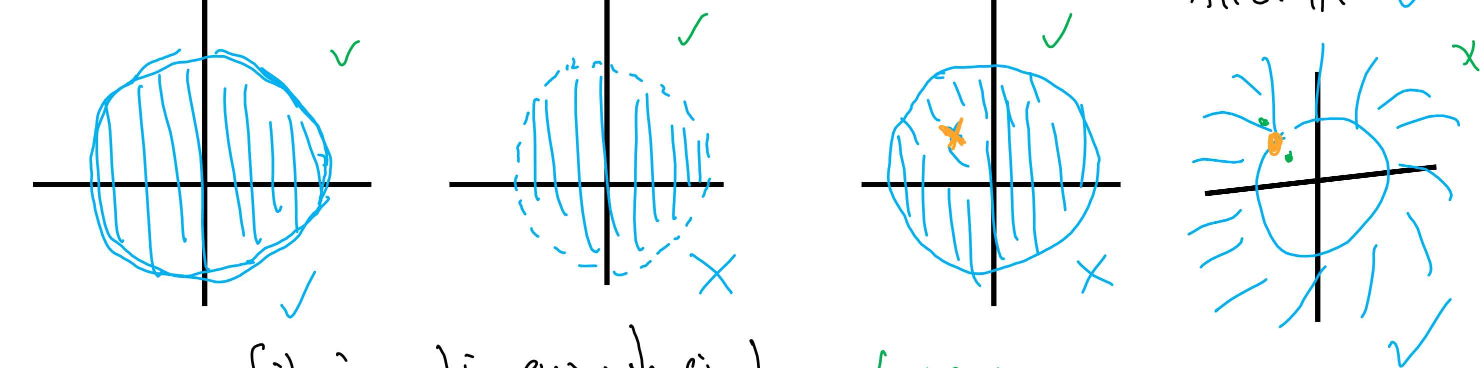
So by the 2^{nd} derivative test f has a local min at $(1,1)$

A global maximum of $f(x, y)$ is like a local maximum, except we must have $f(a, b) \geq f(x, y)$ for **all** (x, y) in the domain of f. A global minimum is defined similarly.

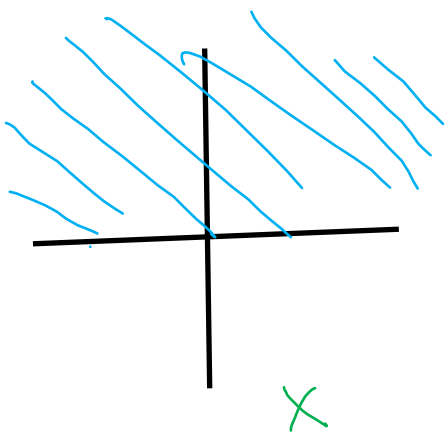
Theorem 66. *On a closed & bounded domain, any continuous function $f(x, y)$ attains a global minimum & maximum.*

Closed: Contains its boundary ✓ or x

• All of $\mathbb{R}^2$ ✓ x

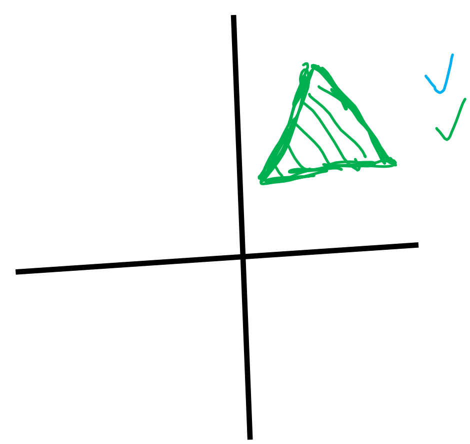


Bounded: fits in a big enough circle ✓ or x



$\{x>0\}$

x



Strategy for finding global min/max of $f(x, y)$ on a closed & bounded domain R

1. Find all critical points of f inside R.
2. Find all critical points of f on the boundary of R
3. Evaluate f at each critical point as well as at any endpoints on the boundary.
4. The smallest value found is the global minimum; the largest value found is the global maximum.

Example 67. Find the global minimum and maximum of $f(x, y) = 4x^2 - 4xy + 2y$ on the closed region R bounded by $y = x^2$ and $y = 4$.

1) Look here

2) Look here

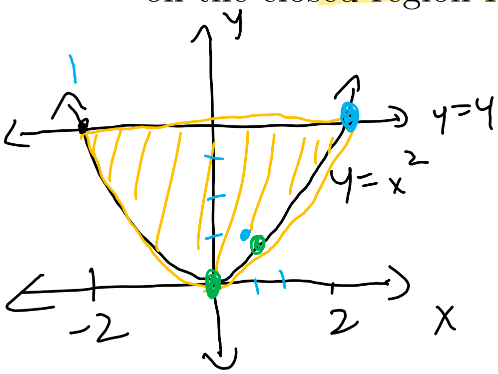
3) Check values at all of these points plus these corners

Strategy for finding global min/max of $f(x,y)$ **on a closed & bounded domain** R

cts

1. Find all critical points of f inside R.
2. Find all critical points of f on the boundary of R
3. Evaluate f at each critical point as well as at any endpoints on the boundary.
   where edges meet
4. The smallest value found is the global minimum; the largest value found is the global maximum.

Example 67. Find the global minimum and maximum of $f(x,y) = 4x^2 - 4xy + 2y$ on the closed region R bounded by $y = x^2$ and $y = 4$.



Domain of optimization

1) Find crit pts of f inside R

$Df = [\,8x - 4y \quad -4x+2\,] = [\,0 \quad 0\,]$

① $8x - 4y = 0$ $\qquad 8(\frac{1}{2}) - 4y = 0$ so $y = 1$

② $-4x + 2 = 0 \to 4x = 2 \to x = \frac{1}{2}$

• Do not need to classify

2) Find crit pts on boundary

• Use equations of boundary to reduce to 1 var

a) On $y=4$: $f(x,4) = 4x^2 - 4x(4) + 2(4)$

$g(x) = 4x^2 - 16x + 8$

$-2 \le x \le 2$

$g'(x) = 8x - 16 = 0$

$x = 2 \quad y = 4$

b) On $y = x^2$: $f(x, x^2) = 4x^2 - 4x(x^2) + 2(x^2)$

$h(x) = 6x^2 - 4x^3, \quad -2 \le x \le 2$

3) Evaluate

Test pts	f
$(\frac{1}{2}, 1)$	1
$\boxed{(2,4)}$	$\boxed{-8}$
$(0,0)$	0
$(1,1)$	2
$\boxed{(-2,4)}$	$\boxed{56}$

Global min is -8 at (2,4)

Global max is 56 at (-2,4)

- Parameterizing boundary is useful for circles/ellipses

$h'(x) = 12x - 12x^2 = 0$

$12x(1-x) = 0$

$x = 0$ $\quad$ $x = 1$

$y = 0^2 = 0$ $\quad$ $y = 1^2 = 1$

Daily Announcements & Reminders:

- HW 14.7 & 14.8 due T
- Do warmup
- No class next M, T

Goals for Today: Sections 14.8, 15.1

- Apply the method of Lagrange multipliers to find extreme values of functions of two or more variables subject to one or more constraints
- Introduce double and iterated integrals for functions of two variables on rectangles
- Use Fubini's Theorem to change the order of integration of a iterated integral

Constrained Optimization

Goal: Maximize or minimize $f(x, y)$ (or $f(x, y, z)$) subject to a *constraint*, $g(x, y) = c$ (or $g(x, y, z) = c$).

Example 68. A new hiking trail has been constructed on the hill with height $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$, above the points $y = -0.5x^2 + 3$ in the xy-plane. What is the highest point on the hill on this path?

Objective function: Thing we are optimizing

- $h(x,y)$

Constraint equation: Restriction on objectives

- $y = \cancel{-0.5}x^2 + 3$ Rewrite: $\underbrace{y + 0.5x^2}_{g(x,y)} = 3$ | $g(x,y) = 3$

Idea: Find all points $\overset{(x,y)}{\downarrow}$ where 1) ∇h, ∇g are parallel

$\iff \nabla h = \lambda \nabla g$

2) $g(x,y) = c$

Ex: $\langle -\frac{1}{2}x, -\frac{1}{2}y \rangle = \lambda \langle x, 1 \rangle$

$$\begin{cases} -\frac{1}{2}x = \lambda x & ① \\ -\frac{1}{2}y = \lambda & ② \\ y + \frac{1}{2}x^2 = 3 & ③ \end{cases}$$

start ①: $\lambda x + \frac{1}{2}x = 0$

$x(\lambda + \frac{1}{2}) = 0$

$x = 0$

Plug into 3

$y + \frac{1}{2}(0)^2 = 3$

$y = 3$

$\lambda + \frac{1}{2} = 0$

$\lambda = -\frac{1}{2}$

Plug into 2

$-\frac{1}{2}y = -\frac{1}{2}$

$y = 1$

Plug into 3

$1 + \frac{1}{2}x^2 = 3$

$x^2 = 4$

$x = \pm 2$

(x,y)	$f(x,y)$
$(0,3)$	$7/4$
$(\pm 2, 1)$	$11/4$ ← max height

Method of Lagrange Multipliers: To find the maximum and minimum values attained by a function $f(x, y, z)$ subject to a constraint $g(x, y, z) = c$, find all points where $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = c$ and compute the value of f at these points.

If we have more than one constraint $g(x, y, z) = c_1, h(x, y, z) = c_2$, then find all points where $\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$ and $g(x, y, z) = c_1, h(x, y, z) = c_2$.

$\nabla f \in \text{Span}(\nabla g, \nabla h, \dots)$

Example 69. Find the points on the surface $z^2 = xy + 4$ that are closest to the origin.

Objective function: Minimize distance btwn (x,y,z) & $(0,0,0)$

$$d(x,y,z) = \sqrt{x^2+y^2+z^2}$$

Actually: $f = x^2+y^2+z^2 = \text{distance}^2$

Constraint function

$\hookrightarrow z^2 - xy = 4$

$$g(x,y,z) = z^2 - xy$$

After class

- $\nabla f = \lambda \nabla g$, $g = 4$ $\Rightarrow$ $\langle 2x, 2y, 2z\rangle = \lambda\langle -y, -x, 2z\rangle$, $z^2 - xy = 4$

$$\begin{cases} 2x = -\lambda y & (1) \\ 2y = -\lambda x & (2) \\ 2z = 2\lambda z & (3) \\ z^2 - xy = 4 & (4) \end{cases}$$

Start with (3):

$z - \lambda z = 0$

$z(1-\lambda) = 0$

$z = 0$:

Plug into (4)

$-xy = 4$ (5)

- This means $x \neq 0, y \neq 0$ so we can safely multiply (1) by x and (2) by y:

$2x^2 = -\lambda xy = 4\lambda$

$2y^2 = -\lambda xy = 4\lambda$

So $x^2 = 2\lambda = y^2$,

so either $x = y$ or $x = -y$

$x=y$: $-y^2 = 4$ impossible

$x=-y$: $y^2 = 4$, $y = \pm 2 \rightarrow (2,-2,0)$, $(-2,2,0)$ are solutions

$1 - \lambda = 0$

$\lambda = 1$

Plug into (1) & (2):

$2x = -y$ (6)

$2y = -x$ (7)

Plug (6) into (7):

$2(-2x) = -x$

$-4x = -x$

$x = 0$

so $y = 0$

Plug into (4):

$z^2 - 0 = 4$

$z = \pm 2$

So $(0,0,2)$, $(0,0,-2)$ are solutions

Finally, check values (using d, not f)

$d(0,0,\pm 2) = 2$

$d(\pm 2, \mp 2, 0) = \sqrt{8}$

So the closest points on $z^2 = xy + 4$ to the origin are $(0,0,\pm 2)$.

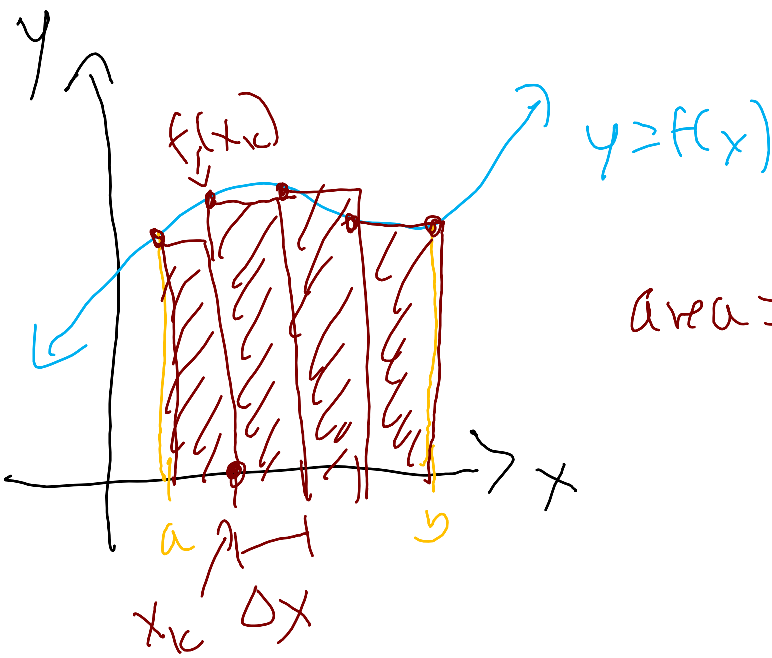
Daily Announcements & Reminders:

- HW 14.7/14.8 due tonight
- Exam 2 on T 10/24 : 14.2–14.8 & 15.1–15.4
- Do the warmup
- No office hours today

Goals for Today: Sections 15.1, 15.2

- Introduce double and iterated integrals for functions of two variables on rectangles
- Use Fubini's Theorem to change the order of integration of a iterated integral
- Be able to set up & evaluate a double integral over a general region
- Change the order of integration for general regions

Recall: Riemann sum and the definite integral from single-variable calculus.



$$\text{area} \approx \sum_{k=1}^{n} f(x_k)\,\Delta x$$

$$\text{area} = \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x$$

Double Integrals

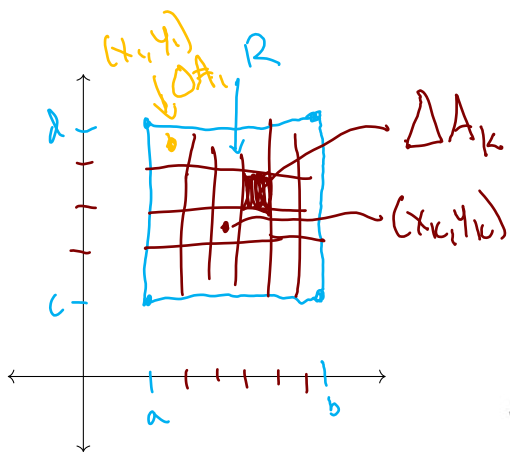
Volumes and Double integrals Let R be the closed rectangle defined below:

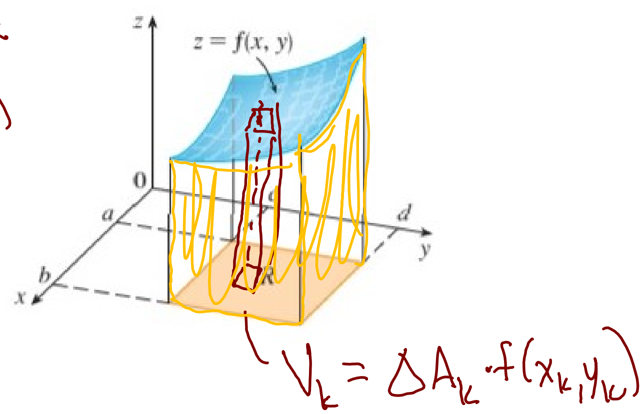
y range

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 | a \leq x \leq b, c \leq y \leq d\}$$

x range

Let $f(x, y)$ be a function defined on R such that $f(x, y) \geq 0$. Let S be the solid that lies above R and under the graph f.





Question: How can we estimate the volume of S?

$$\text{Volume}(S) \approx \sum_{k=1}^{n} f(x_k, y_k) \Delta A_k$$

Definition 70. The <u>double integral</u> of $f(x, y)$ over a rectangle R is

$$\iint_R f(x, y)\, dA = \lim_{|P|\to 0} \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k$$

R ↑ region of integration

↑ $|P|$ is biggest size of all rectangles in your subdivision

if this limit exists.

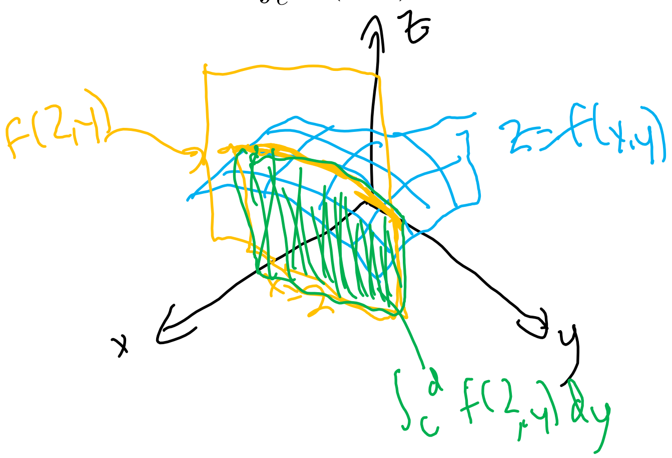
- $\iint_R f(x,y)\, dA$ = signed volume between $z=f(x,y)$ & $z=0$ above R
- If f is cts on R, the limit exists
- Some discontinuous f are integrable

Question: How can we compute a double integral?

Answer: Iterated Integrals

Suppose that f is a function of two variables that is integrable on the rectangle $R = [a, b] \times [c, d]$.

What does $\int_c^d f(2, y)dy$ represent?



- $\int_c^d f(2,y)\,dy$ is the cross-sectional area of the cross-section of the solid at $x=2$

What about $\underbrace{\int_c^d f(x,y)dy}_{\text{(treat } x \text{ as constant)}}$? area of the cross-section of the solid for any x

Let $A(x) = \int_c^d f(x,y)dy$. Then,

$$\text{Volume} = \int_a^b A(x)dx = \int_a^b \left(\int_c^d f(x,y)\, dy \right) dx$$

This is called an iterated integrals.

Q: Does it matter which order we integrate?

