MATH 2551-D FINAL EXAM
PART 1
VERSION A
FALL 2024
COVERS SECTIONS 12.1-12.6, 13.1-13.4, 14.1-2

EXAM KEY

Honor code statement: I will abide strictly by the Georgia Tech honor code at all times. I will not use a calculator. I will not reference any website, application, or other CAS-enabled service. I will not consult with my notes or anyone during this exam. I will not provide aid to anyone else during this exam.

Initial Here: (   ) I attest to my integrity and will not discuss this exam with anyone until **Saturday December 14**

Read all instructions carefully before beginning.

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- When you complete this portion of the exam, you may turn it in and work on another portion of the exam.
- You may not use aids of any kind.
- The Learning Outcomes for this part of the exam are listed on the back of this front cover.
- The Formula Sheet is on the final page and may be removed. Do not remove any other pages.
- Show your work. Answers without work shown will receive little or no credit.
- Good luck! Write yourself a message of encouragement on the front page!

Question:	1	2	3	4	5	6	7	Total
Points:	1	1	3	6	10	10	9	40

Learning Outcomes

- **G1: Lines.** Describe lines using the vector equation of a line. Find the intersections of lines. Describe the relationships of lines to each other. Solve problems with lines and planes.
- **G2: Planes.** Describe planes using the general equation of a plane. Find the intersections of planes. Find the equations of planes using a point and a normal vector. Describe the relationships of planes to each other. Solve problems with lines and planes.
- **G3: Tangent Vectors to Curves.** Compute tangent vectors to parametric curves; velocity, speed, and acceleration. Find equations of tangent lines to parametric curves.
- **G4: Arc Length.** Compute the arc length of a curve in two or three dimensions. Use arc length to solve problems.
- **G5: Curvature.** Compute normal vectors and curvature for curves in two and three dimensions. Interpret these objects geometrically and in application.
- **G6: Surfaces.** Graph cylinders from equations. Describe and find traces on surfaces. Identify standard quadric surfaces including: spheres, ellipsoids, elliptic paraboloid, hyperboloids, cones, hyperbolic paraboloid. Sketch quadric surfaces. Graph functions of two variables and determine their domains and ranges. Graph level curves.
- **G7: Limits of Functions.** Calculate the limits of functions of two variables or determine if they do not exist. Apply the Squeeze Theorem for functions of two variables. Define continuity of a function of two or more independent variables.

For problems 1-2 choose whether each statement is true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false. Be sure to neatly fill in the bubble corresponding to your answer choice.

1. (1 point) **[G6: Surfaces]** The domain of the function $f(x, y, z) = \sqrt{x^2 - 2} + yz + 4$ is $[-\sqrt{2}, \sqrt{2}]$

   ◯ **TRUE** √ **FALSE**

2. (1 point) **[G1: Lines]** If two lines in $\mathbb{R}^3$ are each orthogonal to a third line, then they must be parallel to each other.

   ◯ **TRUE** √ **FALSE**

3. (3 points) **[G4: Arc Length]** Which statement below about arc length is **not** true?

   ◯ **A)** The length of a curve C is always positive.

   ◯ **B)** The length of a curve C may be equal to the distance between its endpoints.

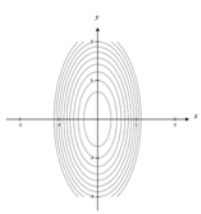
   √ **C) The length of a curve C depends on the parameterization we use to compute it.**

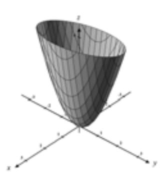
   ◯ **D)** The length of a curve C parameterized by arc length can be greater than 1.

   ◯ **E)** Every smooth curve can be parameterized by arc length.

4. (6 points) **[G5: Surfaces]** Match the contour plots and graphs below with the given functions of two variables.

Function	Graph	Contour Plot
$z = \sin(3x)\sin(3y)$	(J)	(C)
$z = 2e^{-(x^2+y^2)}$	(D)	(E)
$z = 4x^2 + y^2$	(B)	(A)


(A)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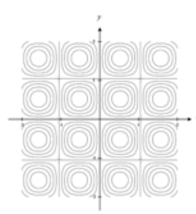
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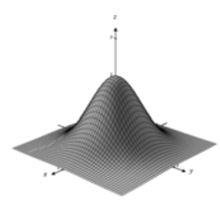
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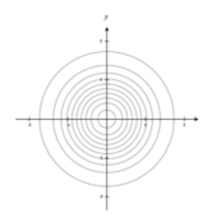
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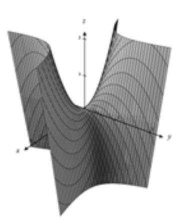
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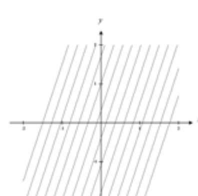
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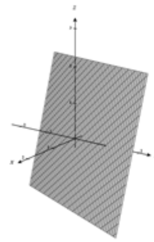
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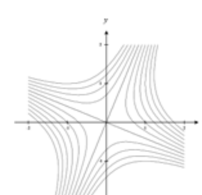
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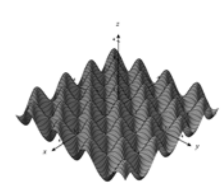
(J)

5. **[G1: Lines], [G2: Planes]** Consider the lines

$$\ell_1(t) = \langle 1, -2, 1\rangle t + \langle 0, 2, -1\rangle \text{ and } \ell_2(t) = \langle 2, 0, -1\rangle t + \langle 3, 0, -1\rangle.$$

(a) (3 points) Find the point of intersection of the lines.

