MATH 2551-C/HP MIDTERM 3
VERSION A
SPRING 2025
COVERS SECTIONS 15.5-15.8, 16.1-16.8

EXAM KEY

Honor code statement: I will abide strictly by the Georgia Tech honor code at all times. I will not use a calculator. I will not reference any website, application, or other CAS-enabled service. I will not consult with my notes or anyone during this exam. I will not provide aid to anyone else during this exam.

Initial Here: (   ) I attest to my integrity and will not discuss this exam with anyone until **Wednesday April 23.**

Read all instructions carefully before beginning.

- Print your name and GT ID neatly above.
- You have 75 minutes to take the exam.
- You may not use aids of any kind.
- The Learning Outcomes for the exam are listed on the back of this front cover. The Formula Sheet is on the final page and may be removed.
- Show your work. Answers without work shown will receive little or no credit.
- Good luck! Write yourself a message of encouragement on the front page!

Question:	1	2	3	4	5	6	7	8	9	10	11	12	Total
Points:	2	2	2	2	4	4	4	4	4	6	8	8	50

Learning Outcomes

- **I1: Double & Triple Integrals.** Set up double and triple integrals as iterated integrals over any region. Sketch regions based on a given iterated integral. Interpret geometric meanings of double and triple integrals.
- **I2: Iterated Integrals.** Compute iterated integrals of two and three variable functions. Apply Fubini's Theorem to change the order of integration of an iterated integral.
- **I3: Change of Variables.** Use polar, cylindrical, and spherical coordinates to transform double and triple integrals. Sketch regions based on given polar, cylindrical, and spherical iterated integrals. Use general change of variables to transform double and triple integrals for easier calculation. Choose the most appropriate coordinate system to evaluate a specific integral.
- **I4: Integral Applications.** Use multiple integrals to solve physical problems, such as finding area, average value, volume, mass of a lamina, or solid.
- **V1: Line Integrals.** Define line integrals of a scalar function over a smooth curve. Evaluate scalar line integrals in two and three dimensions. Define line integrals of a vector field over a smooth curve. Evaluate vector line integrals in two and three dimensions.
- **V2: Conservative Vector Fields.** Test for conservative vector fields. Find potential functions. State and apply the Fundamental Theorem of Line Integrals.
- **V3: Applications of Vector Calculus.** Interpret mass, center of mass, work, flow, circulation, flux, and surface area in terms of line and/or surface integrals, as appropriate. Use line and surface integrals to solve physical problems.
- **V4: Generalizations of the FTC.** State and apply Green's Theorem, Stokes' Theorem and the Divergence Theorem to solve problems in two and three dimensions. Compute curl and divergence of vector fields.
- **V5: Surfaces & Surface Integrals.** Parameterize common surfaces. Define and compute surface integrals for scalar and vector valued functions.

For problems 1-2 choose whether each statement is true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false. Be sure to neatly fill in the bubble corresponding to your answer choice.

1. (2 points) [**I3: Change of Variables**] A coordinate transformation should be invertible and differentiable at every point in its domain.

   ◯ **TRUE** $\surd$ **FALSE**

2. (2 points) [**V2: Conservative Vector Fields**] If $\int_C f\ ds = 0$, where C is a circle in $\mathbb{R}^2$, then f is conservative.

   ◯ **TRUE** $\surd$ **FALSE**

3. (2 points) [**V3: Applications of Vector Calculus**] Given two circles centered at the origin, oriented counterclockwise, and any vector field $\mathbf{F}$, then the circulation of $\mathbf{F}$ is larger around the circle with larger radius.

   ◯ **TRUE** $\surd$ **FALSE**

4. (2 points) [**V5: Surfaces & Surface Integrals**] If

$$\iint_S \mathbf{F} \cdot \mathbf{n}\ d\sigma > 0$$

   then the angle between $\mathbf{F}$ and $\mathbf{n}$ is acute somewhere on S.

   $\surd$ **TRUE** ◯ **FALSE**

5. (4 points) [**I3: Change of Variables**] Which coordinate transformation below maps the circle $u^2 + v^2 = 1$ onto the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$?