Example 71. Evaluate $\int_1^2 \int_3^4 6x^2 y\, dy\, dx$. $\overset{?}{=} \int_3^4 \int_1^2 6x^2y\, dx\, dy$

$$= \int_1^2 \left(6x^2 \cdot \frac{y^2}{2} \Big|_{y=3}^{y=4} \right) dx = \int_1^2 6x^2 \cdot 8 - 6x^2 \cdot \frac{9}{2}\, dx$$

$$= \int_1^2 21x^2\, dx$$

$$= 7x^3 \Big|_1^2$$

$$= 7(8-1) = \boxed{49}$$

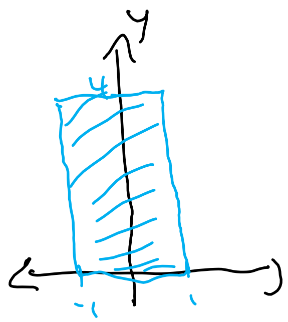
The volume above the rectangle $[1,2]\times[3,4]$ under $z = 6x^2y$ is 49.

Theorem 72 (Fubini's Theorem)**.** *If f is continuous on the rectangle $R = [a,b] \times [c,d]$, then*

$$\int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy = \iint_R f(x,y)\, dA$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Example 73. Compute $\iint_R xe^{e^{e^y}}\, dA$, where R is the rectangle $[-1,1]\times[0,4]$.



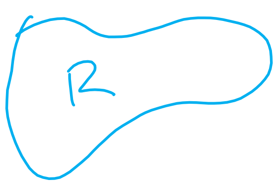
$$V = \int_{-1}^{1}\int_{0}^{4} xe^{e^{e^y}}\, dy\, dx \leftarrow \text{HARD}$$

$$V = \int_{0}^{4}\int_{-1}^{1} xe^{e^{e^y}}\, dx\, dy \leftarrow \text{better}$$

$$= \int_0^4 \frac{1}{2}x^2\left(e^{e^{e^y}}\right)\Big|_{x=-1}^{x=1} dy$$

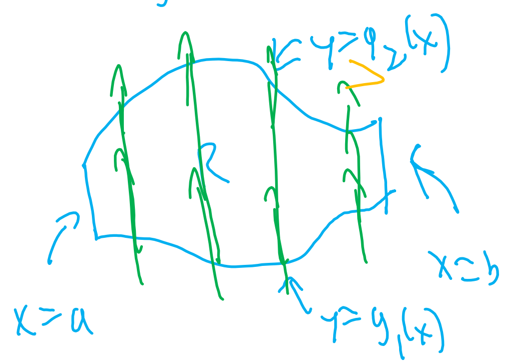
$$= \int_0^4 0\, dy = 0$$

Question: What if the region R we wish to integrate over is not a rectangle?

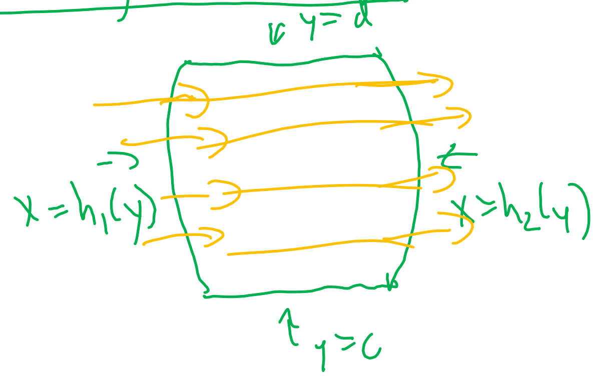


Answer: Repeat same procedure - it will work if the boundary of R is smooth and f is continuous.

Vertically simple region



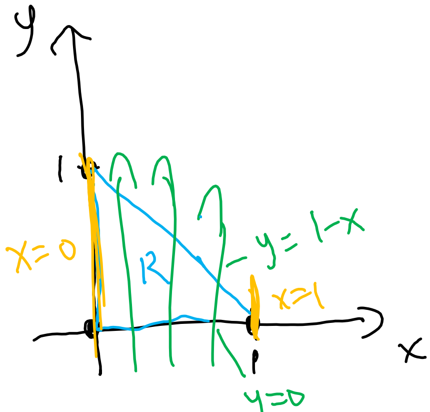
Horizontally simple region

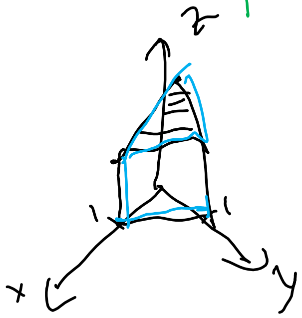


Example 74. Compute the volume of the solid whose base is the triangle with vertices $(0,0),(0,1),(1,0)$ in the xy-plane and whose top is $z = 2 - x - y$.

Vertically simple:

Fubini: $\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$





$$V = \iint_R 2-x-y\,dA$$

no y!

$$= \int_0^1 \int_0^{1-x} 2-x-y\,dy\,dx$$

ALWAYS CONSTANTS

$$= \int_0^1 2y - xy - \frac{y^2}{2}\Big|_{y=0}^{y=1-x} dx$$

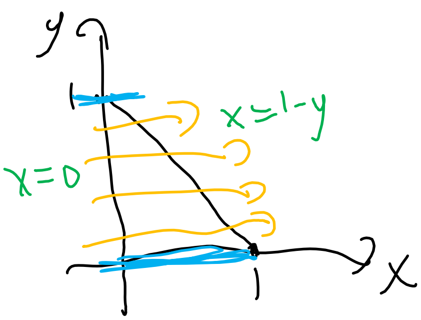
$$= \int_0^1 2(1-x) - x(1-x) - \frac{(1-x)^2}{2} - (0-0-0)\,dx$$

$$= \int_0^1 \frac{3}{2} - 2x + \frac{1}{2}x^2\,dx$$

$$= \frac{3}{2}x - x^2 + \frac{1}{6}x^3\Big|_0^1 = \frac{3}{2} - 1 + \frac{1}{6} - 0$$

$$= \boxed{\frac{2}{3}}$$

Horizontally simple:



$$V = \int_0^1 \int_0^{1-y} 2-x-y\,dx\,dy = \frac{2}{3}$$

Example 75. Write the two iterated integrals for $\iint_R 1\ dA$ for the region R which is bounded by $y = \sqrt{x}, y = 0$, and $x = 9$.

Example 76. Set up an iterated integral to evaluate the double integral $\iint_R 6x^2 y\ dA$, where R is the region bounded by $y = 1, x = 0, x = 2$, and $y = x$.

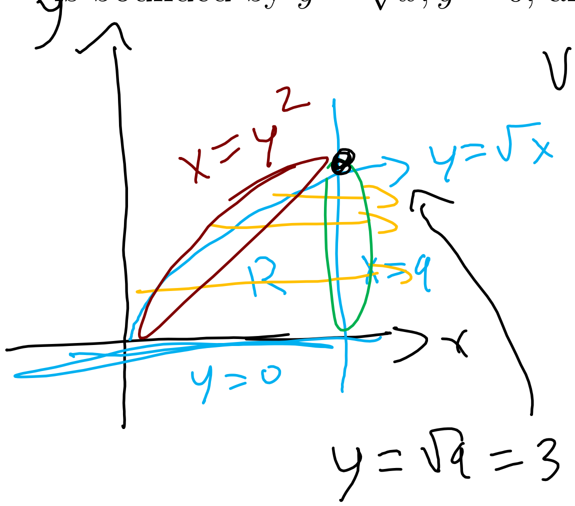
Daily Announcements & Reminders:

- HW 15.1 due tonight
- Quiz 6 tomorrow on 15.1/15.2
- See Canvas for midterm 2 info

Goals for Today: Sections 15.3, 15.4

- Compute areas of general regions in the plane
- Compute the average value of a function of two variables
- Introduce the polar coordinate system
- Convert double integrals to iterated polar integrals
- Compute iterated polar integrals

Example 75. Write the two iterated integrals for $\iint_R 1\, dA$ for the region R which is bounded by $y = \sqrt{x}, y = 0$, and $x = 9$.

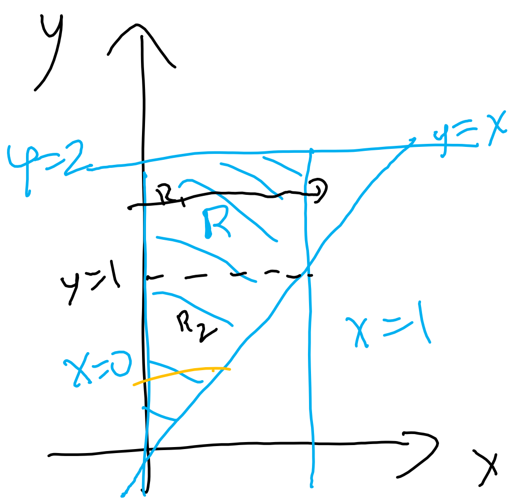


V.S.: $\iint_R 1\, dA = \int_0^9 \int_0^{\sqrt{x}} 1\, dy\, dx$

H.S.: $\iint_R 1\, dA = \int_0^3 \int_{y^2}^{9} 1\, dx\, dy$

Example 76. Set up an iterated integral to evaluate the double integral $\iint_R 6x^2y\, dA$, where R is the region bounded by ~~$y = 1, x = 0, x = 2$, and y =~~ x.

$x=1$ $x=0$ $y=2$ $y=x$



- Only v.s.

- $\iint_R 6x^2y\, dA = \iint_{R_1} 6x^2y\, dA + \iint_{R_2} 6x^2y\, dA$

independent of order but is only needed for $dx\, dy$

V.S: $\int_0^1 \int_x^2 6x^2y\, dy\, dx$

OR $\int_1^2 \int_0^1 6x^2y\, dx\, dy + \int_0^1 \int_0^y 6x^2y\, dx\, dy$

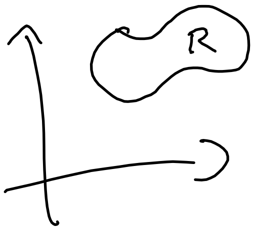
Area & Average Value

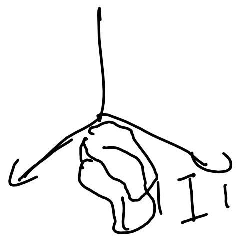
Two other applications of double integrals are computing the area of a region in the plane and finding the average value of a function over some domain.

no units

Area: If R is a region bounded by smooth curves, then

$$\text{Area}(R) = \iint_R 1\, dA$$





z = f(x,y) has units of length

$$\boxed{V = \iint_R 1\, dA} = \text{area}(R)\cdot 1$$

Example 77. Find the area of the region R bounded by $y = \sqrt{x}$, $y = 0$, and $x = 9$.

$$\text{area}(R) = \iint_R 1\, dA = \int_0^3 \int_{y^2}^9 1\, dx\, dy$$

$$= \int_0^3 9 - y^2\, dy$$

$$= 9y - y^3/3 \Big|_0^3 = 27 - 9 = 18$$

Average Value: The average value of $f(x,y)$ on a region R contained in $\mathbb{R}^2$ is

$$f_{avg} = \frac{1}{\text{area}} \iint_R f(x,y)\, dA$$

$f_{avg} = 5{,}00$ ppl / sq mile on a region with area 100 ~~sq miles~~

How many ppl? $f_{avg} \cdot \text{area} = 500 \cdot 100 = 50{,}000$

Example 78. Find the average temperature on the region R in the previous example if the temperature at each point is given by $T(x, y) = 4xy^2$. °C

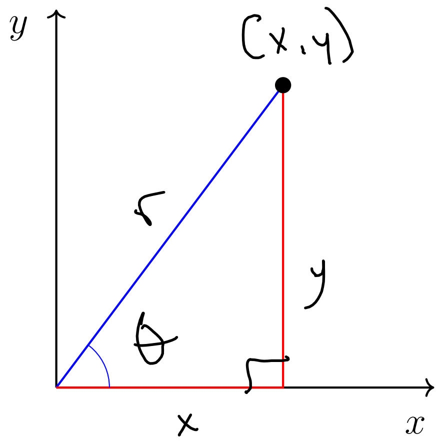
$$T_{avg} = \frac{1}{\text{area}} \iint_R 4xy^2 \, dA$$

$$= \frac{1}{18} \int_0^3 \int_{y^2}^9 4xy^2 \, dx \, dy$$

$$\vdots$$

$$=$$

Polar Coordinates:



Cartesian coordinates: Give the distances in $\hat{i}$ and $\hat{j}$ directions from (0,0)

Polar coordinates:

- r = distance from (0,0) to (x,y)
- θ = angle between the ray $\overrightarrow{OP}$ and the positive x-axis (measured CCW)

We can use trigonometry to go back and forth.

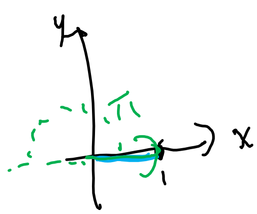
Polar to Cartesian:

$$x = r\cos(\theta) \qquad y = r\sin(\theta)$$

Cartesian to Polar:

$$r^2 = x^2 + y^2 \qquad \tan(\theta) = \frac{y}{x}$$

• not unique



$(x,y) = (1,0) \Longleftrightarrow (r,\theta) = (1,0)$

OR $(r,\theta) = (-1,\pi)$

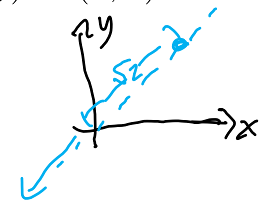
When we integrate: $r \geq 0$, $0 \leq \theta \leq 2\pi$

Example 79. a) Find a set of polar coordinates for the point $(x, y) = (1, 1)$.

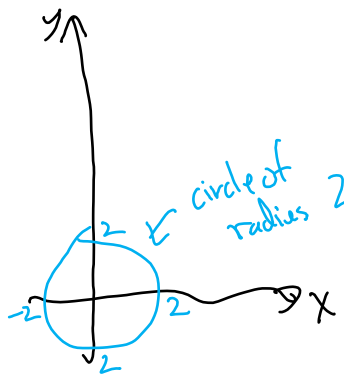
$r^2 = 1^2 + 1^2 = 2$ $\tan(\theta) = y/x = 1/1 = 1$

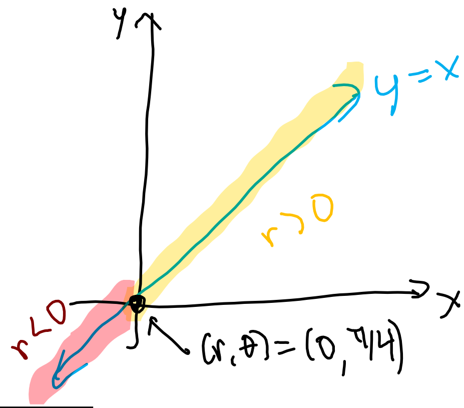
$r = \sqrt{2}$ $\theta = \pi/4$

$r = -\sqrt{2}$ $\theta = 5\pi/4$ ✓



b) Graph the set of points (x, y) that satisfy the equation $r = 2$ and the set of points that satisfy the equation $\theta = \pi/4$ **in the xy-plane.**





c) Write the function $f(x, y) = \sqrt{x^2 + y^2}$ in polar coordinates.

$$f(r,\theta) = \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2} = \sqrt{r^2(\underbrace{\cancel{\cos^2\theta + \sin^2\theta}}_{1})} = |r|$$

$$f(r,\theta) = \sqrt{r^2} = |r|$$

d) [Itempool] Write a Cartesian equation describing the points that satisfy $r = 2\sin(\theta)$.

$r = 2\sin\theta \iff \underbrace{\quad}_{\text{eqn with } x\text{'s } \& \ y\text{'s}}$

$r^2 = 2r\sin\theta$

$\boxed{x^2 + y^2 = 2y}$

$x^2 + y^2 - 2y + 1 = 1$

$x^2 + (y-1)^2 = 1$

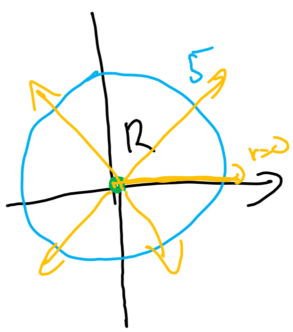
• circle of radius 1 centered at $(0,1)$

Double Integrals in Polar Coordinates

Goal: Given a region R in the xy-plane described in polar coordinates and a function $f(r,\theta)$ on R, compute $\iint_R f(r,\theta)\, dA$.

Example 80. Compute the area of the disk of radius 5 centered at $(0,0)$.

$dA = r\, dr\, d\theta$



Radially simple

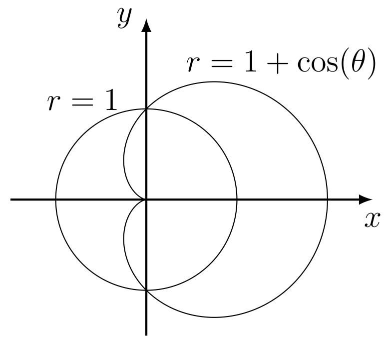
$$\text{Area} = \iint_R 1\, dA = \int_0^{2\pi}\int_0^5 1\, r\, dr\, d\theta = \int_0^{2\pi} \frac{1}{2}r^2\Big|_0^5 d\theta = \int_0^{2\pi} \frac{25}{2}\, d\theta = 25\pi$$

Remember: In polar coordinates, the area form $dA =$________________

Example 81. Compute $\iint_D e^{-(x^2+y^2)}\, dA$ on the washer-shaped region $1 \le x^2 + y^2 \le 4$.

Example 82. Compute the area of the smaller region bounded by the circle $x^2 + (y-1)^2 = 1$ and the line $y = x$.

Example 83 (Itempool)**.** Write an integral for the volume under $z = x$ on the region between the cardioid $r = 1 + \cos(\theta)$ and the circle $r = 1$, where $x \geq 0$.



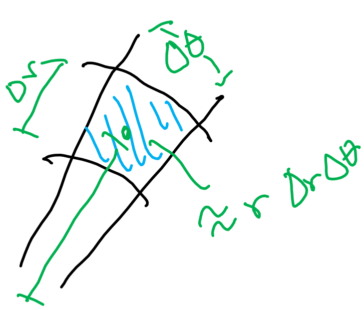
Daily Announcements & Reminders:

- 15.2, 15.3 HW due tonight, 15.4 due Tuesday
- Exam 2 on T, see Canvas
- No quiz next week or during Thanksgiving break
  - Quiz 10 is the M 11/27

Goals for Today: Sections 15.5, 15.6

- Practice computing polar double integrals
- Define triple integrals
- Learn how to write triple integrals as iterated integrals.
- Compute triple iterated integrals
- Change the order of integration in a triple iterated integral.

