MATH 2551-C/HP MIDTERM 1
VERSION A
SPRING 2025
COVERS SECTIONS 12.1-12.6, 13.1-13.4, 14.1-2

EXAM KEY

Honor code statement: I will abide strictly by the Georgia Tech honor code at all times. I will not use a calculator. I will not reference any website, application, or other CAS-enabled service. I will not consult with my notes or anyone during this exam. I will not provide aid to anyone else during this exam.

(   ) I attest to my integrity.

Read all instructions carefully before beginning.

- Print your name and GT ID neatly above.
- You have 75 minutes to take the exam.
- You may not use aids of any kind.
- The Learning Outcomes for the exam are listed on the back of this front cover. The Formula Sheet is on the final page and may be removed.
- Show your work. Answers without work shown will receive little or no credit.
- Good luck! Write yourself a message of encouragement on the front page!

Question	Points
1	1
2	1
3	2
4	6
5	10
6	10
7	10
8	10
Total:	50

Learning Outcomes

- **G1: Lines and Planes.** Describe lines using the vector equation of a line. Describe planes using the general equation of a plane. Find the equations of planes using a point and a normal vector. Find the intersections of lines and planes. Describe the relationships of lines and planes to each other. Solve problems with lines and planes.
- **G2: Calculus of Curves.** Compute tangent vectors to parametric curves; velocity, speed, and acceleration. Find equations of tangent lines to parametric curves. Solve initial value problems for motion on parametric curves.
- **G3: Geometry of Curves.** Compute the arc length of a curve in two or three dimensions. Use arc length to solve problems. Compute normal vectors and curvature for curves in two and three dimensions. Interpret these objects geometrically and in application.
- **G4: Surfaces.** Graph cylinders from equations. Describe and find traces on surfaces. Identify standard quadric surfaces including: spheres, ellipsoids, elliptic paraboloid, hyperboloids, cones, hyperbolic paraboloid. Sketch quadric surfaces. Graph functions of two variables and determine their domains and ranges. Graph level curves.
- **G5: Limits of Functions.** Calculate the limits of functions of two variables or determine if they do not exist. Apply the Two-Path Test to show limits do not exist. Define continuity of a function of two or more variables.

For problems 1-2 choose whether each statement is true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false. Be sure to neatly fill in the bubble corresponding to your answer choice.

1. (1 point) [**G2: Calculus of Curves**] For any smooth curve C in space parameterized by a function $\mathbf{r}(t)$, $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}''(t)$.

   ◯ **TRUE** √ **FALSE**

2. (1 point) [**G1: Lines and Planes**] Two planes orthogonal to a third plane are parallel to each other.

   ◯ **TRUE** √ **FALSE**

3. (2 points) [**G4: Surfaces**] The domain of the function

$$f(x, y) = \frac{\ln(4 - y^2)}{\sqrt{1 - x^2}}$$

   is which of the following?

   ◯ **A)** All of $\mathbb{R}^2$
   ◯ **B)** The region on one side of a line
   ◯ **C)** The interior of a disk
   √ **D) The interior of a rectangle**
   ◯ **E)** None of the above

Solution: From the numerator we have $4 - y^2 > 0$, so $-2 < y < 2$. From the denominator we have $1 - x^2 > 0$, so $-1 < x < 1$. Therefore the domain is the rectangle formed by combining these restrictions.

4. (6 points) **[G4: Surfaces]** Create a contour plot for the function

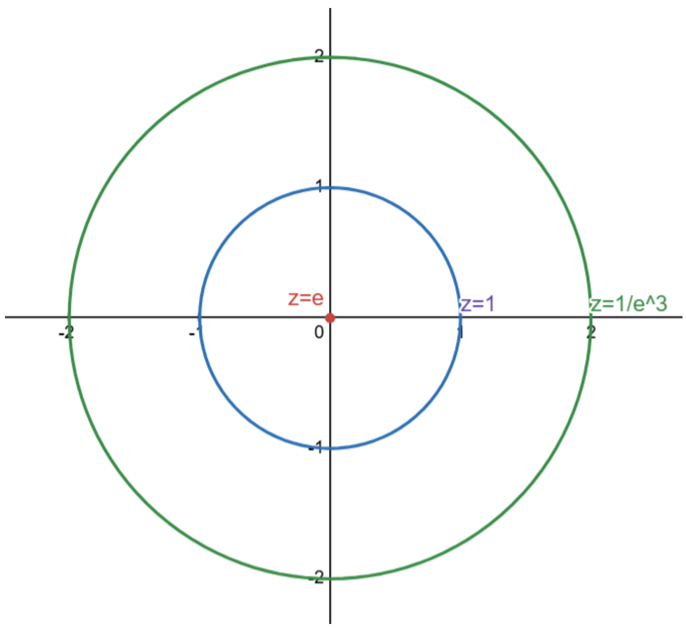
$$f(x, y) = e^{1-x^2-y^2}.$$

Your plot should include at least three level curves, be carefully labelled, and you should include work showing how you determined the level curves for full credit. Choose your level curves carefully!