   ◯ **A)** $\mathbf{T}(u, v) = \langle 2u\cos(v), 3u\sin(v)\rangle$

   $\surd$ **B)** $\mathbf{T}(u, v) = \langle 2u, 3v\rangle$

   ◯ **C)** $\mathbf{T}(u, v) = \left\langle \dfrac{u}{2}, \dfrac{v}{3} \right\rangle$

   ◯ **D)** $\mathbf{T}(u, v) = \langle 4u, 9v\rangle$

   ◯ **E)** $\mathbf{T}(u, v) = \left\langle \dfrac{u}{4}, \dfrac{v}{9} \right\rangle$

6. (4 points) **[V2, V3, V4]** Consider $\mathbf{F}(x, y, z) = \langle \sin(z^2 + \pi) + 2026z, e^{x^{2025}}, 2026y \rangle$ and S be the surface which is the part of the sphere $x^2 + y^2 + z^2 = 4$ between $z = 1$ and $z = 2$, oriented away from the origin. For each integration theorem or strategy below, bubble in "Yes" if it is appropriate to use to compute the flux of $\mathbf{F}$ across S and "No" otherwise.

√ **Yes** ◯ **No** **Parameterization and direct calculation**

◯ Yes ✓ No Fundamental Theorem of Line Integrals

◯ Yes ✓ No Green's Theorem

√ **Yes** ◯ **No** **Stokes' Theorem**

◯ Yes ✓ No Divergence Theorem

7. (4 points) **[V1: Line Integrals]** Select the statement below which is **not** true about line integrals.

◯ **A)** Line integrals of scalar functions over a curve are independent of parameterization of the curve.

√ **B) If f is a scalar function, C a curve, and $-C$ the oppositely oriented curve, then**

$$\int_C f\ ds = -\int_{-C} f\ ds.$$

◯ **C)** If a curve C is divided into two curves C_1 and C_2, then

$$\int_C \mathbf{F} \cdot \mathbf{T}\ ds = \int_{C_1} \mathbf{F} \cdot \mathbf{T}\ ds + \int_{C_2} \mathbf{F} \cdot \mathbf{T}\ ds.$$

◯ **D)** If $\mathbf{F}$ is a vector field, C a curve, and $-C$ the oppositely oriented curve, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_{-C} \mathbf{F} \cdot d\mathbf{r}.$$

8. (4 points) [**V4: Generalizations of the FTC**] Select the line integral below which is equivalent to the surface integral

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma$$

where S is the portion of the sphere $x^2 + y^2 + z^2 = 6$ below $z = -\sqrt{2}$, oriented away from the origin and $\mathbf{F} = \langle y^2, x, 1\rangle$.

○ **A)** $\int_0^{2\pi} -8\sin^3(t) + 4\cos^2(t)\; dt$

○ **B)** $\int_0^{2\pi} -\sin^3(-t) + \cos^2(-t)\; dt$

○ **C)** $\int_0^{2\pi} 8\cos^3(t) + 4\sin^2(t)\; dt$

√ **D)** $\int_0^{2\pi} 8\sin^3(-t) - 4\cos^2(-t)\; dt$

9. (4 points) [**V5: Surfaces & Surface Integrals**] Select **all** of the functions below which parameterize the surface which is the portion of the paraboloid $z = x^2 + y^2$ with $z \leq 9$.

○ **A)** $\mathbf{r}(s,t) = \langle s, t, s^2 + t^2\rangle, \quad (s,t) \in \mathbb{R}^2$

○ **B)** $\mathbf{r}(s,t) = \langle s, t, s^2 + t^2\rangle, \quad 0 \leq s, t \leq 3$

√ **C)** $\mathbf{r}(s,t) = \langle s, t, s^2 + t^2\rangle, \quad -3 \leq s \leq 3, -\sqrt{9 - s^2} \leq t \leq \sqrt{9 - s^2}$

○ **D)** $\mathbf{r}(s,t) = \langle s\cos(t), s\sin(t), s^2\rangle, \quad s \leq 3, 0 \leq t \leq 2\pi$

√ **E)** $\mathbf{r}(s,t) = \langle s\cos(t), s\sin(t), s^2\rangle, \quad 0 \leq s \leq 3, 0 \leq t \leq 2\pi$

10. (6 points) **[I1, I3, I4]** Set up an integral for the mass of the solid which has density

$$\delta(x, y, z) = \frac{10}{1 + x^2 + y^2 + z^2}$$

and occupies the region in the first octant below the sphere $x^2 + y^2 + z^2 = 4$ and above the plane $z = \sqrt{2}$.