Remember: In polar coordinates, the area form $dA =$ $r\,dr\,d\theta$



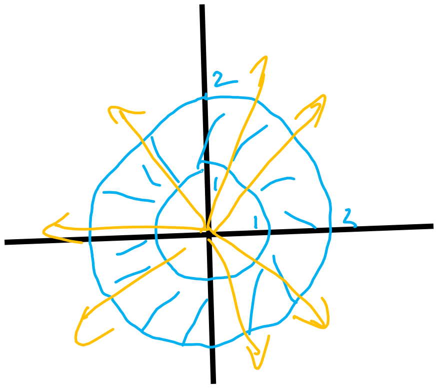
- Good for regions that are radially simple
- Integrand is simpler in polar coordinates
  often: x^2+y^2 is in integrand
- When we are integrating: $0 \le r$, $0 \le \theta \le 2\pi$ *

Example 81. Compute $\iint_D e^{-(x^2+y^2)}\, dA$ on the washer-shaped region $1 \le x^2 + y^2 \le 4$.

Q: Is D H.S or V.S.? No.

$1 \le r^2 \le 4$

$1 \le r \le 2$



1) Describe D in polar coords

$1 \le r \le 2$; $0 \le \theta \le 2\pi$

2) Convert integrand

e^{-r^2}

$du = -2r\,dr$

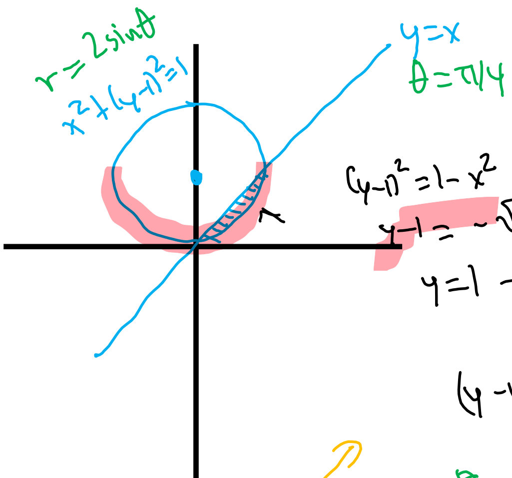
$u = -r^2$

$$\iint_D e^{-(x^2+y^2)} dA = \int_0^{2\pi} \int_1^2 e^{-r^2} \underbrace{r\, dr\, d\theta}_{dA} = \left(\int_0^{2\pi} d\theta\right) \cdot \left(\int_1^2 e^{-r^2} r\, dr\right)$$

$$= 2\pi \left(\int_{-1}^{-4} -\tfrac{1}{2} e^u\, du \right)$$

$$= 2\pi \left(\tfrac{1}{2}(e^{-1} - e^{-4}) \right)$$

Example 82. Compute the area of the smaller region bounded by the circle $x^2 + (y-1)^2 = 1$ and the line $y = x$.



• region is simple in all ways

$$A = \iint_R 1\, dA = \int_0^{\pi/4} \int_0^{2\sin\theta} r\, dr\, d\theta$$

$(y-1)^2 = 1 - x^2$

$y - 1 = -\sqrt{1-x^2}$

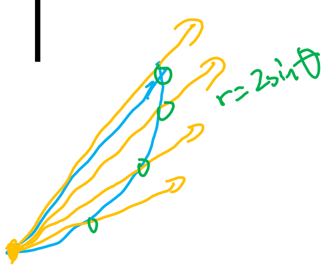
$y = 1 - \sqrt{1-x^2}$

$$= \int_0^1 \int_{1-\sqrt{1-x^2}}^{x} 1\, dy\, dx$$

$(y-1)^2 = 1 - y^2$

$\vdots$

$y = 1, 0$



$$= \int_0^{\pi/4} \tfrac{1}{2} (2\sin\theta)^2\, d\theta$$

$2\sin^2\theta = 2\left(\tfrac{1}{2}(1-\cos(2\theta))\right)$

$$= \int_0^{\pi/4} 1 - \cos(2\theta)\, d\theta$$

$$= \theta - \tfrac{1}{2}\sin(2\theta) \Big|_0^{\pi/4}$$

$$= \left(\tfrac{\pi}{4} - \tfrac{1}{2}\right) - (0 - 0)$$

$\sin^2(\theta) + \cos^2(\theta) = 1$

15.5 Triple Integrals

Idea: Suppose D is a solid region in $\mathbb{R}^3$. If $f(x, y, z)$ is a function on D, e.g. mass density, electric charge density, temperature, etc., we can approximate the total value of f on D with a Riemann sum.

$$\sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta V_k,$$

by breaking D into small rectangular prisms ΔV_k. Taking the limit gives a

triple integral : $\iiint_D f(x, y, z)\, dV$

Important special case:

$$f_{\text{avg on } D} = \frac{\iiint_D f\, dV}{\text{volume of } D}$$

$$\iiint_D 1\, dV = \text{volume of } D$$

Again, we have Fubini's theorem to evaluate these triple integrals as iterated integrals.

Computationally, this is straightforward.

CONSTANTS — only outer variable — only outer 2 variables

Example 83. 1. **Mechanics:** Compute $\int_0^1 \int_0^{2-x} \int_0^{2-x-y} 1 \, dz \, dy \, dx.$

$$= \int_0^1 \int_0^{2-x} z \Big|_{z=0}^{z=2-x-y} dy\, dx$$

$$= \int_0^1 \int_0^{2-x} 2-x-y \, dy\, dx$$

$$= \int_0^1 (2-x)y - \frac{y^2}{2} \Big|_{y=0}^{y=2-x} dx$$

$$= \int_0^1 (2-x)^2 - \frac{(2-x)^2}{2} \, dx$$

$$= \int_0^1 \frac{(2-x)^2}{2} \, dx$$

$$= -\frac{(2-x)^3}{6} \Big|_0^1$$

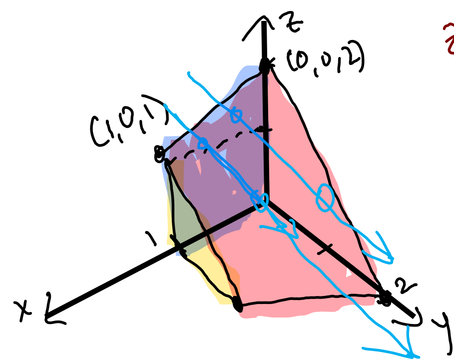
$$= -\frac{1}{6} + \frac{8}{6} = \boxed{\frac{7}{6}}$$

2. **Interpretation:** What shape is this the volume of?

$0 \le z \le 2-x-y$

$0 \le y \le 2-x$

$0 \le x \le 1$



$z = 2-x-y$ plane

$\Updownarrow$

$y = 2-x-z$

3. **Rearrange:** Write an equivalent iterated integral in the order $dy\, dz\, dx$.

$$\int_0^1 \int_0^{2-x} \int_0^{2-x-z} 1 \; dy \; dz \; dx$$

1) Find inner bounds by drawing arrows

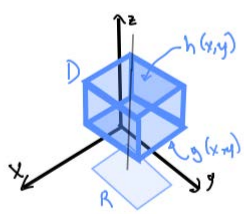
2) Sketch shadow and set up double integral

We will think about converting triple integrals to iterated integrals in terms of the shadow of D on one of the coordinate planes.

Case 1: z**-simple**) region. If R is the shadow of D on the xy-plane and D is bounded above and below by the surfaces $z = h(x, y)$ and $z = g(x, y)$, then

$$\iiint_D f(x,y,z)\ dV = \iint_R \left(\int_{g(x,y)}^{h(x,y)} f(x,y,z)\ dz \right)\ dy\ dx$$

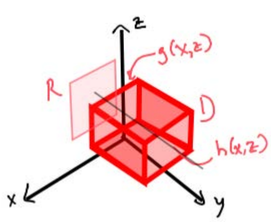
$dx\,dy$



Case 2: y**-simple**) region. If R is the shadow of D on the xz-plane and D is bounded right and left by the surfaces $y = h(x, z)$ and $y = g(x, z)$, then

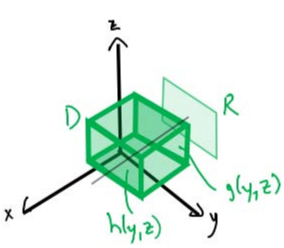
$$\iiint_D f(x,y,z)\ dV = \iint_R \left(\int_{g(x,z)}^{h(x,z)} f(x,y,z)\ dy \right)\ dz\ dx$$

$dx\,dz$



Case 3: x**-simple**) region. If R is the shadow of D on the yz-plane and D is bounded front and back by the surfaces $x = h(y, z)$ and $x = g(y, z)$, then

$$\iiint_D f(x,y,z)\ dV = \iint_R \left(\int_{g(y,z)}^{h(y,z)} f(x,y,z)\ dx \right)\ dz\ dy$$



Example 84. Write an integral for the volume of the solid in the first octant bounded by $z = 3 - x^2 - y^2$ and $z = 2y$ treating the solid as a) z-simple and b) x-simple. Is the solid also y-simple?

1st octant: $x \geq 0, y \geq 0, z \geq 0$

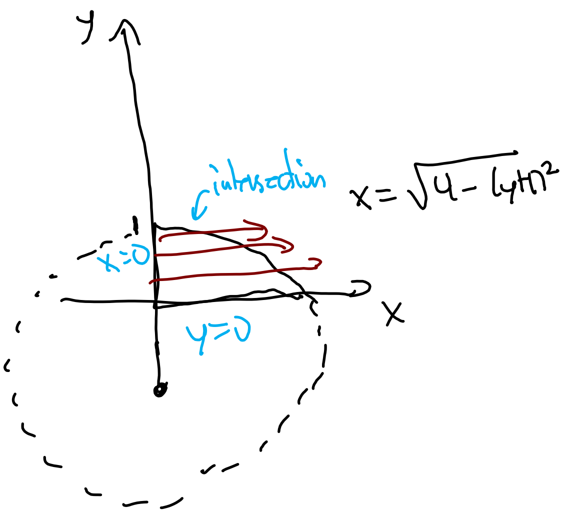
a) z-simple $\displaystyle\int_0^1 \int_0^{\sqrt{4-(y+1)^2}} \int_{2y}^{3-x^2-y^2} 1 \, dz \, dx \, dy$

- z-bounds: enter at plane $z = 2y$
  exit at paraboloid $z = 3 - x^2 - y^2$

$2y = 3 - x^2 - y^2$

$x^2 + (y+1)^2 = 4$

- Sketch shadow



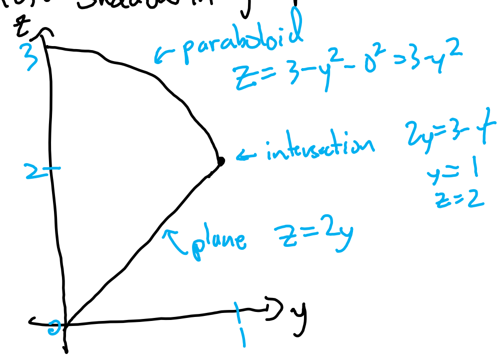
Example 84 (cont.) $\boxed{\text{Post Class}}$

$0 \le x$
$0 \le y$
$2y \le z \le 3 - x^2 - y^2$

b) x-simple: $\displaystyle\int_0^1 \int_{2y}^{3-y^2} \int_0^{\sqrt{3-y^2-z}} 1 \, dx \, dz \, dy$

1) x-bounds: arrows enter the region at $x=0$
and leave at paraboloid: $z = 3 - x^2 - y^2$
rearrange: $x^2 = 3 - y^2 - z$
$x = \sqrt{3 - y^2 - z}$

2) Sketch shadow in yz-plane $(x=0)$



• Just vertically simple, so we use $dz\,dy$

Daily Announcements & Reminders:

- HW 15.5 due tonight (15.6)
- Exam scores will be released tomorrow
  - not graded work until Tuesday

Goals for Today: Sections 15.6, 15.7

- Apply our work to find the mass and center of mass of objects in $\mathbb{R}^2$ and $\mathbb{R}^3$
- Be able to convert between Cartesian, cylindrical, and spherical coordinate systems in $\mathbb{R}^3$
- Compute triple integrals expressed in cylindrical coordinates
- Compute triple integrals expressed in spherical coordinates

15.6: Applications

Suppose $\delta(x, y, z)$ is the mass density (mass/unit volume in $\mathbb{R}^3$, mass/unit area in $\mathbb{R}^2$). Then one could approximate the mass of a volume D by breaking D into small rectangular prisms of volume ΔV_k and computing

$$\text{mass}(D) \approx \sum_{k=1}^{n} \delta(x_k, y_k, z_k)\Delta V_k \quad \Rightarrow \quad \text{mass}(D) = \iiint_D \delta(x, y, z)\, dV,$$

by taking the limit. Similarly, one can find formulas for the moment of D around each different coordinate plane and therefore a formula for the center of mass of D.

TABLE 15.1 Mass and first moment formulas

THREE-DIMENSIONAL SOLID

Mass: $M = \iiint\limits_D \delta\, dV$ $\quad \delta = \delta(x, y, z)$ is the density at (x, y, z).

First moments about the coordinate planes:

$$M_{yz} = \iiint\limits_D x\,\delta\, dV, \qquad M_{xz} = \iiint\limits_D y\,\delta\, dV, \qquad M_{xy} = \iiint\limits_D z\,\delta\, dV$$

Center of mass:

$$\bar{x} = \frac{M_{yz}}{M}, \qquad \bar{y} = \frac{M_{xz}}{M}, \qquad \bar{z} = \frac{M_{xy}}{M}$$

TWO-DIMENSIONAL PLATE

Mass: $M = \iint\limits_R \delta\, dA$ $\quad \delta = \delta(x, y)$ is the density at (x, y).

First moments: $M_y = \iint\limits_R x\,\delta\, dA, \qquad M_x = \iint\limits_R y\,\delta\, dA$

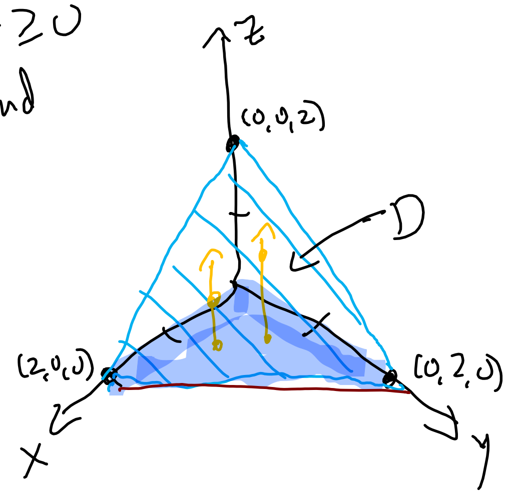
Center of mass: $\bar{x} = \dfrac{M_y}{M}, \qquad \bar{y} = \dfrac{M_x}{M}$

Example 85 (Itempool)**.** A solid region in the first octant is bounded by the plane $x + y + z = 2$. The density of the solid is $\delta(x, y, z) = 2x$. Sketch the solid, then compute its mass and give integral expressions for the coordinates $\bar{x}, \bar{y}, \bar{z}$ of the center of mass.

- $x \geq 0$, $y \geq 0$, $z \geq 0$

$\boxed{x+y+z=2}$ bound

$\hookrightarrow z = 2-x-y$



$$\text{mass} = \iiint_D \delta \, dV$$

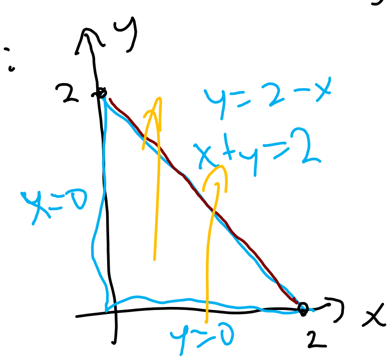
$$= \iiint_D 2x \, dV$$

$$= \underbrace{\int_0^2 \int_0^{2-x}}\int_0^{2-x-y} 2x \, dz \, dy \, dx$$

$$= \frac{4}{3}$$

- Choose direction of integration (x-simple, y-simple, z-simple)
- Find inner bounds
- Find shadow

Shadow:



$$x_{cm} = \frac{\iiint_D x \cdot \delta \, dV}{\iiint_D \delta \, dV}$$

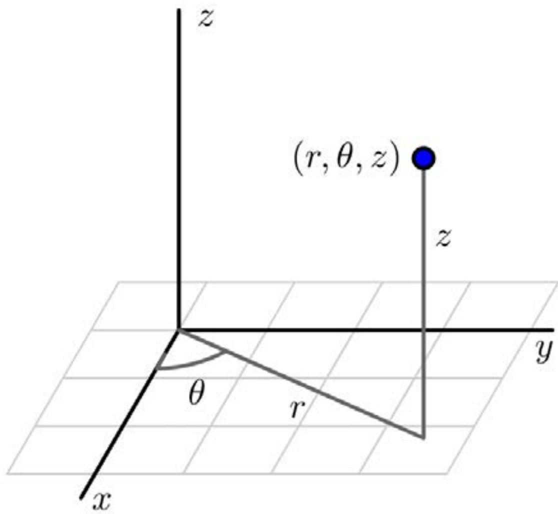
$$= \frac{\int_0^2 \int_0^{2-x} \int_0^{2-x-y} x \cdot 2x \, dz \, dy \, dx}{4/3}$$

CM: (x_{cm}, y_{cm}, z_{cm})

$$y_{cm} = \frac{3}{4} \int_0^2 \int_0^{2-x} \int_0^{2-x-y} y \cdot 2x \, dz \, dy \, dx$$

$$z_{cm} = \frac{3}{4} \int_0^2 \int_0^{2-x} \int_0^{2-x-y} z \cdot 2x \, dz \, dy \, dx$$

Cylindrical Coordinate System



Cylindrical to Cartesian:

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad z = z$$

Cartesian to Cylindrical:

$$r^2 = x^2 + y^2, \quad \tan(\theta) = \frac{y}{x}, \quad z = z$$

- Polar with z-component

For uniqueness:

$r \geq 0$, $\theta \in [0, 2\pi]$

Example 86. a) Find cylindrical coordinates for the point with Cartesian coordinates $(-1, \sqrt{3}, 3)$.