Solution: We set the line equations equal (using different parameter variables) to find their intersction.

$$t = 2s + 3 \qquad 2 - 2t = 0 \qquad t - 1 = -s - 1.$$

Solving this system gives $t = 1, s = -1$ and so we see the lines meet at the point $(1, 0, 0)$.

(b) (3 points) Find a vector orthogonal to the direction vectors of both lines.

Solution: A vector orthogonal to both direction vectors is their cross product:

$$\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{j} \\ 1 & -2 & 1 \\ 2 & 0 & -1 \end{vmatrix} = \langle 2, 3, 4\rangle.$$

(c) (2 points) Give an equation for the plane containing both lines.

Solution: Combining our work in (a) and (b) yields the plane

$$2(x - 1) + 3y + 4z = 0.$$

(d) (2 points) Give a third line which is orthogonal to both lines, passing through their intersection point.

Solution: Such a line is

$$\ell_3(t) = \langle 2, 3, 4\rangle t + \langle 1, 0, 0\rangle.$$

6. (a) (6 points) **[G5: Curvature]** Find the curvature $\kappa(t)$ of the plane curve

$$\mathbf{r}(t) = \langle \cos^3(t), \sin^3(t) \rangle, 0 \le t \le \pi/2.$$

Solution: We compute

$$\begin{aligned} \mathbf{r}'(t) &= \langle -3\cos^2(t)\sin(t), 3\sin^2(t)\cos(t) \rangle, \text{ so} \\ \|\mathbf{r}'(t)\| &= \langle \sqrt{9\cos^4(t)\sin^2(t) + 9\sin^4(t)\cos^2(t)} \\ &= \sqrt{9\cos^2(t)\sin^2(t)(\cos^2(t)+\sin^2(t))} \\ &= 3\cos(t)\sin(t). \end{aligned}$$

Therefore

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \langle -\cos(t), \sin(t) \rangle.$$

Now we compute

$$\mathbf{T}'(t) = \langle \sin(t), \cos(t) \rangle \quad \text{and} \quad \|\mathbf{T}'(t)\| = 1$$

and we can combine these to get the curvature.

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{1}{3\cos(t)\sin(t)}$$

(b) (4 points) **[G3: Tangent Vectors to Curves]** Find the tangent line to the space curve parameterized by

$$\mathbf{r}(t) = \langle 2\sqrt{2}\cos^3(t), 2\sqrt{2}\sin^3(t), t - \frac{\pi}{4} \rangle$$

at the point where $t = \pi/4$.

Solution: We have $\mathbf{r}(\pi/4) = \langle 1, 1, 0 \rangle$ and

$$\mathbf{r}'(t) = \langle -6\sqrt{2}\cos^2(t)\sin(t), 6\sqrt{2}\sin^2(t)\cos(t), 1 \rangle.$$

Then $\mathbf{r}'(t) = \langle -3, 3, 1 \rangle$ and so the tangent line to the curve at this point is

$$\langle -3, 3, 1 \rangle t + \langle 1, 1, 0 \rangle.$$

7. [**G7: Limits of Functions**]

(a) (4 points) Compute

$$\lim_{(x,y)\to(1,-1)} \frac{2x^2+3xy+y^2}{x+y}$$

or show that this limit does not exist.

Solution: Evaluating the limit directly gives the indeterminate form 0/0, but we can simplify.

$$\begin{aligned}\lim_{(x,y)\to(1,-1)} \frac{2x^2+3xy+y^2}{x+y} &= \lim_{(x,y)\to(1,-1)} \frac{(x+y)(2x+y)}{x+y} \\ &= \lim_{(x,y)\to(1,-1)} 2x+y \\ &= 1\end{aligned}$$

(b) (5 points) Use the Squeeze Theorem to show that

$$\lim_{(x,y)\to(0,0)} \frac{5x^2y^3}{x^2y^2+y^4} = 0.$$

It may help to show that

$$\frac{5x^2}{x^2+y^2} \leq 5.$$

Solution: First, since $x^2 < x^2+y^2$, we have

$$5\frac{x^2}{x^2+y^2} < 5(1) = 5.$$

We can see that

$$\frac{5x^2y^3}{x^2y^2+y^4} = \frac{y^3}{y^2}\cdot\frac{5x^2}{x^2+y^2},$$

so we let $g(x,y) = \frac{y^3}{y^2}$ and $h(x,y) = \frac{5x^2}{x^2+y^2}$ in the statement of the Squeeze Theorem. Since

$$\lim_{(x,y)\to(0,0)} \frac{y^3}{y^2} = \lim_{y\to 0} y = 0$$

and we showed above that $\left|\frac{5x^2}{x^2+y^2}\right| \leq 5$, we have by the Squeeze Theorem that