Solution: This region is nicely described in any of our major coordinate systems. In cylindrical coordinates it is

$$\sqrt{2} \le z \le \sqrt{4 - r^2}, \quad 0 \le r \le \sqrt{2}, \quad 0 \le \theta \le \pi/2,$$

since the sphere and plane meet when $x^2 + y^2 = 2$. Since this occurs when $r = z$, we have $\varphi = \pi/4$ at this point as well and so the spherical coordinate description of the region is

$$\sqrt{2}\sec(\varphi) \le \rho \le 2, \quad 0 \le \varphi \le \pi/4, \quad 0 \le \theta \le \pi/2.$$

Therefore

$$\begin{aligned}
\text{mass} &= \iiint_D \delta(x, y, z)\ dV \\
&= \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} \int_{\sqrt{2}}^{\sqrt{4-x^2-y^2}} \frac{10}{1 + x^2 + y^2 + z^2}\ dz\ dy\ dx \\
&= \int_0^{\pi/2} \int_0^{\sqrt{2}} \int_{\sqrt{2}}^{\sqrt{4-r^2}} \frac{10r}{1 + r^2 + z^2}\ dz\ dr\ d\theta \\
&= \int_0^{\pi/2} \int_0^{\pi/4} \int_{\sqrt{2}\sec(\varphi)}^{2} \frac{10\rho^2 \sin(\varphi)}{1 + \rho^2}\ d\rho\ d\varphi\ d\theta
\end{aligned}$$

11. (8 points) **[V2, V3]** Show that the force

$$\mathbf{F}(x,y) = \langle y^2+1, 2xy-3\rangle N$$

is conservative and find the work it does on an object moving along the line segment from $(1,-1)$ to $(2,1)$, measured in meters.

Solution: This force is conservative because we have

$$\nabla \times \mathbf{F} = \langle 0,0,2y-2y\rangle = \langle 0,0,0\rangle$$

and $\mathbf{F}$ is defined on all of $\mathbb{R}^2$.

A potential function is

$$f(x,y) = \int P\ dx = xy^2 + x + C(y).$$

Taking the y-partial derivative we see

$$2xy - 3 = f_y = 2xy + C'(y)$$

so a choice for $C(y)$ is $-3y$ and we have the potential

$$f(x,y) = xy^2 + x - 3y.$$

Finally, applying the Fundamental Theorem of Line Integrals we have

$$\begin{aligned}\text{work } &= \int_{(1,-1)}^{(2,1)} \mathbf{F}\cdot d\mathbf{r}\\ &= f(2,1) - f(1,-1)\\ &= (2+2-3) - (1+1+3)\\ &= -4\ N\cdot m\end{aligned}$$

Alternatively, we may work harder. A parameterization of the line is $\mathbf{r}(t) = \langle 1,2\rangle t + \langle 1,-1\rangle, 0 \le t \le 1$, so $\mathbf{r}'(t) = \langle 1,2\rangle$. Then

$$\begin{aligned}\text{work } &= \int_C \mathbf{F}\cdot d\mathbf{r}\\ &= \int_0^1 \langle (2t-1)^2+1, 2(2t-1)(t+1)-3\rangle \cdot \langle 1,2\rangle\ dt\\ &= \int_0^1 \langle 4t^2-4t+2, 4t^2+2t-5\rangle\cdot\langle 1,2\rangle\ dt\\ &= \int_0^1 4t^2-4t+2+8t^2+4t-10\ dt\\ &= \int_0^1 12t^2-8\ dt\\ &= 4t^3-8t|_0^1\\ &= -4N\cdot m\end{aligned}$$

12. (8 points) **[I1, I2, I3, V4, V5]** Compute the flux of the vector field

$$\mathbf{F}(x,y,z) = \langle x^3 + z^{1000}, y^3 + \sin(x), z + (x+4)^{y^2} \rangle$$

out of the surface which bounds the cylindrical solid with $x^2 + y^2 \leq 2$ and $0 \leq z \leq 3$.

Solution: This solid is closed and the derivatives of this field are continuously differentiable, so we may apply the Divergence Theorem.

$$\begin{aligned}
\text{flux} &= \iint_S \mathbf{F} \cdot \mathbf{n}\ d\sigma \\
&= \iiint_D \nabla \cdot \mathbf{F}\ dV \\
&= \iiint_D 3x^2 + 3y^2 + 1\ dV \\
&= \int_0^{2\pi} \int_0^{\sqrt{2}} \int_0^3 (3r^2 + 1) r\ dz\ dr\ d\theta \\
&= \int_0^{2\pi} \int_0^{\sqrt{2}} 9r^3 + 3r\ dr\ d\theta \\
&= \int_0^{2\pi} 12\ d\theta \\
&= 24\pi
\end{aligned}$$

SCRATCH PAPER - PAGE LEFT BLANK

SCRATCH PAPER - PAGE LEFT BLANK

FORMULA SHEET

Rules for Triple Integrals for the Sketching Impaired

1: Choose a variable appearing exactly twice for the next integral.