$x \quad y \quad z$

$$(r, \theta, z) = \left(2, \underbrace{\tan^{-1}\left(\frac{\sqrt{3}}{-1}\right)}_{\text{"}5\pi/6\text{"}}, 3\right)$$

$2 = \sqrt{1^2 + \sqrt{3}^2}$

$5\pi/6$

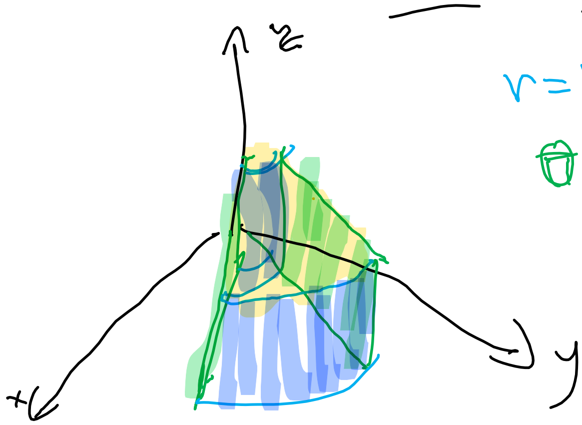
b) Find Cartesian coordinates for the point with cylindrical coordinates $(2, 5\pi/4, 1)$.

$r \quad \theta \quad z$

$$(x, y, z) = \left(2 \cdot \frac{-1}{\sqrt{2}}, 2 \cdot \frac{-1}{\sqrt{2}}, 1\right)$$

Example 87. In xyz-space sketch the *cylindrical box*

$$B = \{(r, \theta, z) \mid 1 \le r \le 2,\ \pi/6 \le \theta \le \pi/3,\ 0 \le z \le 2\}.$$



$r=1$, $r=2$ are cylinders

$\theta=\pi/6$, $\theta=\pi/3$ vert. planes

$z=0$, $z=2$ are horiz. planes

Triple Integrals in Cylindrical Coordinates

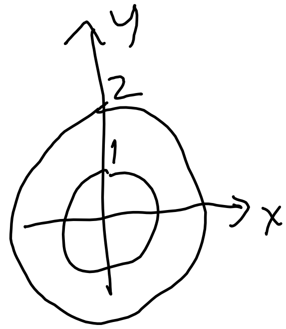
We have $dV =$ $r\,dz\,dr\,d\theta$ or $r\,dr\,dz\,d\theta$

Example 88. Set up a iterated integral in cylindrical coordinates for the volume of the region D lying below $z = x + 2$, above the xy-plane, and between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

$z = x+2$

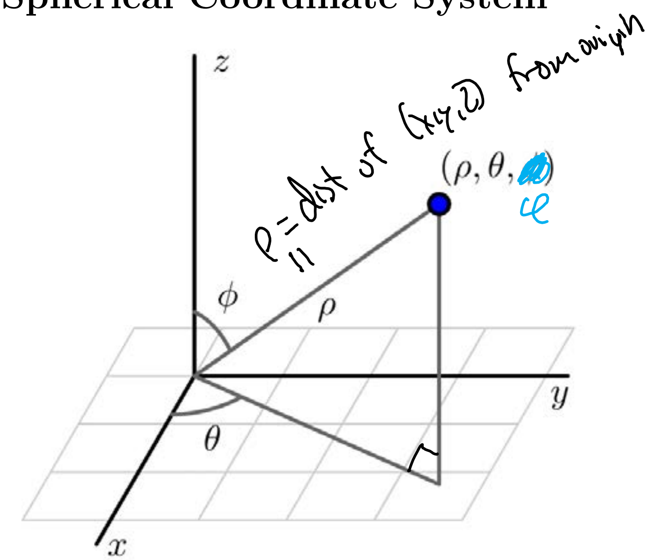
$= r\cos\theta + 2$

$$V = \iiint_D 1\, dV = \int_0^{2\pi} \int_1^2 \int_0^{r\cos\theta+2} 1 \cdot r\,dz\,dr\,d\theta$$



Example 89 (Itempool)**.** Suppose the density of the cone defined by $r = 1 - z$ with $z \geq 0$ is given by $\delta(r, \theta, z) = z$. Set up an iterated integral in cylindrical coordinates that gives the mass of the cone.

Spherical Coordinate System



For uniqueness:

$$\boxed{\rho} \geq 0, \quad \boxed{\theta} \in [0, 2\pi], \quad \boxed{\begin{matrix}\phi \\ \| \\ \varphi\end{matrix}} \in [0, \pi]$$

rho theta phi

Spherical to Cartesian:

$$x = \rho \sin(\varphi) \cos(\theta)$$
$$y = \rho \sin(\varphi) \sin(\theta)$$
$$z = \rho \cos(\varphi)$$

Cartesian to Spherical:

$$\rho^2 = x^2 + y^2 + z^2$$
$$\tan(\theta) = \frac{y}{x}$$
$$\tan(\varphi) = \frac{\sqrt{x^2 + y^2}}{z}$$

Example 90. a) Find spherical coordinates for the point with Cartesian coordinates $(-2, 2, \sqrt{8})$.

x y z

$$\rho = \sqrt{4 + 4 + 8} = 4$$
$$\theta = \arctan\left(\frac{2}{-2}\right) = 3\pi/4$$
$$\varphi = \arctan\left(\frac{\sqrt{8}}{\sqrt{8}}\right) = \pi/4$$

b) Find Cartesian coordinates for the point with spherical coordinates $(2, \pi/2, \pi/3)$.

ρ θ φ

$$x = 2 \sin\frac{\pi}{3} \cos\frac{\pi}{2} = 2 \cdot \frac{\sqrt{3}}{2} \cdot 0 = 0$$
$$y = 2 \sin\frac{\pi}{3} \sin\frac{\pi}{2} = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$$
$$z = 2 \cos\left(\frac{\pi}{3}\right) = 2 \cdot \frac{1}{2} = 1$$

Example 91. In *xyz*-space sketch the *spherical box*

$$B = \{(\rho, \varphi, \theta) \mid 1 \leq \rho \leq 2,\ 0 \leq \varphi \leq \pi/4,\ \pi/6 \leq \theta \leq \pi/3\}.$$

- $1 = \rho$, $2 = \rho$ spheres w/ radius 1, 2
- $\pi/6 = \theta$, $\theta = \pi/3$ are vertical planes
- $0 = \varphi$ (z-axis)
  $\pi/4 = \varphi$ (cone)

Triple Integrals in Spherical Coordinates

We have $dV =$ $\rho^2 \sin\varphi \; d\rho \, d\varphi \, d\theta$

Example 92. Write an iterated integral for the volume of the "ice cream cone" D bounded above by the sphere $x^2+y^2+z^2 = 1$ and below by the cone $z = \sqrt{3}\sqrt{x^2+y^2}$.

Example 93 (Itempool)**.** Write an iterated integral for the volume of the region that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and outside the cylinder $x^2 + y^2 = 1$.

Daily Announcements & Reminders:

- HW 15.6 due tomorrow, 15.7 due Th
- Quiz 7 tomorrow over 15.5 – 15.7 (not spherical coords)
- Graded exams released by tomorrow night

Goals for Today: Section 15.8

- Integrate in spherical coordinates
- Change variables in multiple integrals
- Identify choices for changing variables in a given integration problem

Example 92. Write an iterated integral for the volume of the "ice cream cone" D bounded above by the sphere $x^2+y^2+z^2=1$ and below by the cone $z=\sqrt{3}\sqrt{x^2+y^2}$.

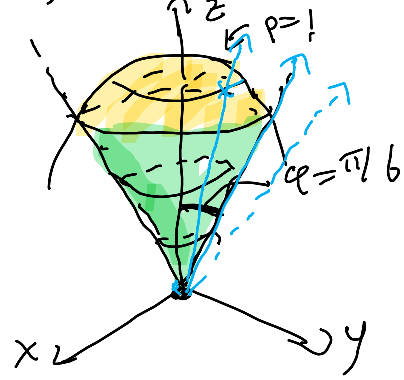
$x = \rho \sin\varphi \cos\theta$
$y = \rho \sin\varphi \sin\theta$
$z = \rho\cos\varphi$

$dV = \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$

$r = \rho\sin\varphi$

$x^2+y^2+z^2=\rho^2$

0) Picture



1) Convert equations to spherical

$\rho^2 = 1$

$\rho = 1$

$\rho\cos\varphi = \sqrt{3}\, r$
$= \sqrt{3}\,\rho\sin\varphi$

$\cos\varphi = \sqrt{3}\sin\varphi$

$\tan\varphi = \frac{1}{\sqrt{3}}$

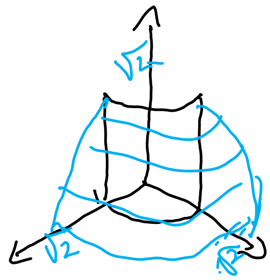
$\varphi = \pi/6$

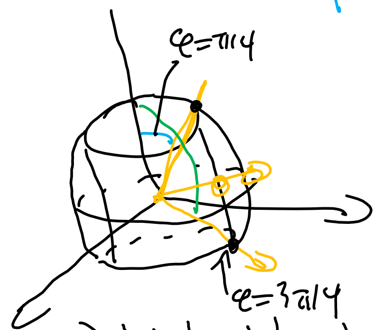
2) Write down integral

$$\int_0^{2\pi}\int_0^{\pi/6}\int_0^1 1\cdot \rho^2\sin\varphi \, d\rho \, d\varphi \, d\theta$$

Example 93 (Itempool)**.** Write an iterated integral for the volume of the region that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and outside the cylinder $\boxed{x^2 + y^2} = 1$.

$\rho \sin\varphi = r = \sqrt{x^2+y^2}$





1) Convert to spherical/cylindrical

$\rho = \sqrt{2}$ $\quad \boxed{\rho^2 \sin^2\varphi = 1}$

$\rho \sin\varphi = 1$

$\rho = \csc\varphi$

2) Find intersections

$\csc\varphi = \sqrt{2}$

$\frac{1}{\sqrt{2}} = \sin\varphi$

$\varphi = \pi/4$ or $3\pi/4$

3) Write integral

$$\int_0^{2\pi} \int_{\pi/4}^{3\pi/4} \int_{\csc\varphi}^{\sqrt{2}} \underbrace{\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta}_{dV}$$

Thinking about single variable calculus: Compute $\displaystyle\int_1^{\sqrt{3}/2} \frac{1}{\sqrt{1-x^2}}\, dx$

$$\int_1^{\sqrt{3}/2} \frac{1}{\sqrt{1-x^2}}\, dx$$

$$= \int_{\pi/2}^{2\pi/3} \frac{1}{\sqrt{1-\sin^2\theta}} \cos\theta \, d\theta$$

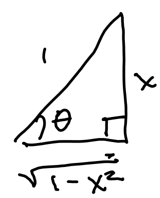
$$= \int_{\pi/2}^{2\pi/3} 1 \, d\theta$$

$x = \sin\theta$

$dx = \cos\theta \, d\theta$

$1 = \sin\theta \rightarrow \theta = \pi/2$

$\frac{\sqrt{3}}{2} = \sin\theta \rightarrow \theta = \frac{2\pi}{3}$



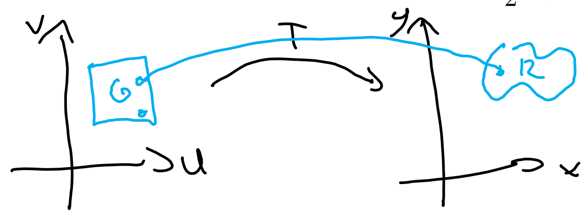
Theorem 94 (Substitution Theorem)**.** *Suppose* $\mathbf{T}(u,v)$ *is a one-to-one, differentiable transformation that maps the region* G *in the uv-plane to the region* R *in the xy-plane. Then*

HARD EASIER

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(\mathbf{T}(u,v))\boxed{|\det(D\mathbf{T}(u,v))|}\,du\,dv.$$

Jacobian - measures distortion
e.g. if Jac $= 2$, area is doubled

Example 95. Evaluate $\displaystyle\int_0^4 \int_{y/2}^{y/2+1} \frac{2x-y}{2}\,dx\,dy$ via the transformation $u = \frac{2x-y}{2}$, $v = \frac{y}{2}$.



1. **Find T:**

$$T(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \end{bmatrix} = \begin{bmatrix} u+v \\ 2v \end{bmatrix} \qquad T^{-1}(x,y) = \begin{bmatrix} x - y/2 \\ y/2 \end{bmatrix}$$

$u = x - \frac{y}{2} \quad v = \frac{y}{2}$

$u = x - v$

$x = u+v \quad y = 2v$

2. **Find G and sketch:**

$0 \le y \le 4$

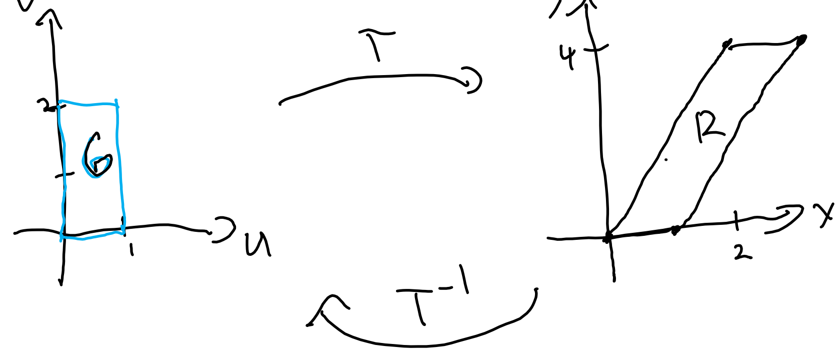
$y/2 \le x \le y/2 + 1$

$0 = y \Rightarrow 0 = 2v \Rightarrow v = 0$

$4 = y \Rightarrow 4 = 2v \Rightarrow v = 2$

$\frac{y}{2} = x \Rightarrow v = u+v \Rightarrow u = 0$

$\frac{y}{2}+1 = x \Rightarrow v+1 = u+v \Rightarrow 1 = u$



$\vec{T} = \begin{bmatrix} u+v \\ 2v \end{bmatrix}$

3. **Find Jacobian:**

$$|\det(DT(u,v))| = \left|\det \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\right|$$
$$= |2-0| = 2$$

4. **Convert and use theorem:**

$$\iint_R \frac{2x-y}{2}\, dA = \int_0^2 \int_0^1 u \cdot 2 \;du\,dv$$
$$= 2$$

$$\frac{2x-y}{2} = \frac{2(u+v)-2v}{2} = u$$

- $T(r,\theta) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$
- $T(\rho, \varphi, \theta) = \begin{bmatrix} \rho \sin\varphi \cos\theta \\ \rho \sin\varphi \sin\theta \\ \rho \cos\varphi \end{bmatrix}$

$DT = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$ $\quad |\det(DT(\rho,\varphi,\theta))| = \rho^2 \sin\varphi$

$\det = r(\cos^2\theta + \sin^2\theta) = r$

Theorem 96 (Derivative of Inverse Coordinate Transformation)**.** *If* $\mathbf{T}(u,v)$ *is a one-to-one differentiable transformation that maps a region* G *in the* uv*-plane to a region* R *in the* xy*-plane, then we have*

$$|\det(\underbrace{D\mathbf{T}(u,v)}_{2\times 2})| = \frac{1}{|\det(\underbrace{D\mathbf{T}^{-1}(x,y)}_{2\times 2})|}$$

$T(u_0, v_0) = (x_0, y_0)$

$$DT(u_0,v_0)\, DT^{-1}(x_0,y_0) = I_2$$

Example 98. a) (Itempool) Find the Jacobian of the transformation

$$x = u + (1/2)v, \ y = v.$$

b) (Itempool) Which transformation seems most suitable for the integral

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3(2x - y)e^{(2x-y)^2} \, dx \, dy?$$

i) $u = x, v = y$

ii) $u = \sqrt{x^2 + y^2}, v = \arctan(y/x)$

iii) $u = 2x - y, \ v = y^3$

iv) $u = 2x - y, \ v = y$

3. **Find Jacobian:**

4. **Convert and use theorem:**

Theorem 96 (Derivative of Inverse Coordinate Transformation)**.** *If* $\mathbf{T}(u, v)$ *is a one-to-one differentiable transformation that maps a region* G *in the* uv*-plane to a region* R *in the* xy*-plane, then we have*

$$|\det(D\mathbf{T}(u, v))| = \frac{1}{|\det(D\mathbf{T}^{-1}(x, y))|}$$

Goal: Use change of variables without needing to calculate $\vec{T}$ if we are given $\vec{T}^{-1}(x,y) = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$.

Example 97. Let's evaluate $\displaystyle\iint_R \frac{y(x+y)}{x^3}\,dA$ where R is the region in the xy-plane bounded by $y = x, y = 3x, y = 1 - x$, and $y = 2 - x$. Consider the coordinate transformation $\underline{u = x + y, v = y/x}$.

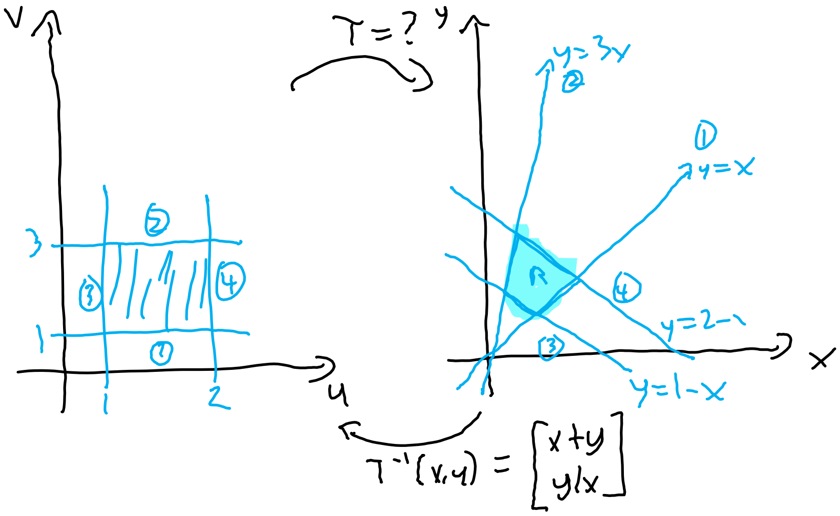
1. **Find the rectangle G in the uv plane that is mapped to R**

$y = x \to \frac{y}{x} = 1 \to v = 1$

$y = 3x \to \frac{y}{x} = 3 \to v = 3$

$y = 1 - x \to x + y = 1 \to u = 1$

$y = 2 - x \to x + y = 2 \to u = 2$



2. **Evaluate $f(\mathbf{T}(u, v))|\det(D\mathbf{T}(u, v))|$ in terms of u and v without directly solving for $\mathbf{T}$ using the theorem above**

$$DT^{-1} = \begin{bmatrix} 1 & 1 \\ -y/x^2 & 1/x \end{bmatrix}$$

$$|\det(DT(u,v))| = \frac{1}{|\det(DT^{-1}(x,y))|} = \frac{1}{\left|\frac{1}{x} + y/x^2\right|} = \frac{1}{\frac{x+y}{x^2}} = \frac{x^2}{x+y}$$

$$f(x,y)\cdot\frac{1}{|\det(DT^{-1}(x,y))|} = \frac{y(x+y)}{x^3}\cdot\frac{x^2}{(x+y)} = \frac{y}{x}$$

So, $f(T(u,v))|\det(DT(u,v))| = v$

3. **Use the Substitution Theorem to compute the integral.**

$$\iint_R \frac{y(x+y)}{x^3}\, dA = \iint_G v\, dA$$

$$= \int_0^3 \int_0^2 v\, du\, dv$$

$$= \int_0^3 uv\Big|_0^2\, dv$$

$$= \int_0^3 2v\, dv$$

$$= v^2\Big|_0^3$$

$$= \boxed{9}$$

Daily Announcements & Reminders:

- HW 15.7 due tonight, 15.8 on T
- Exam 2 regrades open until next W at 5pm

Goals for Today: Section 16.1, 16.2

- Define a line integral for a scalar function $f(x, y)$ or $f(x, y, z)$
- Compute line integrals using parameterizations
- Define and explore vector fields

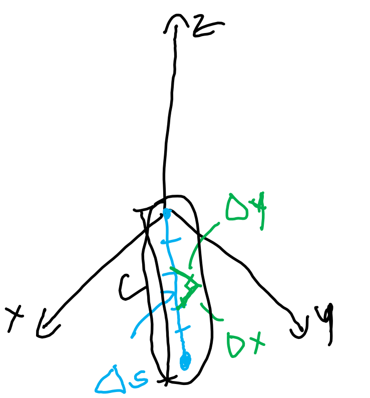
Unit 4: Vector Calculus

Goals:

- Extend 1D / 2D integrals to 1D / 2D objects living in higher-dimensional space
- Extend the fundamental theorem of calculus in new ways

We will use tools from all three units so far to do this: parameterizations, derivatives and gradients, and multiple integrals.