$$\lim_{(x,y)\to(0,0)} \frac{5x^2y^3}{x^2y^2+y^4} = 0.$$

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FORMULA SHEET

- Dot product: $\langle u_1, u_2, u_3\rangle \cdot \langle v_1, v_2, v_3\rangle = u_1v_1 + u_2v_2 + u_3v_3$
- Dot product magnitudes: $\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos(\theta)$
- Cross product: $\langle u_1, u_2, u_3\rangle \times \langle v_1, v_2, v_3\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$
- Cross product magnitude: $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\||\sin(\theta)|$
- Arc length: $L = \displaystyle\int_a^b \|\mathbf{r}'(t)\|\ dt$
- Arc length function: $s(t) = \displaystyle\int_{t_0}^t \|\mathbf{r}'(T)\|\ dT$
- Unit tangent vector: $\mathbf{T} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{d\mathbf{r}}{ds}$
- Curvature: $\kappa = \left\|\dfrac{d\mathbf{T}}{ds}\right\| = \dfrac{1}{\|\mathbf{v}\|}\left\|\dfrac{d\mathbf{T}}{dt}\right\| = \dfrac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|^3}$
- Principal unit normal vector: $\mathbf{N} = \dfrac{1}{\kappa}\dfrac{d\mathbf{T}}{ds} = \dfrac{d\mathbf{T}/dt}{\|d\mathbf{T}/dt\|}$
- Two-Path Test: If there exist two different paths C_1 and C_2 through the point (a, b) along which the limit of $f(x, y)$ takes on different values, then

  $$\lim_{(x,y)\to(a,b)} f(x, y)$$

  does not exist.
- Squeeze theorem: If $f(x, y) = g(x, y)h(x, y)$, where

  $$\lim_{(x,y)\to(a,b)} g(x, y) = 0 \qquad \text{and} \qquad |h(x, y)| \le C \text{ near } (a, b)$$

  for some constant $C > 0$, then

  $$\lim_{(x,y)\to(a,b)} f(x, y) = 0.$$

MATH 2551-D FINAL EXAM
PART 2
VERSION A
FALL 2024
COVERS SECTIONS 14.3-14.8, 15.1-15.4

EXAM KEY

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Question:	1	2	3	4	5	6	7	8	Total
Points:	1	1	3	3	8	8	8	8	40

Learning Outcomes

- **D1: Computing Derivatives.** Compute partial derivatives, total derivatives, directional derivatives, and gradients.
- **D2: Chain Rule.** Apply the Chain Rule for multivariable functions to compute derivatives of composite functions.
- **D3: Gradients.** Interpret the meaning of the gradient and directional derivative of a function at a given point in a specified direction, including in the context of a graph or a contour plot.
- **D4: Tangent Planes and Linear Approximations.** Find equations for tangent planes to surfaces and linear approximations of functions at a given point and use these to solve problems.
- **D5: Optimization.** Find local maxima and minima by using derivative tests for functions of two or more independent variables. Analyze and locate critical points. Find absolute maxima and minima on closed bounded sets. Use maxima and minima to solve application problems.
- **D6: Constrained Optimization.** Define and compute Lagrange multipliers in two and three variables. Use the method of Lagrange multipliers to maximize and minimize functions subject to constraints. Use maxima and minima to solve application problems.
- **I1: Double Integrals.** Set up double integrals as iterated integrals over any region. Sketch regions based on a given iterated integral. Interpret geometric meanings of double integrals.
- **I2: Iterated Integrals.** Compute iterated integrals of two and three variable functions. Apply Fubini's Theorem to change the order of integration of an iterated integral.
- **I3: Polar, Cylindrical, and Spherical Integrals.** Use polar, cylindrical, and spherical coordinates to transform double and triple integrals. Sketch regions based on given polar, cylindrical, and spherical iterated integrals.

For problems 1-2 choose whether each statement is true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false. Be sure to neatly fill in the bubble corresponding to your answer choice.

1. (1 point) **[D1: Computing Derivatives]** Suppose we have a function $f(x, y)$ and $\mathbf{u} \in \mathbb{R}^2$ is a unit vector. Then $D_{\mathbf{u}} f(a, b)$ is a vector.

   ○ **TRUE** √ **FALSE**

2. (1 point) **[D3: Gradients]** The rate of change of a differentiable function f at any point P in the direction of $\nabla f(P)$ is always positive if P is not a critical point of f.

   √ **TRUE** ○ **FALSE**

3. (3 points) **[D5: Optimization]** Suppose $f(x, y)$ is a differentiable function on $\mathbb{R}^2$. Which statement below is **not** true?

   ○ **A)** f must attain a minimum and maximum on any closed and bounded region of $\mathbb{R}^2$.

   ○ **B)** f may have multiple local maxima and no local minima.

   ○ **C)** A critical point of f may be neither a local maximum nor a local minimum.

   √ **D) If** $\det(Hf(a, b)) > 0$ **and** $f_{xx}(a, b) > 0$ **then** $f(a, b)$ **is a local minimum for** f

   ○ **E)** f may not attain a global minimum or maximum.