2: After setting up an integral, cross out any constraints involving the variable just used.

3: Create a new constraint by setting the lower limit of the preceding integral less than the upper limit.

4: A squared variable counts twice.

5: The argument of a square root must be nonnegative.

6: If you do not know which is the upper limit and which is the lower, guess and be prepared to backtrack.

- Trig identities: $\sin^2(x) = \frac{1}{2}(1-\cos(2x))$, $\cos^2(x) = \frac{1}{2}(1+\cos(2x))$
- Volume$(D) = \iiint_D dV$, $\quad f_{avg} = \dfrac{\iiint_D f(x,y,z)dV}{\text{volume of } D}$, Mass: $M = \iiint_D \delta \; dV$
- Coordinate Transforms:
  - Cylindrical: $x = r\cos(\theta)$, $\quad y = r\sin(\theta)$, $z = z$, $dV = r \; dz \; dr \; d\theta$
  - Spherical: $x = \rho\sin(\phi)\cos(\theta)$, $\quad y = \rho\sin(\phi)\sin(\theta)$, $\quad z = \rho\cos(\phi)$, $dV = \rho^2\sin(\phi) \; d\rho \; d\phi \; d\theta$
- Substitution Theorem: If R is the image of G under a coordinate transformation $\mathbf{T}(u,v) = \langle x(u,v), y(u,v)\rangle$ then

$$\iint_R f(x,y) \; dx \; dy = \iint_G f(\mathbf{T}(u,v))|\det D\mathbf{T}(u,v)| \; du \; dv.$$

- Line integrals:
  - $\int_C f(x,y,z) \; ds = \int_a^b f(\mathbf{r}(t))\|\mathbf{r}'(t)\|dt$
  - $\int_C \mathbf{F}\cdot\mathbf{T} \; ds = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t) \; dt$
  - $\int_C \mathbf{F}\cdot\mathbf{n} \; ds = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\langle y'(t), -x'(t)\rangle \; dt.$
- Fundamental Theorem of Line Integrals: $\int_C \nabla f\cdot d\mathbf{r} = f(B) - f(A)$ if C is a path from A to B
- Curl (Mixed Partials) Test: On a simply-connected region $\mathbf{F} = \nabla f \Leftrightarrow \text{curl } \mathbf{F} = \mathbf{0}$
- $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle$ $\qquad$ div $\mathbf{F} = \nabla\cdot\mathbf{F}$ $\qquad$ curl $\mathbf{F} = \nabla\times\mathbf{F}$
- Green's Theorem: If C is a simple closed curve with positive orientation, R is the simply-connected region it encloses, and $\mathbf{F}$ is a vector field whose components have continuous partial derivatives on R then

$$\int_C \mathbf{F}\cdot\mathbf{T} \; ds = \iint_R (\nabla\times\mathbf{F})\cdot\mathbf{k} \; dA \qquad \int_C \mathbf{F}\cdot\mathbf{n} \; ds = \iint_R (\nabla\cdot\mathbf{F}) \; dA.$$

FORMULA SHEET CONTINUED

- Surface integrals:
  - $\iint_S f(x,y,z)\ d\sigma = \iint_R f(\mathbf{r}(u,v))\ \|\mathbf{r}_u \times \mathbf{r}_v\| dA$
  - $\iint_S \mathbf{F}\cdot\mathbf{n}\ d\sigma = \iint_S \mathbf{F}\cdot d\boldsymbol{\sigma} = \iint_R \mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u \times \mathbf{r}_v)\ dA$

- Stokes' Theorem: If S is a piecewise smooth oriented surface bounded by a piecewise smooth curve C and $\mathbf{F}$ is a vector field whose components have continuous partial derivatives on an open region containing S, then

$$\int_C \mathbf{F}\cdot\mathbf{T}\ ds = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\ d\sigma.$$

- Divergence Theorem: If S is a piecewise smooth closed oriented surface enclosing a volume D and $\mathbf{F}$ is a vector field whose components have continuous partial derivatives on D, then

$$\iint_S \mathbf{F}\cdot\mathbf{n}\ d\sigma = \iiint_D \nabla\cdot\mathbf{F}\ dV.$$