Example 99. Suppose we build a wall whose base is the straight line from $(0,0)$ to $(1,1)$ in the xy-plane and whose height at each point is given by $h(x,y) = 2x + y^2$ meters. What is the area of this wall?



$\Delta s = \sqrt{\Delta x^2 + \Delta y^2}$

$$\text{area} = \lim_{n\to\infty} \sum_{k=1}^{n} h(x_k, y_k)\,\Delta s_k$$

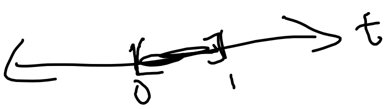
$$\simeq \int_C h(x,y)\,ds$$

$$\boxed{ds = |\vec{r}\,'(t)|\,dt}$$

1) Parameterize C

$\vec{r}(t) = \langle 1,1\rangle t + \langle 0,0\rangle \qquad 0 \le t \le 1$

$= \langle t, t\rangle$



2) Substitute

$$\int_0^1 \underbrace{(2t + t^2)}_{h(\vec{r}(t))} \, |\langle 1,1\rangle|\, dt = \int_0^1 (2t+t^2)\sqrt{2}\,dt$$

$$= \left(t^2 + \tfrac{1}{3}t^3\right)\sqrt{2}\,\Big|_0^1$$

$$= \frac{4\sqrt{2}}{3}$$

Definition 100. The **line integral** of a scalar function $f(x,y)$ over a curve C in $\mathbb{R}^2$ is

$$\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k)\,\Delta s_k$$

$$= \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt \quad \text{if } \vec{r}(t) \text{ parameterizes } C. \text{ with } a \le t \le b$$

What things can we compute with this?

- If $f = 1$: $\int_C 1\,ds = \text{length of } C = \int_a^b |\vec{r}\,'(t)|\,dt$
- If $f = \delta$ is a linear density function: $\int_C \delta\,ds = \text{mass of object lying along } C \text{ with density } \delta$

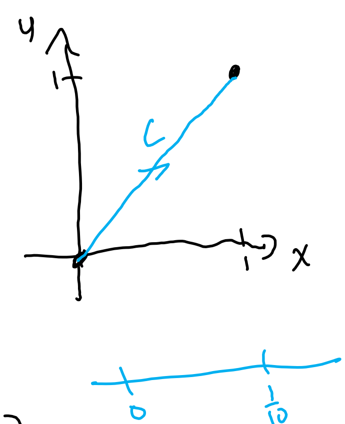
  $\delta = \text{kg/m, g/cm}$
- If f is a moment:

Strategy for computing line integrals:

1. Parameterize the curve C with some $\mathbf{r}(t)$ for $a \le t \le b$
2. Compute $ds = |\mathbf{r}'(t)|\ dt$
3. Substitute: $\int_C f(x,y,z)\ ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\ dt$
4. Integrate

Example 101. (Itempool) Compute $\int_C 2x + y^2\ ds$ along the curve C given by $\mathbf{r}(t) = 10t\mathbf{i} + 10t\mathbf{j}$ for $0 \le t \le \frac{1}{10}$.

$\frac{4\sqrt{2}}{3}$, $\frac{31}{30}$, $\frac{31\sqrt{2}}{300}$



1) Given

2) $|\vec{r}'(t)| = |\langle 10, 10\rangle| = \sqrt{200}$

3) $\int_0^{1/10} \left(2(10t) + (10t)^2\right)\sqrt{200}\ dt$

$= \sqrt{200}\left(10t^2 + \frac{(10t)^3}{30}\right)\Big|_0^{1/10}$

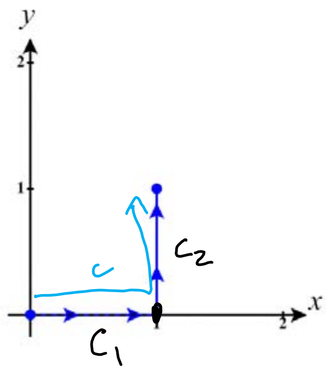
$= \sqrt{200}\left(\frac{1}{10} + \frac{1}{30}\right)$

$= \frac{4}{30}\sqrt{200} = \frac{4}{30}\sqrt{2}\cdot 10 = \frac{4\sqrt{2}}{3}$

- If $\vec{r}_1(t)$ & $\vec{r}_2(t)$ have the same image (are both parametrizations of C)

Then $\int_{a_1}^{b_1} f(\vec{r}_1(t))\,|\vec{r}_1(t)|\,dt = \int_{a_2}^{b_2} f(\vec{r}_2(t))\,|\vec{r}_2(t)|\,dt$

Example 102. Compute $\int_C 2x + y^2 \; ds$ along the curve C pictured below.



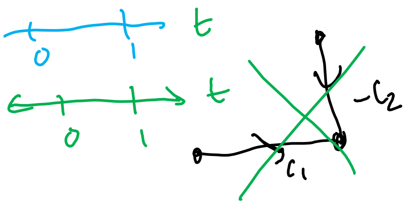
$$\int_C 2x+y^2\,ds = \int_{C_1} 2x+y^2\,ds + \int_{C_2} 2x+y^2\,ds$$

C_1: $\vec{r}_1(t) = \langle 1,0\rangle t + \langle 0,0\rangle \quad 0 \le t \le 1$

$|\vec{r}_1\,'(t)| = |\langle 1,0\rangle| = 1$

C_2: $\vec{r}_2(t) = \langle 0,1\rangle t + \langle 1,0\rangle \quad 0 \le t \le 1$

$|\vec{r}_2\,'(t)| = |\langle 0,1\rangle| = 1$



$$\int_C 2x+y^2\,ds = \int_0^1 \left(2(t)+0^2\right)\cdot 1\,dt + \int_0^1 \left(2(1)+t^2\right)\cdot 1\,dt$$

$$= \int_0^1 2t\,dt + \int_0^1 2+t^2\,dt$$

$$= t^2 + 2t + \frac{t^3}{3}\Big|_0^1 = \boxed{\frac{10}{3}}$$

- Most line integrals are path-dependent

Example 103. (Itempool) Let C be a curve parameterized by $\mathbf{r}(t)$ from $a \le t \le b$. Select all of the true statements below.

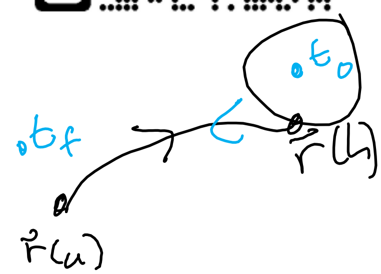
$a-4 \le t \le b-4$

a) $\mathbf{r}(t+4)$ for $a \le t \le b$ is also a parameterization of C with the same orientation

- Domain not shifted to match
- $\vec{r}(a+4)$ to $\vec{r}(b+4)$ is not $\vec{r}(a)$ to $\vec{r}(b)$

b) $\mathbf{r}(2t)$ for $a/2 \le t \le b/2$ is also a parameterization of C with the same orientation

Shifts match



c) $\mathbf{r}(-t)$ for $a \le t \le b$ is also a parameterization of C with the opposite orientation

Shift doesn't match $\quad$ $\vec{r}(-a)$ to $\vec{r}(-b)$

d) $\mathbf{r}(-t)$ for $-b \le t \le -a$ is also a parameterization of C with the opposite orientation

$$\int_C f\,ds = -\int_{-C} f\,ds$$

Shifts match

e) $\mathbf{r}(b-t)$ for $0 \le t \le b-a$ is also a parameterization of C with the opposite orientation

This is d) + correct idea from a)

Daily Announcements & Reminders:

- 15.8 & 16.1 HW due tonight, 16.2 due R
- Quiz 8 tomorrow on 15.8, 16.1

Goals for Today: Section 16.2

- Define and explore vector fields
- Define tangential and normal line integrals for vector fields
- Apply vector line integrals to problems involving work, flow, and flux
- Compute vector line integrals using parameterizations

Vector Fields:

Definition 105. A vector field is a function $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ which associates a vector to every point in its domain.

$\vec{F}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$

Examples:

- Gravity: $\vec{F} = 0\vec{i} + 0\vec{j} - 9.81\vec{k}$
- Electromagnetic Field
- Velocity field for a flowing fluid
- Tangent vectors on a curve/surface
- $\nabla f = \langle f_x, f_y, f_z \rangle$

Graphically: For each point (a, b, c) in the domain of $\mathbf{F}$, draw the vector $\mathbf{F}(a, b, c)$ with its base at (a, b, c).

Tools: CalcPlot3d
Field Play

- $\vec{F}$ is cont. $\iff$ P, Q, R cont.
- $\vec{F}$ is diff. $\iff$ P, Q, R diff.

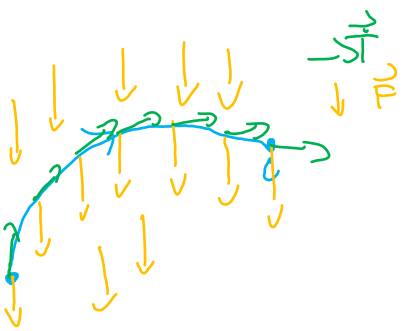
Idea: In many physical processes, we care about the total sum of the strength of that part of a field that lies either in the direction of a curve or perpendicular to that curve.

1. The work done by a field **F** on an object moving along a curve C is given by

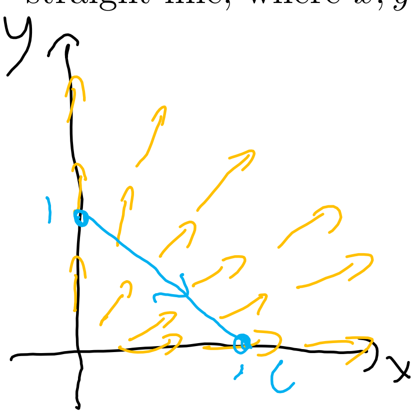
$$\int_C \underbrace{(\vec{F}\cdot\vec{T})}_{\text{component of } \vec{F} \text{ in direction } \vec{T} \text{ along curve}} ds = \int_a^b \left(\vec{F}(\vec{r}(t)) \cdot \underbrace{\frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}}_{\vec{T}(t)}\right) \cancel{|\vec{r}\,'(t)|}\, dt$$

$$= \boxed{\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt}$$

$\cdot\ \vec{r}$ parametrizes C



Example 106. Work Done by a Field. Suppose we have a force field $\mathbf{F}(x, y) = \langle x, y\rangle$ N. Find the work done by **F** on a moving object from $(0, 1)$ to $(1, 0)$ in a straight line, where x, y are measured in meters.



1) Parametrize C: $\vec{r}(t) = \langle 1, -1\rangle t + \langle 0, 1\rangle$
$= \langle t, 1-t\rangle, \quad 0 \le t \le 1$

2) Find $\vec{r}\,'(t)$: $\vec{r}\,'(t) = \langle 1, -1\rangle$

3) Substitute:

$$\int_C (\vec{F}\cdot\vec{T})\, ds = \int_0^1 \langle t, 1-t\rangle \cdot \langle 1, -1\rangle\, dt$$

or

$$\int_C \vec{F}\cdot d\vec{r} \qquad = \int_0^1 t - (1-t)\, dt$$

or

$$= \int_0^1 2t - 1\, dt$$

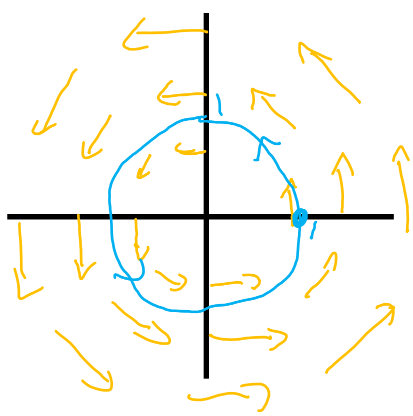
$$\int_C P\,dx + Q\,dy + R\,dz \qquad t^2 - t\Big|_0^1 = \boxed{0}$$

The flow along a curve C of a velocity field $\mathbf{F}$ for a fluid in motion is given by $\int_C \vec{F}\cdot\vec{T}\,ds$

When C is closed, this is called circulation. C is called simple if it does not intersect itself.

	closed	not closed
simple		
not simple		

Example 107. Flow of a Velocity Field. Find the circulation of the velocity field $\mathbf{F}(x, y) = \langle -y, x\rangle$ cm/s around the unit circle, parameterized counterclockwise.



Goal: $\int_C \vec{F}\cdot\vec{T}\,ds$

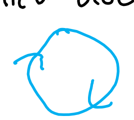
1) Parameterize C: $\vec{r}(t) = \langle \cos(t), \sin(t)\rangle$, $0 \le t \le 2\pi$

2) Compute $\vec{r}\,'(t)$: $\vec{r}\,'(t) = \langle -\sin(t), \cos(t)\rangle$

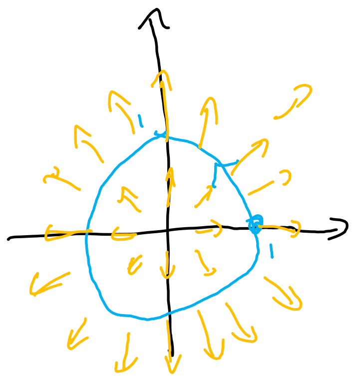
3) Substitute:

$$\text{circulation} = \int_C \vec{F}\cdot\vec{T}\,ds = \int_0^{2\pi} \langle -\sin(t), \cos(t)\rangle \cdot \langle -\sin(t), \cos(t)\rangle\,dt$$
$$= \int_0^{2\pi} \sin^2(t) + \cos^2(t)\,dt$$
$$= \boxed{2\pi}$$

Q: What if C is oriented clockwise? $\Rightarrow$ circulation $= -2\pi$



Example 108. (Itempool) What is the circulation of $\mathbf{F}(x, y) = \langle x, y\rangle$ around the unit circle, parameterized counterclockwise?



$\vec{r}(t) = \langle \cos(t), \sin t\rangle$, $0 \le t \le 2\pi$

$\vec{r}'(t) = \langle -\sin(t), \cos(t)\rangle$

$\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = \langle \cos(t), \sin(t)\rangle \cdot \langle -\sin(t), \cos(t)\rangle$

$= -\cos(t)\sin(t) + \cos(t)\sin(t)$

$= 0$

so $\int_C \vec{F} \cdot \vec{T}\, ds = 0$

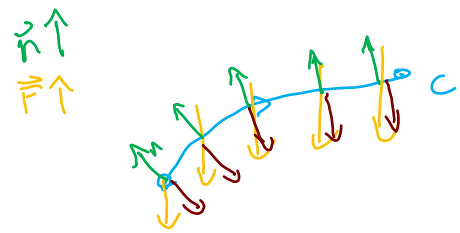
($\vec{F}$ is orthogonal to C)

Strategy for computing tangential component line integrals

e.g. work, flow, circulation integrals

1. Find a parameterization $\mathbf{r}(t)$, $a \le t \le b$ for the curve C.
2. Compute $\mathbf{r}'(t)$.
3. Substitute: $\int_C \mathbf{F} \cdot \mathbf{T}\; ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \underset{\text{dot product}}{\cdot} \mathbf{r}'(t)\; dt$
4. Integrate

Idea: flux across a plane curve of a 2D-vector (velocity) field measures the flow of the field across that curve (instead of along it).



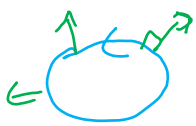
We compute this with the integral

$$\int_C \underbrace{\mathbf{F}\cdot\mathbf{n}}_{\text{portion of } \vec{F} \text{ directly orthogonal to } C}\, ds.$$

The sign of the flux integrals tells us whether the net flow of the field across the curve is in the direction of $\vec{n}$ or in the opposite direction.

$$\int_C \vec{F}\cdot\vec{n}\,ds = \int_a^b \vec{F}(\vec{r}(t))\cdot\frac{\langle y'(t), -x'(t)\rangle}{\cancel{|\vec{r}'(t)|}}\,\cancel{|\vec{r}'(t)|}\,dt$$

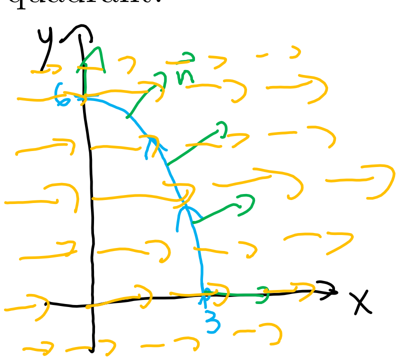
We can choose **n** to be either of



(usual) $\dfrac{\langle y'(t), -x'(t)\rangle}{|\vec{r}'(t)|}$ or $\dfrac{\langle -y'(t), x'(t)\rangle}{|\vec{r}'(t)|}$

if $\vec{r}(t) = \langle x(t), y(t)\rangle$ parameterizes C

Example 109. Flux of a Velocity Field. Compute the flux of the velocity field $\mathbf{v} = \langle 3 + 2y - (y^2/3), 0\rangle$ cm/s across the quarter of the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{36} = 1$ in the first quadrant.