4. (3 points) **[I3: Polar, Cylindrical, and Spherical Integrals]** What is the area of the region enclosed by one loop of the rose $r = \cos(3\theta)$?

   ○ **A)** $\displaystyle\int_{\pi/6}^{\pi/2} \int_0^{\cos(3\theta)} dr\, d\theta$

   ○ **B)** $\displaystyle\int_{\pi/2}^{3\pi/2} \int_0^{\cos(3\theta)} r\, dr\, d\theta$

   ○ **C)** $\displaystyle\int_0^{\pi/6} \int_0^{\cos(3\theta)} r\, dr\, d\theta$

   ○ **D)** $\displaystyle\int_0^{2\pi} \int_0^{\cos(3\theta)} r\, dr\, d\theta$

   √ **E)** $\displaystyle\int_{\pi/6}^{\pi/2} \int_0^{\cos(3\theta)} r\, dr\, d\theta$

5. (8 points) [**D2: Chain Rule**] Use the multivariable Chain Rule to compute the total derivative of the composition $f = g \circ h$ where $g : \mathbb{R}^2 \to \mathbb{R}$ and $h : \mathbb{R}^3 \to \mathbb{R}^2$ are given by

$$g(x, y) = x^2 + 2y^2 + 4 \quad \text{and} \quad h(s, t, u) = \langle 2s + t + u^2, stu \rangle$$

at the point $(s, t, u) = (1, 1, 1)$.

Solution: We have $Df(1, 1, 1) = Dg(h(1, 1, 1))Dh(1, 1, 1)$ so we need only compute the total derivatives and substitute. We have

$$Dg(x, y) = \begin{bmatrix} 2x & 4y \end{bmatrix} \quad Dh(s, t, u) = \begin{bmatrix} 2 & 1 & 2u \\ tu & su & st \end{bmatrix} \quad h(1, 1, 1) = \langle 4, 1 \rangle.$$

So we get

$$Dg(h(1, 1, 1)) = Dg(4, 1) = \begin{bmatrix} 8 & 4 \end{bmatrix} \qquad Dh(1, 1, 1) = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}.$$

Therefore

$$\begin{aligned} Df(1, 1, 1) &= Dg(h(1, 1, 1))Dh(1, 1, 1) \\ &= \begin{bmatrix} 8 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 20 & 12 & 20 \end{bmatrix}. \end{aligned}$$

6. **[D4: Tangent Planes and Linear Approximations]** Consider the level surface $f = 1$ of the function $f(x, y, z) = \sqrt{x^2 - y^2 + z}$.

(a) (4 points) Compute the tangent plane to the point $(1, 1, 1)$ on this surface.

Solution: We have

$$\nabla f = \langle \frac{x}{\sqrt{x^2 - y^2 + z}}, -\frac{y}{\sqrt{x^2 - y^2 + z}}, \frac{1}{2\sqrt{x^2 - y^2 + z}} \rangle$$

so $\nabla f(1, 1, 1) = \langle 1, -1, 1/2 \rangle$. Then the tangent plane to the surface $f = 1$ at the point $(1, 1, 1)$ is

$$(x - 1) - (y - 1) + \frac{1}{2}(z - 1) = 0.$$

(b) (4 points) Use the linearization of f at $(1, 1, 1)$ to estimate the value of $f(1, 1, 1.1)$.

Solution: Using our work from part (a) we have

$$L(x, y, z) = f(1, 1, 1) + \nabla f(1, 1, 1) \cdot \langle x - 1, y - 1, z - 1 \rangle = 1 + (x - 1) - (y - 1) + \frac{1}{2}(z - 1).$$

Now

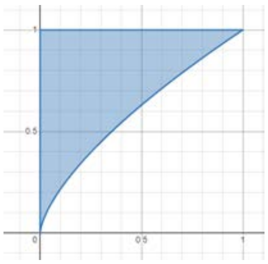
$$f(1, 1, 1.1) \approxeq L(1, 1, 1.1) = 1 + (1 - 1) - (1 - 1) + \frac{1}{2}(1.1 - 1) = 1.05.$$

7. (8 points) **[I1: Double Integrals & I2: Iterated Integrals]** Sketch the domain of integration corresponding to

$$\int_0^1 \int_{x^{2/3}}^1 x\sin(y^4)\ dy\ dx.$$

Then change the order of integration and evaluate. Explain the simplification achieved by changing the order.

Solution: A sketch of the region is below:



We have

$$\begin{aligned}\int_0^1 \int_{x^{2/3}}^1 x\sin(y^4)\ dy\ dx &= \int_0^1 \int_0^{y^{3/2}} x\sin(y^4)\ dx\ dy \\ &= \int_0^1 \frac{1}{2}x^2\sin(y^4)|_{x=0}^{x=y^{3/2}}\ dy \\ &= \int_0^1 \frac{1}{2}y^3\sin(y^4)\ dy \\ &= -\frac{1}{8}\cos(y^4)|_0^1 \\ &= \frac{1}{8}(1-\cos(1))\end{aligned}$$

This simplifies the integral by allowing a u-substitution to deal with the $\sin(y^4)$.

8. (8 points) [**D1: Computing Derivatives & D6: Constrained Optimization**] Find the points on the circle $x^2 + y^2 = 4$ closest to and farthest from the point $(1, -1)$.