Solution: There are many good choices here. All of the level curves are circles, since for any $0 < k \leq e$ we can take the natural log of the equation $k = f(x, y)$ to get

$$\ln(k) = 1 - x^2 - y^2 \implies x^2 + y^2 = 1 - \ln(k).$$

A few particularly nice choices are $k = 1$, resulting in $x^2 + y^2 = 1$, $k = 1/e^3$, resulting in $x^2 + y^2 = 4$, and $k = e$, resulting in $x^2 + y^2 = 0$ (just the origin).



5. **[G1: Lines and Planes]** Let P_1 be the plane $2x+y-z=1$ and P_2 be the plane $x+3z=3$.

(a) (2 points) Show that P_1 and P_2 are intersecting planes.

Solution: A quick check shows that $(3, -5, 0)$ is on both planes.

(b) (2 points) Find normal vectors to both planes.

Solution: From their equations we extract

$$\mathbf{n}_1 = \langle 2, 1, -1 \rangle \quad \mathbf{n}_2 = \langle 1, 0, 3 \rangle.$$

(c) (4 points) Find a parameterization of the line of intersection of the two planes.

Solution: The direction $\mathbf{v}$ of this line is parallel to both planes, so is orthogonal to both normal vectors. Hence we can take

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & 0 & 3 \end{vmatrix} = \langle 3, -7, -1 \rangle.$$

We found a point on the line in (a), so a parameterization for this line is

$$\ell(t) = \langle 3, -7, -1 \rangle t + \langle 3, -5, 0 \rangle, \ t \in \mathbb{R}.$$

(d) (2 points) Find a plane which is orthogonal to both P_1 and P_2 and contains the point $(1, 1, 1)$.

Solution: This plane has normal vector in the direction of $\mathbf{v}$ so its equation is

$$3(x-1) - 7(y-1) - (z-1) = 0.$$

6. **[G2: Calculus of Curves]** A drone has lost access to its GPS mid-flight and is now lost! Thankfully, the pilot can still see the drone's velocity, given by

$$\mathbf{v}(t) = \langle \sin(t) + 4, -2\cos(t) - 2, \cos(2t) + 1 \rangle \ m/s$$

for t in seconds.

(a) (4 points) If the drone started at the origin, find an expression for the position of the drone at time t.

Solution: We have $\mathbf{r}(t) = \int \mathbf{v}(t)\ dt + \mathbf{C}$ so

$$\mathbf{r}(t) = \langle -\cos(t) + 4t, -2\sin(t) - 2t, \frac{1}{2}\sin(2t) + t \rangle + \mathbf{C}.$$

Applying the initial condition $\mathbf{r}(0) = \langle 0, 0, 0 \rangle$ gives

$$\langle 0, 0, 0 \rangle = \langle -1, 0, 0 \rangle + \mathbf{C}.$$

So solving for $\mathbf{C}$ gives

$$\mathbf{r}(t) = \langle -\cos(t) + 4t + 1, -2\sin(t) - 2t, \frac{1}{2}\sin(2t) + t \rangle$$

(b) (2 points) What is the drone's position at time $t = \pi$? In what direction is it flying at time $t = \pi$?

Solution: The drone's position is

$$\mathbf{r}(\pi) = \langle 2 + 4\pi, -2\pi, \pi \rangle$$

and the direction it is flying is

$$\mathbf{v}(\pi) = \langle 4, 0, 2 \rangle.$$

(c) (2 points) Find an equation for the line tangent to the drone's flight path when $t = \pi$.

Solution: From (b) we have that the tangent line is

$$\ell(t) = \langle 4, 0, 2 \rangle (t - \pi) + \langle 2 + 4\pi, -2\pi, \pi \rangle, t \in \mathbb{R}.$$

(d) (2 points) The drone's destination is at $(2 + 40\pi, -2\pi, \pi)\ m$. If the plane loses the ability to steer left and right (in the x and y-directions) at time $t = \pi$, but can still steer up and down (in the z-direction), can the drone make it to its destination? Justify your answer.

Solution: Yes, it can. The drone is currently at the correct y and z-coordinates, so if it steers downward so that $z(t) = 0$ (since $y(t) = 0$ already) and then travels another 9π seconds it will reach the destination.

7. **[G3: Geometry of Curves]** Consider the curve C parameterized by

$$\mathbf{r}(t) = \langle 4\ln(t), 4t^2, 8t\rangle,\ 1 \leq t \leq e^2.$$

(a) (8 points) Compute the length of C.