1) Parameterize: $\vec{r}(t) = \langle 3\cos(t), 6\sin t\rangle$, $0 \le t \le \frac{\pi}{2}$

2) Derivatives $\vec{r}\,'(t) = \langle -3\sin(t), 6\cos(t)\rangle$

so use normal $\langle 6\cos(t), 3\sin(t)\rangle$

3) Substitute:

$$\int_0^{\pi/2} \left\langle 3 + 2(6\sin t) - \frac{(6\sin t)^2}{3}, 0\right\rangle \cdot \langle 6\cos(t), 3\sin(t)\rangle\,dt$$

$= 30$

Strategy for computing normal component line integrals

e.g. flux integrals

1. Find a parameterization $\mathbf{r}(t)$, $a \leq t \leq b$ for the curve C.

2. Compute $x'(t)$ and $y'(t)$ and determine which normal to work with.

3. Substitute: $\int_C \mathbf{F} \cdot \mathbf{n}\ ds = \pm \int_a^b P(\mathbf{r}(t))y'(t) + Q(\mathbf{r}(t))y'(t)\ dt$ (sign based on choice of normal)

4. Integrate

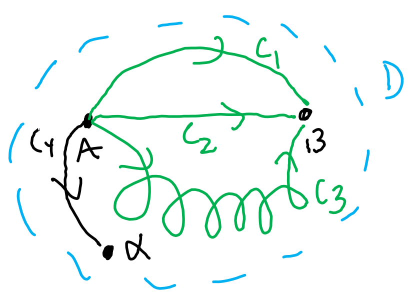
Daily Announcements & Reminders:

- 16.2 HW due tonight

Goals for Today: Section 16.3

- Define conservative vector fields and recognize examples from physics
- Learn how to check if a field is conservative
- Compute potential functions
- Apply the Fundamental Theorem of Line Integrals to compute line integrals of conservative vector fields

Definition 110. A vector field **F** is **path independent** on an open region D if $\int_C \vec{F}\cdot\vec{T}\,ds$ for all paths C in the region that have the same endpoints.



$$\int_{C_1} \vec{F}\cdot\vec{T}\,ds = \int_{C_2} \vec{F}\cdot\vec{T}\,ds = \int_{C_3} \vec{F}\cdot\vec{T}\,ds$$

$$\neq \int_{C_4} \vec{F}\cdot\vec{T}\,ds$$

- gravitational fields are path ind.
- electrostatic field " " "
- spring forces are " " !!

When $\mathbf{F}$ is path independent, we can use the simplest path from point A to point B to compute a line integral, and will often denote the line integral with points as bounds, e.g.

$$\int_{(0,1,2)}^{(3,1,1)} \mathbf{F}\cdot\mathbf{T}\,ds \qquad \text{or} \qquad \int_{(a,b)}^{(c,d)} \mathbf{F}\cdot d\mathbf{r}.$$

Example 111. If C is any closed path and $\mathbf{F}$ is path independent on a region containing C, then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = 0$$

$$\int_{C_1} \vec{F}\cdot d\vec{r} = -\int_{C_2} \vec{F}\cdot d\vec{r}$$

Question: Given $\mathbf{F}$, how do we tell if it is path independent on a particular region?

Hard

For example, is $\mathbf{F}(x,y) = \langle x, y\rangle$ a path independent vector field on its domain?

Don't know.

Example 112. (Itempool) Last time, we saw that if C is the unit circle about the origin, oriented counterclockwise, then $\int_C \langle -y, x\rangle\cdot d\mathbf{r} = 2\pi$. From this, we can conclude:

C is closed! so if $\langle -y, x\rangle$ was path ind. around origin the line integral would be 0

But this is false, so $\langle -y, x\rangle$ is not path ind.

A different idea: Suppose $\mathbf{F}$ is a gradient vector field, i.e. $\mathbf{F} = \nabla f$ for some function of multiple variables f. f is called a potential function for $\mathbf{F}$. In this case we also say that $\mathbf{F}$ is **conservative.**

$f = xy \to \nabla f = \langle y, x\rangle$

ex: $\vec{F} = \langle x, y\rangle$ is conservative

If $\vec{F} = \nabla f$, then $\vec{F} = \langle f_x, f_y\rangle$

$\int f_x \, dx = \int x \, dx \to f = \underbrace{\tfrac{1}{2}x^2 + g(y)}$ $= \boxed{f(x,y) = \tfrac{1}{2}x^2 + \tfrac{1}{2}y^2 + C}$ ← potential

$f_y = y$

$\frac{\partial}{\partial y}$: $f_y = 0 + g'(y) = y \to g(y) = \int y \, dy = \frac{1}{2}y^2 + C$

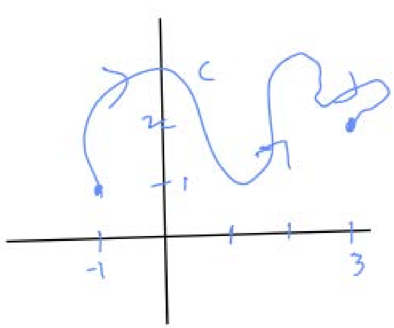
Theorem 113 (Fundamental Theorem of Line Integrals)**.** *If C is a smooth curve from the point A to the point B in the domain of a function f with continuous gradient on C, then*

$$\int_C \nabla f \cdot \mathbf{T}\, ds = f(B) - f(A)$$

$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{T}\, ds$

$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$

Example 114. Compute $\int_C \langle x, y\rangle \cdot d\mathbf{r}$ for the curve C shown below from $(-1, 1)$ to $(3, 2)$.



- $\langle x, y\rangle = \nabla\left(\frac{1}{2}x^2 + \frac{1}{2}y^2\right)$

so by FTofLI

$\int_C \langle x, y\rangle \cdot d\vec{r} = \frac{1}{2}x^2 + \frac{1}{2}y^2 \Big|_{(-1,1)}^{(3,2)}$

$= \frac{1}{2}(9+4) - \frac{1}{2}(1+1)$

$= \frac{11}{2}$

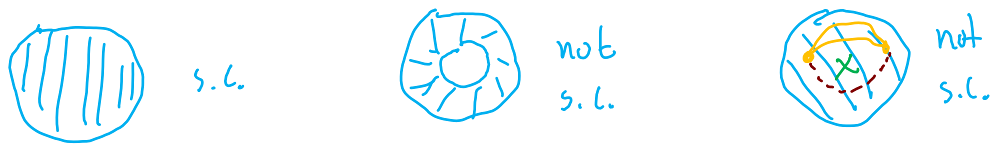
It follows that **every conservative field is path independent.**

$$\left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

In fact, ~~as long as the region we are working on is **simply connected**~~, it is also true that path independent fields are conservative

"no holes"

$\mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^n$ are simply connected



This gives us a better way to test for path-independence and a way to apply the FToLI.

Curl Test for Conservative Fields: Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a vector field. If $\operatorname{curl}\mathbf{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle = \langle 0,0,0\rangle$, then $\mathbf{F}$ is conservative.
if the domain of $\vec{F}$ is simply connected

- If $\mathbf{F}$ is a 2-d vector field, $\operatorname{curl}\mathbf{F} =$ $\langle 0, 0, Q_x - P_y\rangle$
- This is also called the **mixed-partials test**, because

$\operatorname{curl}\vec{F} = \langle 0,0,0\rangle \iff R_y = Q_z$ and $P_z = R_x$ and $Q_x = P_y$

if $\nabla f = \vec{F}$

then $\vec{F} = \langle f_x, f_y, f_z\rangle$

$f_{zy} = f_{yz}$

$f_{xz} = f_{zx}$

$f_{yx} = f_{xy}$

e.g. $\vec{F} = \langle x, y\rangle$

$Q_x = 0$

$P_y = 0$

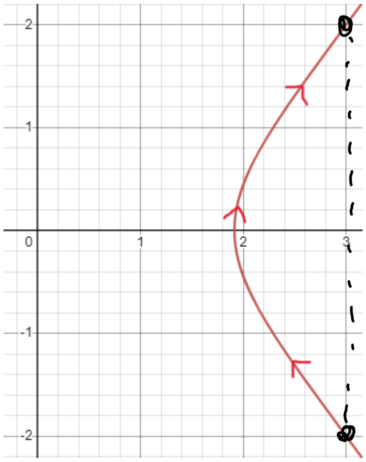
so $\vec{F}$ is conservative

ex: $\vec{F} = \langle -y, x\rangle$

$Q_x = 1$ $\quad$ • $Q_x - P_y = 2 \neq 0$

$P_y = -1$ $\quad$ so $\vec{F}$ is not conservative

Example 115. Evaluate $\int_C (10x^4 - 2xy^3)\, dx - 3x^2y^2\, dy$ where C is the part of the curve $x^5 - 5x^2y^2 - 7x^2 = 0$ from $(3, -2)$ to $(3, 2)$. $\Updownarrow$



$\int_C \vec{F} \cdot \vec{T}\, ds$, where $\vec{F} = \langle \overset{P}{10x^4 - 2xy^3}, \overset{Q}{-3x^2y^2} \rangle$

Q: Is $\vec{F}$ conservative?

$\text{curl}\, \vec{F} = \langle 0, 0, Q_x - P_y \rangle \overset{?}{=} \langle 0,0,0 \rangle$

$= \langle 0, 0, (-6xy^2) - (0 - 6xy^2) \rangle$

$= \langle 0,0,0 \rangle$ ✓

So $\vec{F}$ is conservative.

Q: For what f is $\vec{F} = \nabla f$?

$\int f_x\, dx = \int 10x^4 - 2xy^3\, dx \to f(x,y) = 2x^5 - x^2y^3 + g(y)$

$\int f_y\, dy = \int -3x^2y^2\, dy \to f(x,y) = -x^2y^3 + h(x)$

Comparing: $\boxed{f(x,y) = -x^2y^3 + 2x^5 + C}$

Line integral:

$\int_C \vec{F} \cdot \vec{T}\, ds = f(3,2) - f(3,-2)$

$= -(3)^2(2)^3 + \cancel{2(3)^5} + 3^2(-2)^3 - \cancel{2(3)^5}$

$= \boxed{-144}$

ex: What happens if we try to compute f if $\vec{F}$ is not conservative?

$\vec{F} = \langle -y, x \rangle$ is not conservative

if we ask for $\int f_x\, dx = \int -y\, dx \Rightarrow f(x,y) = -xy + g(y)$

$\int f_y\, dy = \int x\, dy \Rightarrow f(x,y) = xy + h(x)$

Daily Announcements & Reminders:

- 16.3 HW due tonight, 16.4 on Th
- Quiz 9 tomorrow: 16.2 & 16.3
- Correction to last time re:
  - path-indepence $\Leftrightarrow$ conservative
  - curl-free test

Goals for Today: Section 16.4

- Define the divergence and curl of a vector field
- Interpret divergence and curl geometrically
- Apply Green's Theorem to compute line integrals over the boundary of a simply-connected region

Useful notation: $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$

So if $f(x,y,z)$ is a function of three variables, $\nabla f = \left\langle \frac{\partial}{\partial x}(f), \frac{\partial}{\partial y}(f), \frac{\partial}{\partial z}(f) \right\rangle$

If $\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$ is a vector field:

- $\nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle = P_x + Q_y + R_z$ • works for any n
- $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$ • only 3d

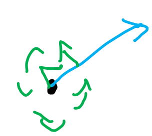
How do we measure the change of a vector field?

1. **Divergence** (in any $\mathbb{R}^n$)

- Tells us expansion / compression
- Measures instantaneous / local flux
- Is a scalar
- Is the instantaneous rate of change of strength of $\vec{F}$ in the direction of flow
- div $\mathbf{F}$ = $\nabla \cdot \vec{F} = P_x + Q_y + R_z$

2. **Curl** (in $\mathbb{R}^3$)

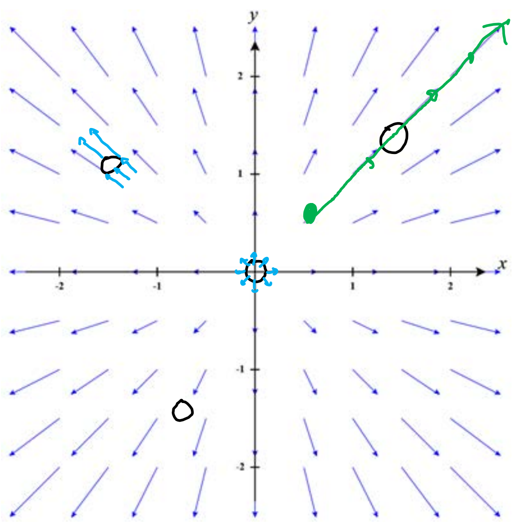
- Tells us circulation
- Measures instantaneous / local circulation
- Is a vector
- Direction gives RHR axis of rotation



- Magnitude gives rate of rotation
- curl $\mathbf{F}$ = $\nabla \times \vec{F}$
- If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$: we use $\nabla \times \mathbf{F} = \nabla \times \langle P, Q, 0 \rangle = \langle 0, 0, Q_x - P_y \rangle$

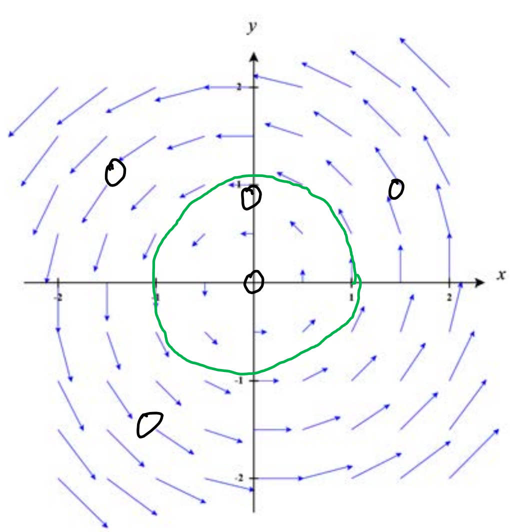
scalar curl $= (\nabla \times \vec{F}) \cdot \hat{k} = Q_x - P_y$

Example 116. Let $\mathbf{F}(x,y) = \langle x, y\rangle$. Based on the visualization of this vector field below, what can we say about the sign (+,-,0) of the divergence and curl of this vector field? Verify by computing the divergence and curl.



- $\operatorname{div}\vec{F} = \nabla\cdot\vec{F} > 0$ b/c more flow out of each small circle
- $\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \vec{0}$ because symmetric force acting on each side of small circle
- $\nabla\cdot\vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\rangle \cdot \langle x, y\rangle$
  $= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) = 1+1 = 2$
- $\nabla\times\vec{F} = \langle 0, 0, Q_x - P_y\rangle$
  $= \langle 0, 0, \frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\rangle = \langle 0,0,0\rangle$

Example 117. (Itempool) Let $\mathbf{F}(x,y) = \langle -y, x\rangle$. Based on the visualization of this vector field below, what can we say about the sign (+,-,0) of the divergence and curl of this vector field? Verify by computing the divergence and curl.



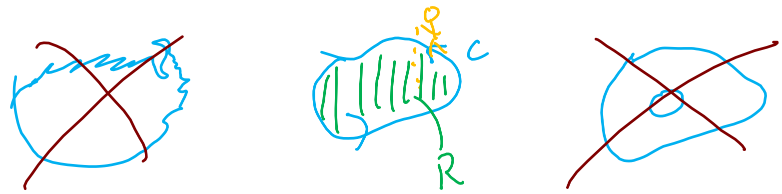
- $\operatorname{div}\vec{F} = 0$ b/c flow out of each small circle = flow in to each small
- $(\nabla\times\vec{F})\cdot\vec{k} > 0$ b/c each circle is pushed more on side away from $(0,0)$ so rotate CCW
- $\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) = 0 + 0 = 0$
- $(\nabla\times\vec{F})\cdot\vec{k} = \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y) = 1-(-1) = 2$

- $\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\rangle$ rotates macroscopically, but it has 0 curl.

Question: How is this useful?

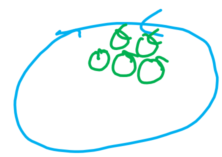
Answer: We can relate rates of change of vector field inside a region to the behavior of the vector field on the boundary of the region.

Theorem 118 (Green's Theorem)**.** *Suppose C is a piecewise smooth, simple, closed curve enclosing on its left a region R in the plane. If $\mathbf{F} = \langle P, Q \rangle$ has continuous partial derivatives around R, then*



a) Circulation form:

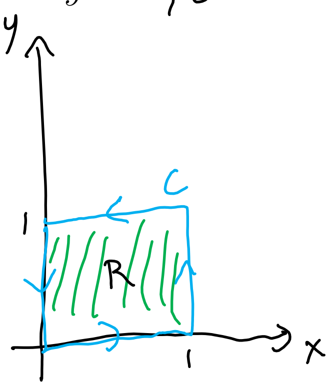
$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C P\, dx + Q\, dy = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA = \iint_R Q_x - P_y\, dA$$



b) Flux form:

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_C P\, dy - Q\, dx = \iint_R (\nabla \cdot \mathbf{F})\, dA = \iint_R P_x + Q_y\, dA$$

Example 119. Evaluate the line integral $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ for the vector field $\mathbf{F} = \langle -y^2, xy\rangle$ where C is the boundary of the square bounded by $x = 0, x = 1, y = 0,$ and $y = 1.$, oriented CCW.