Solution: We apply the method of Lagrange multipliers with objective function $d = \sqrt{(x-1)^2 + (y+1)^2}$ and constraint $g(x, y) = x^2 + y^2 = 4$. For simplicity, we will work with $f = d^2$ since it has extreme values at the same locations as d.

$$\nabla f = \langle 2(x-1), 2(y+1) \rangle, \quad \nabla g = \langle 2x, 2y \rangle$$

Now we get the system

$$\begin{cases} 2(x-1) = 2\lambda x \\ 2(y+1) = 2\lambda y \\ x^2 + y^2 = 4 \end{cases}$$

We can see that neither x or y is 0 (this leads to contradictions immediately in the first and second equation respectively) so we may isolate λ and set the results equal to give

$$\begin{aligned} \frac{x-1}{x} &= \frac{y+1}{y} \\ xy - y &= xy + x \\ y &= -x. \end{aligned}$$

Substituting this into the third equation gives $2x^2 = 4$, so the two possible solution points are $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$. We can see that the first point is closer to $(1, -1)$ (it is in the same quadrant) and the second point is farther from $(1, -1)$.

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FORMULA SHEET - DO NOT WRITE

FORMULA SHEET

- Total Derivative: For $\mathbf{f}(x_1, \ldots, x_n) = \langle f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)\rangle$

$$D\mathbf{f} = \begin{bmatrix} (f_1)_{x_1} & (f_1)_{x_2} & \cdots & (f_1)_{x_n} \\ (f_2)_{x_1} & (f_2)_{x_2} & \cdots & (f_2)_{x_n} \\ \vdots & \ddots & \cdots & \vdots \\ (f_m)_{x_1} & (f_m)_{x_2} & \cdots & (f_m)_{x_n} \end{bmatrix}$$

- Linearization: Near $\mathbf{a}$, $f(\mathbf{x}) \approx L(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a})$
- Chain Rule: If $h = g(f(\mathbf{x}))$ then $Dh(\mathbf{x}) = Dg(f(\mathbf{x}))Df(\mathbf{x})$
- Implicit Differentiation: If z is implicitly given in terms of x and y by $F(x, y, z) = c$, then $\dfrac{\partial z}{\partial x} = \dfrac{-F_x}{F_z}$ and $\dfrac{\partial z}{\partial y} = \dfrac{-F_y}{F_z}$.
- Directional Derivative: If $\mathbf{u}$ is a unit vector, $D_{\mathbf{u}}f(P) = Df(P)\mathbf{u} = \nabla f(P) \cdot \mathbf{u}$
- The tangent plane to a level surface of $f(x, y, z)$ at (a, b, c) is

$$0 = \nabla f(a, b, c) \cdot \langle x - a, y - b, z - c\rangle.$$

- Hessian Matrix: For $f(x, y)$, $Hf(x, y) = \begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix}$
- Second Derivative Test: If (a, b) is a critical point of $f(x, y)$ then
  1. If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) < 0$ then f has a local maximum at (a, b)
  2. If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) > 0$ then f has a local minimum at (a, b)
  3. If $\det(Hf(a, b)) < 0$ then f has a saddle point at (a, b)
  4. If $\det(Hf(a, b)) = 0$ the test is inconclusive
- Area/volume: $\text{area}(R) = \iint_R dA$
- Trig identities: $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$, $\quad \cos^2(x) = \frac{1}{2}(1 + \cos(2x))$
- Average value: $f_{avg} = \dfrac{\iint_R f(x, y)dA}{\text{area of } R}$
- Polar coordinates: $x = r\cos(\theta)$, $\quad y = r\sin(\theta)$, $\quad dA = r\ dr\ d\theta$

MATH 2551-D FINAL EXAM
PART 3
VERSION A
FALL 2024
COVERS SECTIONS 15.5-8, 16.1-8

EXAM KEY

Honor code statement: I will abide strictly by the Georgia Tech honor code at all times. I will not use a calculator. I will not reference any website, application, or other CAS-enabled service. I will not consult with my notes or anyone during this exam. I will not provide aid to anyone else during this exam.

Initial Here: (   ) I attest to my integrity and will not discuss this exam with anyone until **Saturday December 14**

Read all instructions carefully before beginning.

- Print your name and GT ID neatly above.
- You may use any portion of our 2 hour and 50 minute final exam time to work on this portion of the exam.
- When you complete this portion of the exam, you may turn it in and work on another portion of the exam.
- You may not use aids of any kind.
- The Learning Outcomes for the exam are listed on the back of this front cover.
- The Formula Sheet is on the final page and may be removed. Do not remove any other pages. Do not write any work on the Formula Sheet.
- Show your work. Answers without work shown will receive little or no credit.
- Good luck! Write yourself a message of encouragement on the front page!

