MATH 2551-C/HP MIDTERM 2
VERSION A
SPRING 2025
COVERS SECTIONS 14.3-14.8, 15.1-15.4

EXAM KEY

Honor code statement: I will abide strictly by the Georgia Tech honor code at all times. I will not use a calculator. I will not reference any website, application, or other CAS-enabled service. I will not consult with my notes or anyone during this exam. I will not provide aid to anyone else during this exam.

(   ) I attest to my integrity.

Read all instructions carefully before beginning.

- Print your name and GT ID neatly above.
- You have 75 minutes to take the exam.
- You may not use aids of any kind.
- The Learning Outcomes for the exam are listed on the back of this front cover. The Formula Sheet is on the final page and may be removed.
- Show your work. Answers without work shown will receive little or no credit.
- Good luck! Write yourself a message of encouragement on the front page!

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	1	1	2	2	4	4	6	10	10	10	50

Learning Outcomes

- **D1: Computing Derivatives.** Compute partial derivatives, total derivatives, directional derivatives, and gradients. Apply the Chain Rule for multivariable functions to compute derivatives of composite functions.
- **D2: Interpreting Derivatives.** Interpret the meaning of the gradient and directional derivative of a function at a given point in a specified direction, including in the context of a graph or a contour plot.
- **D3: Tangent Planes and Linear Approximations.** Find equations for tangent planes to surfaces and linear approximations of functions at a given point and use these to solve problems.
- **D4: Optimization.** Find local maxima and minima by using derivative tests for functions of two or more independent variables. Analyze and locate critical points. Find absolute maxima and minima on closed bounded sets. Define and compute Lagrange multipliers in two and three variables. Use the method of Lagrange multipliers to maximize and minimize functions subject to constraints. Use maxima and minima to solve application problems.
- **I1: Double & Triple Integrals.** Set up double and triple integrals as iterated integrals over any region. Sketch regions based on a given iterated integral. Interpret geometric meanings of double and triple integrals.
- **I2: Iterated Integrals.** Compute iterated integrals of two and three variable functions. Apply Fubini's Theorem to change the order of integration of an iterated integral.
- **I3: Change of Variables.** Use polar, cylindrical, and spherical coordinates to transform double and triple integrals. Sketch regions based on given polar, cylindrical, and spherical iterated integrals. Use general change of variables to transform double and triple integrals for easier calculation. Choose the most appropriate coordinate system to evaluate a specific integral.
- **I4: Integral Applications.** Use multiple integrals to solve physical problems, such as finding area, average value, volume, mass of a lamina, or solid.

For problems 1-2 choose whether each statement is true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false. Be sure to neatly fill in the bubble corresponding to your answer choice.

1. (1 point) [**D1: Computing Derivatives**] If $f : \mathbb{R}^2 \to \mathbb{R}$, then both $f_x(a,b)$ and $f_y(a,b)$ can exist without f being differentiable at (a,b).

   $\surd$ **TRUE** ◯ **FALSE**

2. (1 point) [**D4: Optimization**] Let $f, g : \mathbb{R}^2 \to \mathbb{R}$. If the solutions to the system

$$\nabla f(x,y) = \lambda \nabla g(x,y), \qquad g(x,y) = c$$

   are $(1,1)$ and $(2,0)$ and $f(1,1) = 3$ and $f(2,0) = 5$, then the maximum value of f subject to the constraint $g(x,y) = c$ is 5.

   ◯ **TRUE** $\surd$ **FALSE**

3. (2 points) [**D2: Interpreting Derivatives**] Suppose the number of cars produced each year is a function $n = f(c,d)$ of the cost of materials c and the demand d of cars by consumers. If we know that $f_c(10000, 50) < 0$, then

   ◯ **A)** The number of cars produced will go up if the cost of materials increases from $10,000$

   ◯ **B)** The number of cars produced will go up if the demand increases from 50

   $\surd$ **C) The number of cars produced will drop if the cost of materials increases from** $10,000$

   ◯ **D)** The number of cars produced will drop if the demand increases from 50

4. (2 points) [**D2: Interpreting Derivatives**] Select the statement below which is **NOT** true about gradients and directional derivatives for a function $f : \mathbb{R}^2 \to \mathbb{R}$.