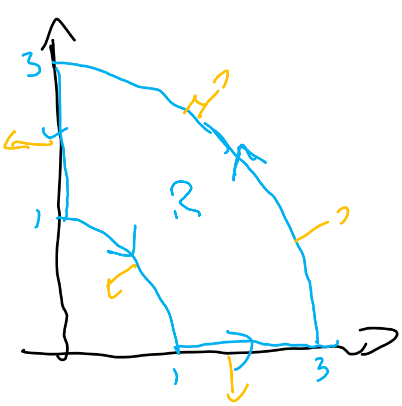


$$\int_C \vec{F}\cdot\vec{T}\, ds = \iint_R Q_x - P_y\, dA$$
$$= \int_0^1 \int_0^1 \frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(-y^2)\, dy\, dx$$
$$= \int_0^1 \int_0^1 y - (-2y)\, dy\, dx$$
$$= \int_0^1 \frac{3}{2}y^2 \Big|_0^1 dx$$
$$= \int_0^1 \frac{3}{2}\, dx = \frac{3}{2}$$

If CW: add a negative sign

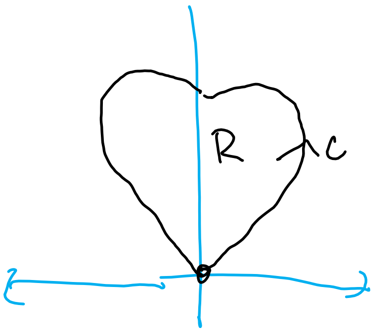
Example 120. Compute the flux out of the region R which is the portion of the annulus between the circles of radius 1 and 3 in the first octant for the vector field $\mathbf{F} = \langle \frac{1}{3}x^3, \frac{1}{3}y^3\rangle$.

$$\text{flux} = \int_C \vec{F}\cdot\vec{n}\, ds = \iint_R \operatorname{div}\vec{F}\, dA$$



$$= \iint_R P_x + Q_y\, dA$$
$$= \iint_R x^2 + y^2\, dA$$
$$= \int_0^{\pi/2} \int_1^3 r^3\, dr\, d\theta$$
$$= \int_0^{\pi/2} \frac{1}{4}(81-1)\, d\theta$$
$$= \int_0^{\pi/2} 20\, d\theta$$
$$= \boxed{10\pi}$$

Example 121. Let R be the region bounded by the curve $\mathbf{r}(t) = \langle \sin(2t), \sin(t)\rangle$ for $0 \le t \le \pi$. Find the area of R, using Green's Theorem applied to the vector field $\mathbf{F} = \frac{1}{2}\langle x, y\rangle$.



$$\text{area}(R) = \iint_R 1 \, dA = \iint_R \text{div}\vec{F} \, dA$$

$$\hat{n} = \langle y', -x' \rangle$$

$$\text{div}\vec{F} = \frac{1}{2} + \frac{1}{2} = 1$$

Greens thm

$$\overset{\text{Greens thm}}{=} \int_C \vec{F} \cdot \vec{n} \, ds$$

$$= \int_0^\pi \frac{1}{2}\langle \sin(2t), \sin(t)\rangle \cdot \langle \cos(t), -2\cos(2t)\rangle \, dt$$

$$= \int_0^\pi \frac{1}{2}\sin(2t)\cos(t) - \sin(t)\cos(2t) \, dt$$

$$= \int_0^\pi \sin(t)\cos^2(t) - \sin(t)\left(2\cos^2(t) - 1\right) dt$$

$u = \cos(t)$ $\quad du = -\sin(t)\,dt$

$$= \int_1^{-1} -u^2 + (2u^2 - 1) \, du$$

$$= \int_{-1}^1 -u^2 + 1 \, du$$

$$= -\frac{1}{3}u^3 + u \Big|_{-1}^1$$

$$= \left(-\frac{1}{3} + 1\right) - \left(\frac{1}{3} - 1\right)$$

$$= \boxed{\frac{4}{3}}$$

Note: This is the idea behind the operation of the measuring instrument known as a planimeter.

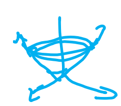
Daily Announcements & Reminders:

- 16.4 HW due probably tonight
- No quiz next week

Goals for Today: Sections 16.5/16.6

- Describe surfaces in $\mathbb{R}^3$ parametrically
- Define and compute surface integrals
- Use surface integrals to compute meaningful quantities: surface areas, masses, flux, etc.

Different ways to think about curves and surfaces:

	Curves	Surfaces
Explicit:	$y = f(x)$ e.g. $y = \sin(x)$	$z = f(x, y)$ e.g. $z = \sqrt{x^2 + y^2}$ 
Implicit:	$F(x, y) = 0$ e.g. $\sin(x) - y = 0$ $x^2 + y^2 = 1$	$F(x, y, z) = 0$ e.g. $\dfrac{\sqrt{x^2 + y^2}}{z} - 1 = 0$ $x^2 + y^2 + z^2 - 1 = 0$
Parametric Form:	$\mathbf{r}(t) = \langle x(t), y(t) \rangle$ e.g. $\mathbf{r}(t) = \langle t, \sin(t) \rangle, t \in \mathbb{R}$	$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$ e.g. $\vec{r}(u,v) = \langle u, v, \sqrt{u^2+v^2} \rangle$ $u, v \in \mathbb{R}$

Example 122. Give parameteric representations for the surfaces below.

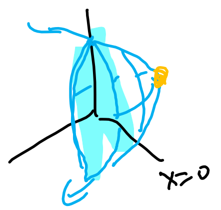
a) $x = y^2 + \frac{1}{2}z^2 - 2$ • an elliptic paraboloid opening in pos. x-direction

✓ $\vec{r}(u,v) = \langle u^2 + \frac{1}{2}v^2 - 2,\ u,\ v \rangle$ $u, v \in \mathbb{R}$

✗ $\vec{r}(s,t) = \langle s^4 + \frac{1}{2}t^4 - 2,\ s^2,\ t^2 \rangle$ (Does not work b/c no neg. y/z coords are produced)

✓ $\vec{r}(s,t) = \langle s^6 + \frac{1}{2}t^6 - 2,\ s^3,\ t^3 \rangle$ $s, t \in \mathbb{R}$

b) The portion of the surface $x = y^2 + \frac{1}{2}z^2 - 2$ which lies behind the yz-plane.



Need: $x < 0$

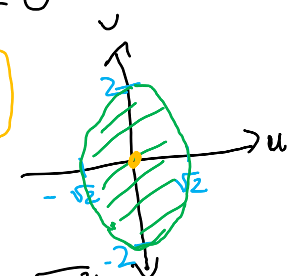
$\vec{r}(u,v) = \langle u^2 + \frac{1}{2}v^2 - 2,\ u,\ v \rangle$

$u^2 + \frac{1}{2}v^2 - 2 < 0$

$\frac{u^2}{2} + \frac{v^2}{4} < 1$

$-\sqrt{2} \le u \le \sqrt{2}$

$-2\sqrt{1 - u^2/2} \le v \le 2\sqrt{1 - u^2/2}$



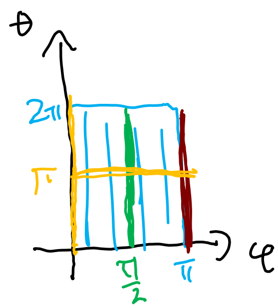
c) $x^2 + y^2 + z^2 = 9$

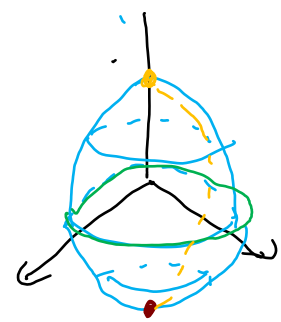
$\rho = 3$

$\vec{r}(\varphi, \theta) = \langle 3 \sin\varphi \cos\theta,\ 3 \sin\varphi \sin\theta,\ 3 \cos\varphi \rangle$

$0 \le \varphi \le \pi$

$0 \le \theta \le 2\pi$





d) $x^2 + y^2 = 25$ $r = 5$

$\vec{r}(\theta, z) = \langle 5 \cos\theta,\ 5 \sin\theta,\ z \rangle$

$0 \le \theta \le 2\pi$

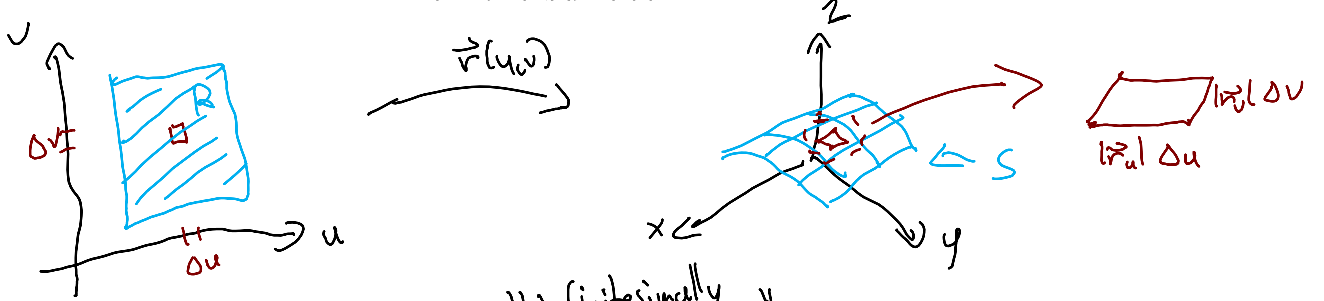
$z \in \mathbb{R}$

What can we do with this? $\vec{r}(u,v)$ parameterizing surface S

$\vec{r}_u(u,v) = \langle x_u(u,v), y_u(u,v), z_u(u,v)\rangle$

If our parameterization is **smooth** ($\mathbf{r}_u, \mathbf{r}_v$ not parallel in the domain), then:

- $\mathbf{r}_u \times \mathbf{r}_v$ is a normal vector to surface
- A rectangle of size $\Delta u \times \Delta v$ in the uv-domain is mapped to a ~~rectangle~~ parallelogram of size $|\vec{r}_u \times \vec{r}_v|\,\Delta u \Delta v$ on the surface in $\mathbb{R}^3$. $|\vec{w}_1 \times \vec{w}_2| = |\vec{w}_1||\vec{w}_2|\sin\theta$



$d\sigma =$ "infinitesimally small surface area"

- Thus, $Area(S) = \iint_S 1\, d\sigma = \iint_R \underbrace{|\vec{r}_u \times \vec{r}_v|\, dA}_{= d\sigma}$

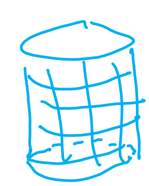
Example 123. (Itempool) Find the area of the portion of the cylinder $x^2 + y^2 = 25$ between $z = 0$ and $z = 1$.

$\vec{r}(\theta, z) = \langle 5\cos\theta, 5\sin\theta, z\rangle$

$0 \le \theta \le 2\pi$

$0 \le z \le 1$

• Does not include disks to cap top & bottom



$\vec{r}_\theta = \langle -5\sin\theta, 5\cos\theta, 0\rangle$

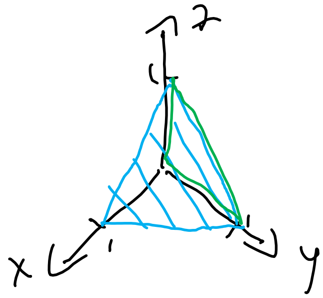
$\vec{r}_z = \langle 0, 0, 1\rangle$

$\vec{r}_\theta \times \vec{r}_z = \langle 5\cos\theta, 5\sin\theta, 0\rangle$

$|\vec{r}_\theta \times \vec{r}_z| = \sqrt{25\cos^2\theta + 25\sin^2\theta} = 5$

$Area(S) = \iint_S 1\, d\sigma = \int_0^{2\pi}\int_0^1 5\, dz\, d\theta = 10\pi$

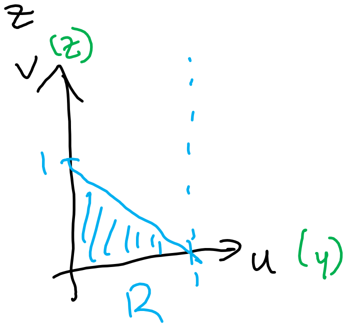
Example 124. Suppose the density of a thin plate S in the shape of the portion of the plane $x + y + z = 1$ in the first octant is $\delta(x, y, z) = 6xy$. Find the mass of the plate. kg/m²



$$\text{mass} = \iint_S \delta(x,y,z)\, d\sigma$$

1) Parameterize S: $x = 1 - y - z$

$$\vec{r}(u,v) = \langle 1-u-v,\ u,\ v \rangle$$
$$0 \le u \le 1$$
$$0 \le v \le 1-u$$



2) Find $|\vec{r}_u \times \vec{r}_v|$

$$\vec{r}_u = \langle -1, 1, 0 \rangle$$
$$\vec{r}_v = \langle -1, 0, 1 \rangle$$
$$\vec{r}_u \times \vec{r}_v \overset{(u,v)}{=} \langle 1, 1, 1 \rangle$$
$$|\vec{r}_u \times \vec{r}_v| = \sqrt{3}$$

3) Plug in & evaluate: $\iint_R \delta(\vec{r}(u,v))\, |\vec{r}_u \times \vec{r}_v|\, dA$

$$\text{mass} = \iint_S \delta(x,y,z)\, d\sigma = \iint_S 6xy\, d\sigma$$
$$= \int_0^1 \int_0^{1-u} 6(1-u-v)u \cdot \sqrt{3}\, dv\, du$$
$$= \sqrt{3}/4$$

Goal: If $\mathbf{F}$ is a vector field in $\mathbb{R}^3$, find the total flux of $\mathbf{F}$ through a surface S.

Note: If the flux is positive, that means the net movement of the field through S is in the direction of ______________________

If $\mathbf{r}(u, v)$ is a smooth parameterization of S with domain R, we have

$$\text{flux of } \mathbf{F} \text{ through } S = \iint_S (\mathbf{F} \cdot \mathbf{n})\, d\sigma = \iint_R \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA.$$

Example 125. Find the flux of $\mathbf{F} = \langle x, y, z \rangle$ through the upper hemisphere of $x^2 + y^2 + z^2 = 4$, oriented away from the origin.

Daily Announcements & Reminders:

- HW 16.5 due Sunday (do it today!)
- Quiz 10 on 16.4/16.5 on Monday
- Exam 3 info later today on Canvas
- Enjoy break
- No office hours today

Goals for Today: Section 16.6/16.7

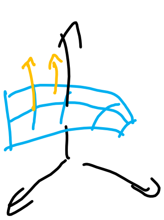
- Compute flux surface integrals
- Interpret the physical significance of flux surface integrals
- Introduce and apply Stokes' Theorem for surface integrals

Goal: If $\mathbf{F}$ is a vector field in $\mathbb{R}^3$, find the total flux of $\mathbf{F}$ through a surface S.

Note: If the flux is positive, that means the net movement of the field through S is in the direction of normal vector to S

If $\mathbf{r}(u,v)$ is a smooth parameterization of S with domain R, we have

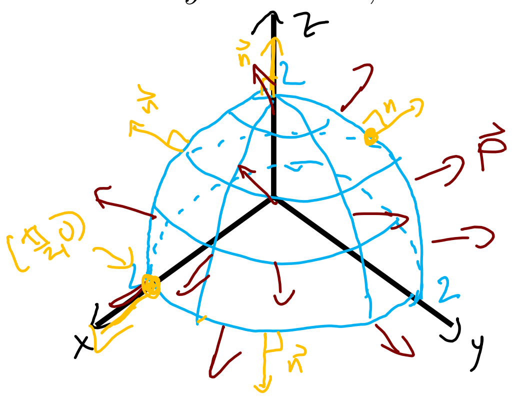
gives us consistent normals



$$\text{flux of } \mathbf{F} \text{ through } S = \iint_S \underbrace{(\mathbf{F}\cdot\mathbf{n})}_{\text{normal component to } S \text{ of } \vec{F}}\, d\sigma = \iint_R \mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u \times \mathbf{r}_v)\, dA.$$

unit normal to S

Example 125. Find the flux of $\mathbf{F} = \langle x, y, z \rangle$ through the upper hemisphere of $x^2 + y^2 + z^2 = 4$, oriented away from the origin.



1) Parameterize S

$-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$
$-2 \le x \le 2$

$\vec{r}(x,y) = \langle x, y, \sqrt{4-x^2-y^2} \rangle$ • not as nice bounds

$\rho = 2, \quad 0 \le \varphi \le \frac{\pi}{2}, \quad 0 \le \theta \le 2\pi$

$\vec{r}(\varphi, \theta) = \langle 2\sin\varphi\cos\theta, 2\sin\varphi\sin\theta, 2\cos\varphi \rangle$

2) Compute $\vec{r}_\varphi \times \vec{r}_\theta$:

$\vec{r}_\varphi = \langle 2\cos\varphi\cos\theta, 2\cos\varphi\sin\theta, -2\sin\varphi \rangle$
$\vec{r}_\theta = \langle -2\sin\varphi\sin\theta, 2\sin\varphi\cos\theta, 0 \rangle$

$\vec{r}_\varphi \times \vec{r}_\theta = \langle 0 + 4\sin^2\varphi\cos\theta, -(0 - 4\sin^2\varphi\sin\theta), 4\sin\varphi\cos\varphi(\underbrace{\cos^2\theta + \sin^2\theta}_{1}) \rangle$

(check orientation) • At $(\varphi, \theta) = (\frac{\pi}{2}, 0)$: $\vec{r}_\varphi \times \vec{r}_\theta = \langle 4(1), 4(0), 4(0)(1) \rangle = \langle 4, 0, 0 \rangle$

• z-component: $4\sin\varphi\cos\varphi = 2\sin(2\varphi) > 0$ if $0 \le \varphi \le \pi/2$

3) Plug in:

$$\text{flux} = \iint_S \vec{F} \cdot \vec{n}\, d\sigma = \int_0^{2\pi} \int_0^{\pi/2} F(\vec{r}(\varphi,\theta)) \cdot (\vec{r}_\varphi \times \vec{r}_\theta)\, d\varphi\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \langle 2\sin\varphi\cos\theta, 2\sin\varphi\sin\theta, 2\cos\varphi \rangle \cdot \langle 4\sin^2\varphi\cos\theta, 4\sin^2\varphi\sin\theta, 4\sin\varphi\cos\varphi \rangle\, d\varphi\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/2} 8\sin^3\varphi\cos^2\theta + 8\sin^3\varphi\sin^2\theta + 8\sin\varphi\cos^2\varphi\, d\varphi\, d\theta$$

$\hookrightarrow 8\sin^3\varphi(\underbrace{\cos^2\theta + \sin^2\theta}_{1}) + 8\sin\varphi\cos^2\varphi$

$\hookrightarrow 8\sin\varphi(\underbrace{\sin^2\varphi + \cos^2\varphi}_{1})$

$$= \int_0^{2\pi} \int_0^{\pi/2} 8\sin\varphi\, d\varphi\, d\theta$$

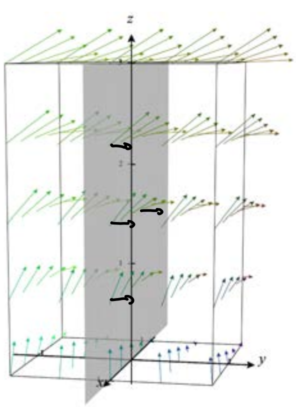
$$= \int_0^{2\pi} -8\cos\varphi \Big|_0^{\pi/2} d\theta$$

$$= 8 \cdot 2\pi = \boxed{16\pi}$$

Example 126. (Itempool) Suppose S is a smooth surface in $\mathbb{R}^3$ and $\mathbf{F}$ is a vector field in $\mathbb{R}^3$. **True or False:** If $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma > 0$, then the angle between $\mathbf{F}$ and $\mathbf{n}$ is acute at all points on S.