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	1	1	3	3	3	3	8	8	10	40

Learning Outcomes

- **I3: Polar, Cylindrical, and Spherical Integrals.** Use polar, cylindrical, and spherical coordinates to transform double and triple integrals. Sketch regions based on given polar, cylindrical, and spherical iterated integrals.
- **I4: Triple Integrals.** Set up triple integrals as iterated integrals over any region. Sketch regions based on a given iterated integral. Interpret geometric meanings of triple integrals.
- **I5: Integral Applications.** Use multiple integrals to solve physical problems, such as finding area, average value, volume, mass of a lamina, or solid.
- **I6: Change of Variables.** Use general change of variables to transform double and triple integrals for easier calculation. Choose the most appropriate coordinate system to evaluate a specific integral.
- **V1: Scalar Line Integrals.** Define line integrals of a scalar function over a smooth curve. Evaluate scalar line integrals in two and three dimensions.
- **V2: Vector Line Integrals.** Define line integrals of a vector-valued function or vector field over a smooth curve. Evaluate vector line integrals in two and three dimensions.
- **V3: Conservative Vector Fields.** Test for conservative vector fields. Find potential functions. Evaluate line integrals using the Fundamental Theorem of Line Integrals.
- **V4: Green's Theorem.** Compute two-dimensional curls and divergence of vector fields. Evaluate integrals using Green's theorem. Solve applied problems using Green's Theorem, such as finding area.
- **V5: Surface Integrals.** Define and compute surface integrals for scalar and vector valued functions.
- **V6: Applications of Vector Calculus.** Interpret work, flow, flux, and surface area in terms of line and/or surface integrals, as appropriate.
- **V7: Generalizations of the FTC.** State and apply Stokes' Theorem and the Divergence Theorem to solve problems in three dimensions.

For problems 1-2 choose whether each statement is true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false. Be sure to neatly fill in the bubble corresponding to your answer choice.

1. (1 point) **[V4: Triple Integrals]** If D is any solid region in space bounded by smooth surfaces, then we can evaluate

$$\iiint_D dV$$

in any order of integration to compute the volume of D.

√ **TRUE** ◯ **FALSE**

2. (1 point) **[V3: Conservative Vector Fields]** Every vector field $\mathbf{F}$ defined on all of $\mathbb{R}^2$ is conservative.

◯ **TRUE** √ **FALSE**

3. (3 points) **[V3, V4, V6, V7]** Consider $\mathbf{G}(x, y, z) = \langle e^z + 2023y, \cos(x^{2024}), 2025x\rangle$ and S be the surface which is the part of the paraboloid $z = x^2 + 4y^2$ between $z = 0$ and $z = 10$, oriented away from the z-axis. For each integration theorem or strategy below, bubble in "Yes" if it is appropriate to use to compute the flux of $\mathbf{G}$ across S and "No" otherwise.

◯ Yes ✓ No Fundamental Theorem of Line Integrals
◯ Yes ✓ No Green's Theorem
√ **Yes** ◯ **No** **Stokes' Theorem**
◯ Yes ✓ No Divergence Theorem
√ **Yes** ◯ **No** **Parameterization and direct calculation**

4. (3 points) **[V3, V4, V6, V7]** Consider the vector field $\mathbf{F}(x, y) = \langle 17x + y, x - 21y\rangle$ in $\mathbb{R}^2$ and the curve C which is the circle $x^2 + y^2 = 256$ oriented counterclockwise with outward normal vector. For each integration theorem or strategy below, bubble in "Yes" if it is appropriate to use to compute the flux of $\mathbf{F}$ across C and "No" otherwise.

◯ Yes ✓ No Fundamental Theorem of Line Integrals
√ **Yes** ◯ **No** **Green's Theorem**
◯ Yes ✓ No Stokes' Theorem
◯ Yes ✓ No Divergence Theorem
√ **Yes** ◯ **No** **Parameterization and direct calculation**

5. (3 points) **[I5: Integral Applications]** The integral below could describe the mass of:

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{\sqrt{x^2+y^2}}^{3}(x^2+y^2+z^2)\,dz\,dy\,dx$$

○ **A)** a solid ball that gets lighter away from the origin
○ **B)** a solid ball that is equally heavy at all points
○ **C)** a solid ball that gets heavier away from the origin
○ **D)** a solid cone that gets lighter away from the origin
√ **E) a solid cone that gets heavier away from the origin**

6. (3 points) **[V5: Surface Integrals]** Which of the following is a parameterization of the surface S which is the portion of the cylinder $y^2+z^2=9$ between the planes $x=0$ and $x=2$?

○ **A)** $\mathbf{r}(z,\theta)=\langle 3\cos(\theta), 3\sin(\theta), z\rangle \quad 0\le z\le 2, 0\le\theta\le 2\pi$
○ **B)** $\mathbf{r}(r,\theta)=\langle r\cos(\theta), r\sin(\theta), 2\rangle \quad 0\le r\le 3, 0\le\theta\le 2\pi$
√ **C)** $\mathbf{r}(x,\theta)=\langle x, 3\cos(\theta), 3\sin(\theta)\rangle \quad 0\le x\le 2, 0\le\theta\le 2\pi$
○ **D)** $\mathbf{r}(x,y)=\langle x, y, \sqrt{9-y^2}\rangle \quad 0\le x\le 2, -3\le y\le 3$
○ **E)** $\mathbf{r}(x,\theta)=\langle x, 2\cos(\theta), 2\sin(\theta)\rangle \quad 0\le x\le 3, 0\le\theta\le 2\pi$