Solution: We have

$$\begin{aligned}
L &= \int_1^{e^2} \|\mathbf{r}'(t)\|\ dt \\
&= \int_1^{e^2} \|\langle 4/t, 8t, 8\rangle\|\ dt \\
&= \int_1^{e^2} \sqrt{16t^{-2} + 64t^2 + 64}\ dt \\
&= \int_1^{e^2} \sqrt{(4t^{-1} + 8t)^2}\ dt \\
&= \int_1^{e^2} |(4t^{-1} + 8t)|\ dt \\
&= \int_1^{e^2} (4t^{-1} + 8t)\ dt \\
&= 4\ln|t| + 4t^2\ |_1^{e^2} \\
&= (8 + 4e^4) - (0 + 4) \\
&= 4 + 4e^4
\end{aligned}$$

(b) (2 points) Is $\mathbf{r}(t)$ an arc-length parameterization of C? Justify your answer.

Solution: It is not an arc-length parameterization of C because it is not unit-speed: $\|\mathbf{r}'(t)\| = 4t^{-1} + 8t \neq 1$.

8. **[G5: Limits of Functions]** Compute each limit below or show that it does not exist.

(a) (2 points) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{e^{x^2-y}}{\ln(x+y)}$

Solution: This function is continuous at $(1,1)$; we compute the limit by direct calculation to get $1/\ln(2)$.

(b) (4 points) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x^2-y^2}{3x^2+2xy-5y^2}$

Solution: We factor and then evaluate.

$$\begin{aligned}\lim_{(x,y)\to(1,1)} \frac{x^2-y^2}{3x^2+2xy-5y^2} &= \lim_{(x,y)\to(1,1)} \frac{(x+y)(x-y)}{(x-y)(3x+5y)} \\ &= \lim_{(x,y)\to(1,1)} \frac{x+y}{3x+5y} \\ &= \frac{2}{8} = \frac{1}{4}.\end{aligned}$$

(c) (4 points) $\displaystyle\lim_{(x,y)\to(0,1)} \frac{2x(y-1)}{x^2+3(y-1)^2}$

Solution: No algebraic substitution is apparent, so we try the Two-Path Test. Along $x=0$ or $y=1$ we get the same limit of 0:

$$\text{Along } x=0: \quad \begin{aligned}\lim_{(x,y)\to(0,1)} \frac{2x(y-1)}{x^2+3(y-1)^2} &= \lim_{y\to1} \frac{0}{0+3(y-1)^2} \\ &= \lim_{y\to1} 0 \\ &= 0.\end{aligned}$$

However, along the line $y=x+1$ (which goes through $(0,1)$) we have $y-1=x$ and so get

$$\text{Along } y=x+1: \quad \begin{aligned}\lim_{(x,y)\to(0,1)} \frac{2x(y-1)}{x^2+3(y-1)^2} &= \lim_{x\to0} \frac{2x(x)}{x^2+3x^2} \\ &= \lim_{x\to0} \frac{1}{2} \\ &= \frac{1}{2}.\end{aligned}$$

Since these limits on two different paths through $(0,1)$ are different, the overall limit does not exist by the Two Path Test.

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FORMULA SHEET - DO NOT WRITE

FORMULA SHEET

- Dot product: $\langle u_1, u_2, u_3\rangle \cdot \langle v_1, v_2, v_3\rangle = u_1v_1 + u_2v_2 + u_3v_3$
- Dot product magnitudes: $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos(\theta)$
- Cross product: $\langle u_1, u_2, u_3\rangle \times \langle v_1, v_2, v_3\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$
- Cross product magnitude: $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\||\sin(\theta)|$
- Tangent line to a curve $\mathbf{r}(t)$ at $t = a$: $\ell(t) = \mathbf{r}(a) + \mathbf{r}'(a)(t - a)$
- Arc length: $L = \int_a^b \|\mathbf{r}'(t)\|\, dt$
- Arc length function: $s(t) = \int_{t_0}^t \|\mathbf{r}'(T)\|\, dT$
- Unit tangent vector: $\mathbf{T} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{d\mathbf{r}}{ds}$
- Curvature: $\kappa = \left\|\frac{d\mathbf{T}}{ds}\right\| = \frac{1}{\|\mathbf{v}\|}\left\|\frac{d\mathbf{T}}{dt}\right\| = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3}$
- Principal unit normal vector: $\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}/dt}{\|d\mathbf{T}/dt\|}$
- Two-Path Test: If there exist two different paths C_1 and C_2 through the point (a, b) along which the limit of $f(x, y)$ takes on different values, then

$$\lim_{(x,y)\to(a,b)} f(x, y)$$

does not exist.