Converse: If angle btwn $\vec{F}$ and $\vec{n}$ is acute at all points on S
then $\iint_S \vec{F}\cdot\vec{n}\, d\sigma > 0$.
is true.

Example 127. (Itempool) Based on the plot of the vector field $\mathbf{F}$ and the surface S below, oriented in the positive y-direction, is the flux integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$ positive, negative, or zero?



At every point on S, $\vec{F}$ has pos. y-component, so
$\vec{F}\cdot\vec{n} = \vec{F}\cdot\vec{j} > 0$
so $\iint_S \vec{F}\cdot\vec{n}\, d\sigma > 0$

Recall: If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field, we defined its:

1. *divergence:* $\nabla \cdot \mathbf{F} = P_x + Q_y + R_z$
2. *curl:* $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$

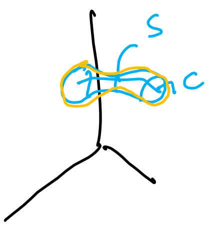
Example 128. (Itempool) Suppose $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field in $\mathbb{R}^3$ with continuous partial derivatives. Compute the divergence of the curl of $\mathbf{F}$, i.e. $\nabla \cdot (\nabla \times \mathbf{F})$.

$$\nabla \cdot (\langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle)$$
$$= \cancel{R_{yx}} - \cancel{Q_{zx}} + \cancel{P_{zy}} - \cancel{R_{xy}} + \cancel{Q_{xz}} - \cancel{P_{yz}}$$
$$= 0$$

- $\nabla \times (\nabla f) = \vec{0}$ (conservative vectorfields have zero curl)
  - $\nabla \times (\nabla \cdot \vec{F})$ not defined

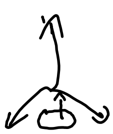
Theorem 129 (Stokes' Theorem)**.** *Let S be a smooth oriented surface and C be its compatibly oriented boundary. Let $\mathbf{F}$ be a vector field with continuous partial derivatives. Then*

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma = \int_C \mathbf{F} \cdot \mathbf{T} \, ds.$$



"flux of the curl of $\vec{F}$ through S" = "circulation of $\vec{F}$ around boundary of S"

- If S is a region R in the xy-plane, then we get:



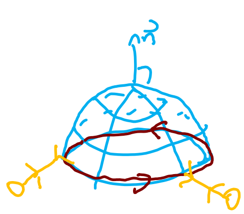
$$\iint_R (\nabla \times \vec{F}) \cdot \vec{k} \, dA = \int_C \vec{F} \cdot \vec{T} \, ds \qquad \text{[Green's Thm]}$$

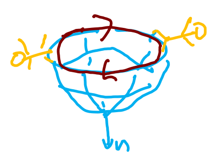
- An **oriented surface** is one where the normal vectors are consistent on all of S

- S and C are oriented compatibly if:

walking along C in its orientation with your head in the direction of $\vec{n}$ to S results in S being on your left

- closed surfaces, e.g. a sphere, have no boundary





Daily Announcements & Reminders:

- HW 16.6 due tonight, 16.7, 16.8 due Th
- Exam 3 is Th, see Canvas (15.5–15.8, 16.1–16.8)
- 175 pts for 100% HW score
- Final exam details next week
- CIOS is open – 85% completed by 12/8 for bonus 4 quiz pts dropped

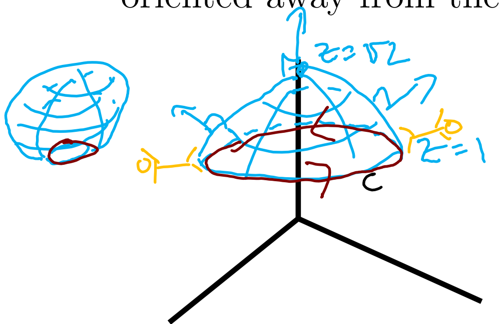
Goals for Today: Section 16.7/16.8

- Apply Stokes' Theorem to flux integral problems.
- Use Stokes' Theorem to simplify flux integrals
- Introduce and apply the Divergence Theorem to flux integral problems

Theorem 131 (Stokes' Theorem)**.** *Let S be a smooth oriented surface and C be its compatibly oriented boundary. Let $\mathbf{F}$ be a vector field with continuous partial derivatives. Then*

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\ \underbrace{d\sigma}_{\text{surface area wrt arc length}} = \int_C \mathbf{F} \cdot \mathbf{T}\ ds.$$

Example 132 (DD)**.** Let $\mathbf{F} = \langle -y, x+(z-1)x^{x\sin(x)}, x^2+y^2\rangle$. Find $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,d\sigma$ over the surface S which is the part of the sphere $x^2+y^2+z^2=2$ above $z=1$, oriented away from the origin.



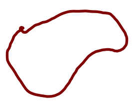
Option 1) Parameterize S, compute curl $\vec{F}$, substitute.

$$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x+(z-1)x^{x\sin(x)} & x^2+y^2 \end{vmatrix}$$

$$= \langle 2y - x^{x\sin(x)}, \ldots, \ldots \rangle$$

boundary: $x^2+y^2+(1)^2=2$

$x^2+y^2=1$



Option 2: Use Stokes' Thm: $\iint_S (\nabla\times\vec{F})\cdot\vec{n}\,d\sigma = \int_C \vec{F}\cdot\vec{T}\,ds$

- Orient C counterclockwise
- parameterize C: $\vec{r}(t) = \langle \cos(t), \sin(t), 1\rangle$, $\quad 0 \le t \le 2\pi$

  $\vec{r}\,'(t) = \langle -\sin(t), \cos(t), 0\rangle$

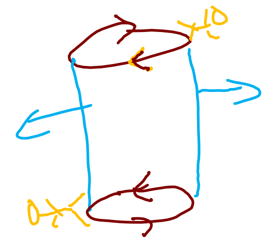
$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

$$= \int_0^{2\pi} \langle -\sin(t), \cos(t)+0, 1\rangle\cdot\langle -\sin(t), \cos(t), 0\rangle\,dt$$

$$= \int_0^{2\pi} \sin^2(t)+\cos^2(t)\,dt$$

$$= \boxed{2\pi}$$

$\langle \cos(t), -\sin(t), 1\rangle$

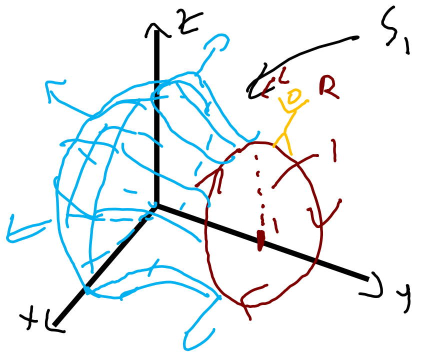


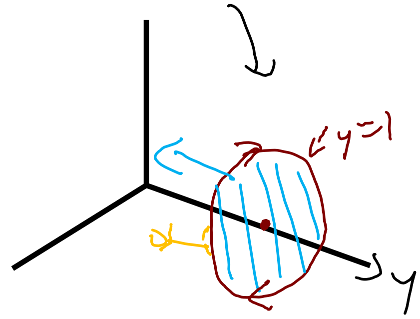
Question: What can we say if two different surfaces S_1 and S_2 have the same oriented boundary C?

$$\iint_{S_1} (\nabla \times \vec{F}) \cdot \vec{n}\, d\sigma = \int_C \vec{F} \cdot T\, ds = \iint_{S_2} (\nabla \times \vec{F}) \cdot \vec{n}\, d\sigma$$

Example 133. Suppose $\operatorname{curl} \mathbf{F} = \langle y^{y^y} \sin(z^2), (y-1)e^{x^{x^x}} + 2, -ze^{x^{x^x}} \rangle$. Compute the net flux of the curl of $\mathbf{F}$ over the surface pictured below, which is oriented outward and whose boundary curve is a unit circle centered on the y-axis in the plane $y = 1$.

S_1 is hard to work with: instead use S_2





$$\iint_{S_1} \operatorname{curl}\vec{F} \cdot \vec{n}\, d\sigma = \iint_{S_2} \operatorname{curl}\vec{F} \cdot \vec{n}\, d\sigma$$

$$= \iint_{S_2} \operatorname{curl}\vec{F} \cdot \langle 0, -1, 0 \rangle\, d\sigma$$

$$= \iint_{S_2} 0 + 2(-1) + 0\, d\sigma$$

$$= \iint_{S_2} -2\, d\sigma = -2(\pi)$$

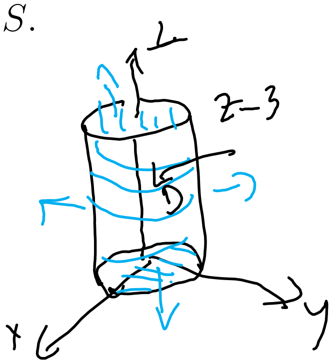
π ← S.A. of disk

Theorem 134 (Divergence Theorem)**.** *Let S be a closed surface oriented outward, D be the volume inside S, and $\mathbf{F}$ be a vector field with continuous partial derivatives. Then*

$$\iint_S \mathbf{F} \cdot \mathbf{n}\; d\sigma = \iiint_D \nabla \cdot \mathbf{F}\; dV.$$

"net flux of F through the closed surface" = "sum of local flux of F inside surface"

Example 135. Let $\mathbf{F} = \langle y^{1234} e^{\sin(yz)}, y - x^{z^x}, z^2 - z \rangle$ and S be the surface consisting of the portion of cylinder of radius 1 centered on the z-axis between $z = 0$ and $z = 3$, together with top and bottom disks, oriented outward. Find the flux of $\mathbf{F}$ through S.



needs 3 integrals, 1 for each side

$$\iint_S \vec{F} \cdot \hat{n}\, d\sigma = \iiint_D \vec{\nabla} \cdot \vec{F}\, dV \rightarrow P_x + Q_y + R_z$$

$$= \iiint_D 0 + 1 + 2z - 1\, dV$$

$$= \iiint_D 2z\, dV$$

$$= \int_0^{2\pi} \int_0^1 \int_0^3 2z \cdot r\, dz\, dr\, d\theta$$

$$= 9\pi$$

Daily Announcements & Reminders:

- Last day to work WeBWORK for points today (160 pt cap)
- Final Exam info in Canvas announcement
- Grades for exam 3 out by tomorrow morning
- Dr. Irvine will proctor our final

Goals for Today:

- Answer student questions about the course/unit.
- Review the core ideas of the course/unit.
- Practice problems from the course/unit.

not relevant to final

- ws 3 #6 , 4#3 ✓
- volume of sphere/paraboloid region
- curvature ✓
- linearization
- limits of 2-variable functions ✓
- Lagrange multipliers

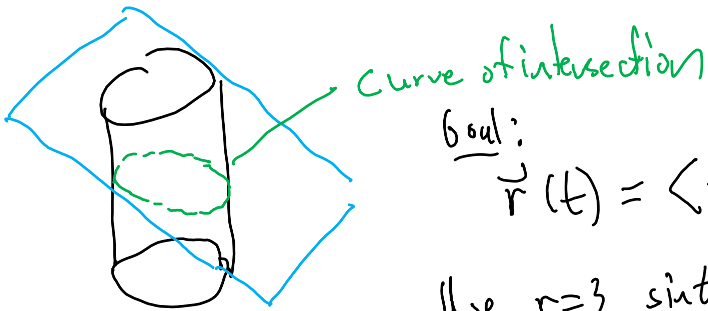
Responses to questions: ws 4 #3

3. Find a vector-valued function for the curve of intersection of the cylinder $x^2 + y^2 = 9$ and the plane $y + z = 2$.

   Hint: How could you parameterize the circle $x^2 + y^2 = 9$ in the plane?

$r^2 = 9$

$r = 3$



Goal:

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$\hookrightarrow z = 2 - y$

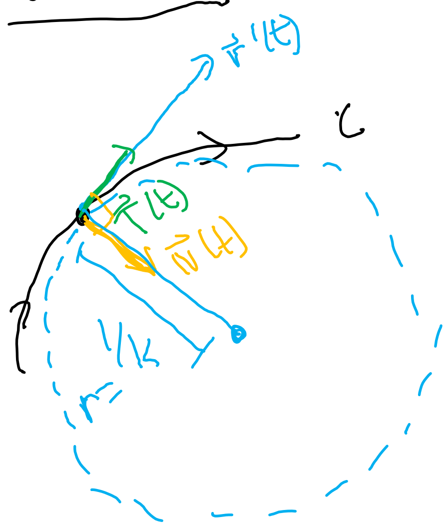
Use $r = 3$, $\sin t$, $\cos t$;

$\vec{r}(t) = \langle 3\cos(t), 3\sin(t), 2 - 3\sin(t) \rangle$

~~$0 \le t \le 1$?~~

$0 \le t \le 2\pi$

Curvature: $\kappa = \dfrac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \dfrac{\text{"rate of change of direction travel"}}{\text{" speed"}}$



principal unit normal vector:

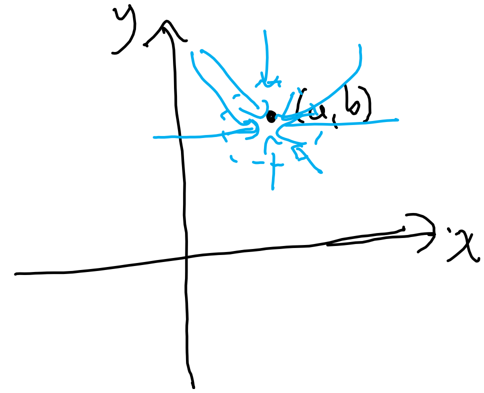
$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$$

Unit tangent:

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

Limits of functions of 2 variables

$$\lim_{(x,y)\to(a,b)} f(x,y)$$



- It works.

- It works, but need to do some algebra.
  - factoring, divide by highest powers, multiply by conjugate, etc.
  - Not L'Hospital's rule

- It doesn't work & we show that by using Two Path Test
  - Find two paths through (a,b) on which the value of the limit differs.

ex: Compute $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8}$ or show the limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8} = \frac{0}{0+0} \quad \text{indeterminate}$$

Try for 2-path test b/c no obvious simplification.

Along x-axis: $y=0$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{xy^4}{x^2+y^8} = \lim_{(x,0)\to(0,0)} \frac{xy^4}{x^2+y^8} = \lim_{x\to 0} \frac{0}{x^2+0} = \lim_{x\to 0} 0 = 0$$

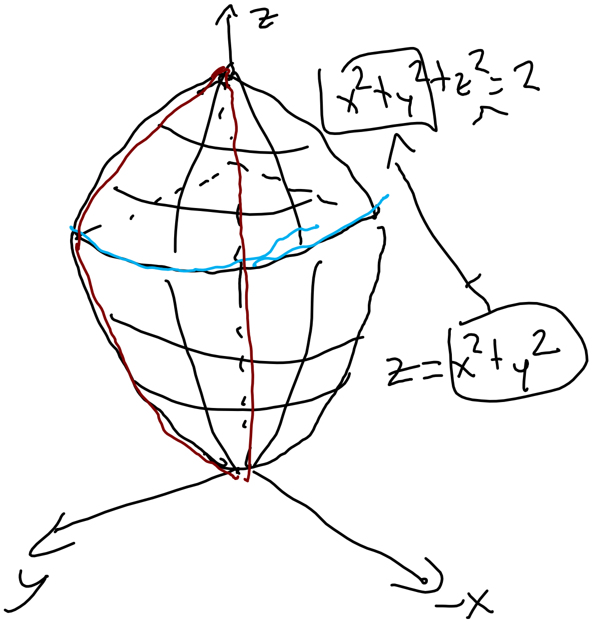
Along y-axis: $x=0$
$\to$ limit is 0

Along $x=y^4$:

$$\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8} = \lim_{(y^4,y)\to(0,0)} \frac{y^8}{y^8+y^8} = \lim_{y\to 0} \frac{1}{2} = \frac{1}{2}$$

These limits are not equal, so the limit does not exist.

8. Consider the volume D in the second octant of $\mathbb{R}^3$ ($x \le 0, y \ge 0, z \ge 0$) which is bounded above by the sphere $x^2+y^2+z^2=2$ and below by the paraboloid $z=x^2+y^2$.



Intersection: $z+z^2=2$

$z^2+z-2=0$

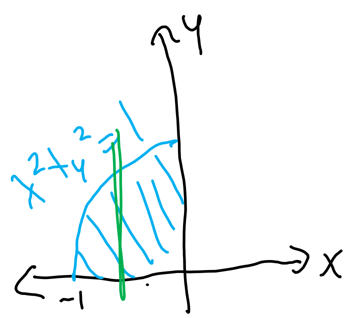
$(z+2)(z-1)=0$

$z=1$, $1=x^2+y^2$

In Cartesian

$$V=\int_{-1}^{0}\int_{0}^{\sqrt{1-x^2}}\int_{x^2+y^2}^{\sqrt{2-x^2-y^2}} dz\,dy\,dx$$

Shadow in xy-plane:



$0 \le r \le 1$

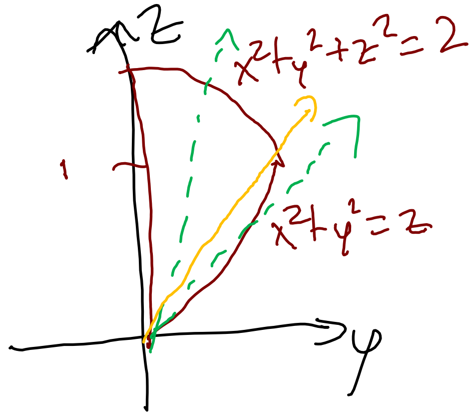
$\pi/2 \le \theta \le \pi$

In Cylindrical

z-Bounds: $z=r^2$ to $z=\sqrt{2-r^2}$

$$V=\int_{\pi/2}^{\pi}\int_{0}^{1}\int_{r^2}^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta$$

Sketch shadow on yz-plane



In spherical coords, need 2 integrals b/c not ρ-simple

Linearization: Idea: mimic $f(x)-f(a) \approx f'(a)(x-a)$

if x is near a

$$f(\vec{x})-f(\vec{a}) \approx Df(\vec{a})(\vec{x}-\vec{a})$$

if $\vec{x}$ is near $\vec{a}$

$$f(x,y) \approx f(a,b) + \begin{bmatrix} f_x(a,b) & f_y(a,b) \end{bmatrix} \begin{bmatrix} x-a \\ y-b \end{bmatrix}$$

$$L(x,y) = \underbrace{f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}_{z = \text{ is tangent plane}}$$