7. (8 points) **[I6: Change of Variables]** Use the transformation

$$\mathbf{T}(u,v) = \begin{bmatrix} \frac{1}{4}(u+v) \\ \frac{1}{4}(v-3u) \end{bmatrix}$$

to change variables in the integral

$$\iint_R 4x + 8y \; dA$$

where R is the parallelogram bounded by the lines

$$x - y = -4, \quad x - y = 4, \quad y + 3x = 0, \quad y + 3x = 8.$$

A complete solution will include a sketch of the new region of integration in the uv-plane. **Do not evaluate the resulting integral in terms of u and v.**

Solution: First, notice that we have

$$x - y = \frac{1}{4}u + \frac{1}{4}v - \frac{1}{4}v + \frac{3}{4}u = u \qquad \text{and} \qquad y + 3x = \frac{1}{4}v - \frac{3}{4}u + \frac{3}{4}u + \frac{3}{4}v = v.$$

So the new region of integration is the rectangle $-4 \leq u \leq 4, 0 \leq v \leq 8$ in the uv-plane.

Next, we have

$$|\det(D\mathbf{T}(u,v)) = |\det \begin{bmatrix} 1/4 & 1/4 \\ -3/4 & 1/4 \end{bmatrix}| = \left|\frac{1}{16} + \frac{3}{16}\right| = \frac{1}{4}.$$

Finally, we have

$$4x + 8y = u + v - 6u + 2v = -5u + 3v.$$

Combining all of this, the Substitution Theorem gives

$$\iint_R 4x + 8y \; dA = \int_{-4}^{4} \int_0^8 \frac{3v - 5u}{4} \; dv \; du.$$

8. **[V7: Generalizations of the FTC.]** Consider the vector field

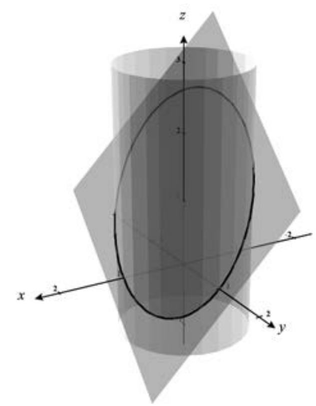
$$\mathbf{F}(x, y, z) = \langle z, x, y \rangle,$$

the surface S which the part of the plane $x + y + z = 1$ inside the cylinder $x^2 + y^2 = 1$, oriented with normal away from the origin, and the curve C which is the boundary of S. The curve and surface are pictured below.

(a) (4 points) Give a parameterization of C which is oriented compatibly with S.

Solution: Since S is oriented with normal away from the origin, we need C to be oriented counterclockwise around the z-axis viewed from above the xy-plane for a compatible orientation. Therefore one compatible parameterization of C is

$$\mathbf{r}(t) = \langle \cos(t), \sin(t), 1 - \cos(t) - \sin(t) \rangle, \quad 0 \leq t \leq 2\pi.$$



(b) (6 points) Apply Stokes' Theorem with your parameterization from (a) to compute the flux of the curl of $\mathbf{F}$ across S.

Solution: We will need $\mathbf{r}'(t)$ and $\mathbf{F}(\mathbf{r}(t))$ to apply Stokes' Theorem here. These are

$$\mathbf{r}'(t) = \langle -\sin(t), \cos(t), \sin(t) - \cos(t) \rangle \qquad \mathbf{F}(\mathbf{r}(t)) = \langle 1 - \cos(t) - \sin(t), \cos(t), \sin(t) \rangle.$$

Now we can apply Stokes' Theorem:

$$\begin{aligned}
\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma &= \int_C \mathbf{F} \cdot d\mathbf{r} \\
&= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\
&= \int_0^{2\pi} \langle 1 - \cos(t) - \sin(t), \cos(t), \sin(t) \rangle \cdot \langle -\sin(t), \cos(t), \sin(t) - \cos(t) \rangle\, dt \\
&= \int_0^{2\pi} -\sin(t) + \sin(t)\cos(t) + \sin^2(t) + \cos^2(t) + \sin^2(t) - \sin(t)\cos(t)\, dt \\
&= \int_0^{2\pi} 1 - \sin(t) + \sin^2(t)\, dt \\
&= \int_0^{2\pi} \frac{3}{2} - \sin(t) - \frac{1}{2}\cos(2t)\, dt \\
&= \frac{3}{2}t + \cos(t) - \frac{1}{4}\sin(2t)\Big|_0^{2\pi} \\
&= 3\pi.
\end{aligned}$$

9. [**V1: Vector Line Integrals, V2: Vector Line Integrals**] Consider the curve C which is the line segment from the point $(2,0)$ to the point $(0,3)$.

(a) (4 points) Compute $\int_C x + y\ ds$.