   ◯ **A)** If a f is differentiable at (a,b), then the directional derivative exists in every direction at (a,b).

   $\surd$ **B) If the directional derivative of** f **exists in every direction at** (a,b)**, then** f **is differentiable at** (a,b)**.**

   ◯ **C)** If f is differentiable, $\nabla f(a,b)$ gives the direction of greatest increase of f at (a,b).

   ◯ **D)** If f is differentiable, $\nabla f(a,b)$ is orthogonal to the level curve $f(x,y) = f(a,b)$.

5. (4 points) [**D1: Computing Derivatives**] Suppose that we have $z = f(x,y)$ and $(x(u,v), y(u,v)) = \mathbf{g}(u,v)$ (so that $f : \mathbb{R}^2 \to \mathbb{R}$ and $\mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^2$) and f and $\mathbf{g}$ are differentiable. Use some of the values in the table below to compute $z_u(2,1)$ for the composite function $z(u,v) = f(\mathbf{g}(u,v))$.

$f_x(2,1) = a$	$f_y(2,1) = b$	$f(2,1) = 3$	$x(2,1) = 1$	$x_u(2,1) = p$	$x_v(2,1) = q$
$f_x(1,0) = c$	$f_y(1,0) = d$	$f(1,0) = 4$	$y(2,1) = 0$	$y_u(2,1) = r$	$y_v(2,1) = s$

Solution: We have

$$\begin{aligned} Dz(u,v) &= \begin{bmatrix} z_u(2,1) & z_v(2,1) \end{bmatrix} \\ &= Df(\mathbf{g}(2,1))D\mathbf{g}(2,1) \\ &= Df(x(2,1), y(2,1))D\mathbf{g}(2,1) \\ &= Df(1,0)D\mathbf{g}(2,1) \\ &= \begin{bmatrix} f_x(1,0) & f_y(1,0) \end{bmatrix} \begin{bmatrix} x_u(2,1) & x_v(2,1) \\ y_u(2,1) & y_v(2,1) \end{bmatrix} \\ &= \begin{bmatrix} c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} \\ &= \begin{bmatrix} cp + dr & cq + ds \end{bmatrix} \end{aligned}$$

So $z_u(2,1) = cp + dr$.

6. (4 points) [**I3: Change of Variables**] Set up but do not evaluate an iterated integral in polar coordinates for the double integral

$$\iint_R 3x - y \, dA$$

where R is the region in the first quadrant enclosed by the circle $x^2 + y^2 = 9$ and the lines $x = 0$ and $y = x$.

Solution: In polar coordinates, this is the region with $0 \leq r \leq 3$ and $\pi/4 \leq \theta \leq \pi/2$, since $y = x$ is the line $\theta = \pi/4$ and $x = 0$ is the line $\theta = \pi/2$. So the integral becomes

$$\int_{\pi/4}^{\pi/2} \int_0^3 (3r\cos(\theta) - r\sin(\theta))r \, dr \, d\theta.$$

7. (6 points) [**D4: Optimization**] Find and classify all critical points for the function

$$f(x,y) = \ln(x+y) + x^2 - y.$$

Solution: We compute the derivative and set it to the zero matrix to find our critical points.

$$Df = \begin{bmatrix} \frac{1}{x+y} + 2x & \frac{1}{x+y} - 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

From the second coordinate we have $x+y = 1$ and so the first coordinate gives $1+2x = 0$. Therefore $x = -1/2$, $y = 3/2$ is the sole critical point. Now

$$Hf = \begin{bmatrix} 2 - \frac{1}{(x+y)^2} & -\frac{1}{(x+y)^2} \\ -\frac{1}{(x+y)^2} & -\frac{1}{(x+y)^2} \end{bmatrix}$$

and we compute

$$Hf(-1/2, 3/2) = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}.$$

Since $\det(Hf(-1/2,3/2) = -2 < 0$, f has a saddle point at $(-1/2, 3/2)$.

8. **[D3: Tangent Planes and Linear Approximations]** Let $f(x, y, z) = e^x + \cos(y + z)$.

(a) (4 points) Find the linearization of f at $(0, \pi/2, 0)$.

Solution: We have

$$Df(x, y, z) = \begin{bmatrix} e^x & -\sin(y + z) & -\sin(y + z) \end{bmatrix}.$$

Now

$$f(0, \pi/2, 0) = 1 + \cos(\pi/2) = 1 \qquad Df(0, \pi/2, 0) = \begin{bmatrix} 1 & -1 & -1 \end{bmatrix}.$$

Therefore

$$L(x, y, z) = 1 + \begin{bmatrix} 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y - \pi/2 \\ z \end{bmatrix} = 1 + x - (y - \pi/2) - z.$$

(b) (3 points) Approximate the value of $f(0.1, \pi/2, -0.1)$.

Solution:

$$\begin{aligned} f(0.1, \pi/2, -0.1) &\approxeq L(0.1, \pi/2, -0.1) \\ &= 1 + 0.1 - (\pi/2 - \pi/2) - (-0.1) \\ &= 1.2 \end{aligned}$$

(c) (3 points) Find a tangent plane to the level surface $1 = e^x + \cos(y + z)$.

Solution: Note that $(0, \pi/2, 0)$ lies on this level surface, so a tangent plane is

$$\nabla f(0, \pi/2, 0) \cdot \langle x, y - \pi/2, z \rangle = 0$$

and by our work above this is

$$x - (y - \pi/2) - z = 0.$$

9. (10 points) [**D4: Optimization**] Find the extreme values of $x^2 + y^2$ on the closed triangular region bounded by $x = 0$, $y = 0$, and $y + 2x = 2$ in the first quadrant.

Solution: First we look for critical points inside the region. $Df = \begin{bmatrix} 2x & 2y \end{bmatrix}$ is the zero matrix exactly when $x = 0, y = 0$. So $(0, 0)$ is a test point since it is in our region. Next we will deal with the boundary.

On $x = 0$**.** Now $f(0, y) = 0^2 + y^2 = g(y)$ and $g'(y) = 2y = 0$ implies $y = 0$, giving us $(0, 0)$ again. We also include the endpoints of this side: $(0, 0)$ (again!) and the intersection of $y + 2x = 2$ and $x = 0$ at $(0, 2)$.

On $y = 0$**.** Now $f(x, 0) = x^2 + 0^2 = h(x)$ and $h'(x) = 2x = 0$ gives us $(0, 0)$ once more. The endpoints of this side are $(0, 0)$ and the intersection of $y + 2x = 2$ and $y = 0$ at $(1, 0)$.

On $y + 2x = 2$**.** Now we have $y = 2 - 2x$, so $f(x, 2 - 2x) = x^2 + (2 - 2x)^2 = k(x)$ and $k'(x) = 2x - 4(2 - 2x) = 0$ implies $10x - 8 = 0$ so $x = 4/5, y = 2/5$. The endpoints of this segment we found earlier at $(1, 0)$ and $(0, 2)$.

Now we test all of the points:

(x, y)	$f(x, y)$
$(0, 0)$	0
$(1, 0)$	1
$(0, 2)$	4
$(4/5, 2/5)$	$4/5$

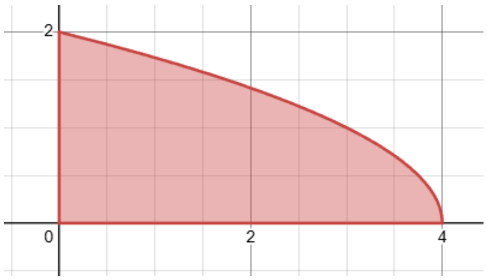
From this we see that the maximum value of f is 4 and the minimum value of f is 0 on this region.