Solution: For both parts, we need a parameterization $\mathbf{r}(t)$ of C and its derivative $\mathbf{r}'(t)$.

$$\mathbf{r}(t) = \langle -2, 3\rangle t + \langle 2, 0\rangle, 0 \le t \le 1 \qquad \mathbf{r}'(t) = \langle -2, 3\rangle.$$

For this part, we also need $\|\mathbf{r}'(t)\| = \sqrt{4+9} = \sqrt{13}$. Now we have

$$\begin{aligned}\int_C x + y\ ds &= \int_0^1 (x(t) + y(t))\|\mathbf{r}'(t)\|\ dt \\ &= \int_0^1 ((2-2t) + 3t)\sqrt{13}\ dt \\ &= \int_0^1 \sqrt{13}(2+t)\ dt \\ &= \sqrt{13}(2t + t^2/2)|_0^1 \\ &= \frac{5\sqrt{13}}{2}.\end{aligned}$$

(b) (4 points) Compute $\int_C (x\mathbf{i} + y\mathbf{j}) \cdot d\mathbf{r}$.

Solution: We will use our parameterization from part (a).

$$\begin{aligned}\int_C (x\mathbf{i} + y\mathbf{j}) \cdot d\mathbf{r}\ ds &= \int_0^1 (x(t)\mathbf{i} + y(t)\mathbf{j}) \cdot \mathbf{r}'(t)\ dt \\ &= \int_0^1 \langle 2-2t, 3t\rangle \cdot \langle -2, 3\rangle\ dt \\ &= \int_0^1 -4 + 13t\ dt \\ &= -4t + \frac{13}{2}t^2|_0^1 \\ &= \frac{5}{2}.\end{aligned}$$

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FORMULA SHEET

Rules for Triple Integrals for the Sketching Impaired

1: Choose a variable appearing exactly twice for the next integral.

2: After setting up an integral, cross out any constraints involving the variable just used.

3: Create a new constraint by setting the lower limit of the preceding integral less than the upper limit.

4: A squared variable counts twice.

5: The argument of a square root must be nonnegative.

- Trig identities: $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$, $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$
- Volume$(D) = \iiint_D dV$, $\quad f_{avg} = \dfrac{\iiint_D f(x, y, z)dV}{\text{volume of } D}$, Mass: $M = \iiint_D \delta \, dV$
- Coordinate Transforms:
  - Cylindrical: $x = r\cos(\theta)$, $\quad y = r\sin(\theta)$, $z = z$, $dV = r \; dz \; dr \; d\theta$
  - Spherical: $x = \rho \sin(\phi)\cos(\theta)$, $\quad y = \rho\sin(\phi)\sin(\theta)$, $\quad z = \rho\cos(\phi)$, $dV = \rho^2 \sin(\phi) \; d\rho \; d\phi \; d\theta$
- Substitution Theorem: If R is the image of G under a coordinate transformation $\mathbf{T}(u, v) = \langle x(u, v), y(u, v)\rangle$ then

$$\iint_R f(x, y) \; dx \; dy = \iint_G f(\mathbf{T}(u, v))|\det D\mathbf{T}(u, v)| \; du \; dv.$$

- Line integrals:
  - $\int_C f(x, y, z) \; ds = \int_a^b f(\mathbf{r}(t))\|\mathbf{r}'(t)\|dt$
  - $\int_C \mathbf{F} \cdot \mathbf{T} \; ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \; dt$
  - $\int_C \mathbf{F} \cdot \mathbf{n} \; ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \langle y'(t), -x'(t)\rangle \; dt.$
- Fundamental Theorem of Line Integrals: $\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A)$ if C is a path from A to B
- Curl (Mixed Partials) Test: On a simply-connected region $\mathbf{F} = \nabla f \Leftrightarrow \text{curl } \mathbf{F} = \mathbf{0}$
- $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$ $\qquad$ $\text{div } \mathbf{F} = \nabla \cdot \mathbf{F}$ $\qquad$ $\text{curl } \mathbf{F} = \nabla \times \mathbf{F}$
- Green's Theorem: If C is a simple closed curve with positive orientation and R is the simply-connected region it encloses, then

$$\int_C \mathbf{F} \cdot \mathbf{T} \; ds = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k} \; dA \qquad \int_C \mathbf{F} \cdot \mathbf{n} \; ds = \iint_R (\nabla \cdot \mathbf{F}) \; dA.$$

FORMULA SHEET CONTINUED

- Surface integrals:
  - $\iint_S f(x,y,z)\ d\sigma = \iint_R f(\mathbf{r}(u,v))\ \|\mathbf{r}_u \times \mathbf{r}_v\| dA$
  - $\iint_S \mathbf{F} \cdot \mathbf{n}\ d\sigma = \iint_S \mathbf{F} \cdot d\boldsymbol{\sigma} = \iint_R \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\ dA$

- Stokes' Theorem: If S is a piecewise smooth oriented surface bounded by a piecewise smooth curve C and $\mathbf{F}$ is a vector field whose components have continuous partial derivatives on an open region containing S, then

$$\int_C \mathbf{F} \cdot \mathbf{T}\ ds = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\ d\sigma.$$

- Divergence Theorem: If S is a piecewise smooth closed oriented surface enclosing a volume D and $\mathbf{F}$ is a vector field whose components have continuous partial derivatives on D, then

$$\iint_S \mathbf{F} \cdot \mathbf{n}\ d\sigma = \iiint_D \nabla \cdot \mathbf{F}\ dV.$$