10. (a) (5 points) **[I1: Double & Triple Integrals, I2: Iterated Integrals]** Sketch the region of integration and write an equivalent iterated integral for

$$\int_0^2 \int_0^{4-y^2} y \, dx \, dy$$

using the other order of integration.

Solution: The region is shown below.



$$\int_0^2 \int_0^{4-y^2} y \, dx \, dy = \int_0^4 \int_0^{\sqrt{4-x}} y \, dy \, dx$$

(b) (3 points) **[I2: Iterated Integrals]** Choose one of these two integrals and evaluate it.

Solution: In this case both integrals are straightforward to compute. Since we went to the trouble of swapping the order of integration we will use the new order:

$$\begin{aligned}\int_0^4 \int_0^{\sqrt{4-x}} y \, dy \, dx &= \int_0^4 \frac{1}{2} y^2 |_0^{\sqrt{4-x}} \, dx \\ &= \int_0^4 2 - \frac{1}{2} x \, dx \\ &= 2x - \frac{1}{4} x^2 |_0^4 \\ &= 8 - 4 \\ &= 4.\end{aligned}$$

(c) (2 points) **[I4: Integral Applications]** Give an integral expression for the area of this region of integration.

Solution:

$$\text{Area} = \int_0^2 \int_0^{4-y^2} 1 \, dx \, dy$$

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- Total Derivative: For $\mathbf{f}(x_1, \ldots, x_n) = \langle f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)\rangle$

$$D\mathbf{f} = \begin{bmatrix} (f_1)_{x_1} & (f_1)_{x_2} & \cdots & (f_1)_{x_n} \\ (f_2)_{x_1} & (f_2)_{x_2} & \cdots & (f_2)_{x_n} \\ \vdots & \ddots & \cdots & \vdots \\ (f_m)_{x_1} & (f_m)_{x_2} & \cdots & (f_m)_{x_n} \end{bmatrix}$$

- Linearization: Near $\mathbf{a}$, $f(\mathbf{x}) \approx L(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a})$

- Chain Rule: If $h = g(f(\mathbf{x}))$ then $Dh(\mathbf{x}) = Dg(f(\mathbf{x}))Df(\mathbf{x})$

- Implicit Differentiation: If z is implicitly given in terms of x and y by $F(x, y, z) = c$, then $\dfrac{\partial z}{\partial x} = \dfrac{-F_x}{F_z}$ and $\dfrac{\partial z}{\partial y} = \dfrac{-F_y}{F_z}$.

- Directional Derivative: If $\mathbf{u}$ is a unit vector, $D_{\mathbf{u}}f(P) = Df(P)\mathbf{u} = \nabla f(P) \cdot \mathbf{u}$

- The tangent plane to a level surface of $f(x, y, z)$ at (a, b, c) is

$$0 = \nabla f(a, b, c) \cdot \langle x - a, y - b, z - c\rangle.$$

- Hessian Matrix: For $f(x, y)$, $Hf(x, y) = \begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix}$

- Second Derivative Test: If (a, b) is a critical point of $f(x, y)$ then
  1. If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) < 0$ then f has a local maximum at (a, b)
  2. If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) > 0$ then f has a local minimum at (a, b)
  3. If $\det(Hf(a, b)) < 0$ then f has a saddle point at (a, b)
  4. If $\det(Hf(a, b)) = 0$ the test is inconclusive

- Lagrange Multipliers: Extreme values of a function $f(x, y, z)$ subject to a constraint $g(x, y, z) = c$ occur at points with $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = c$.

- $\text{Area}(R) = \iint_R 1 \, dA$

- Trig identities: $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$, $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$

- Average value: $f_{avg} = \dfrac{\iint_R f(x, y)dA}{\text{Area}(R)}$

- Polar coordinates: $x = r\cos(\theta)$, $y = r\sin(\theta)$, $dA = r \, dr \, d\theta$