MATH 2551-D MIDTERM 3
VERSION A
FALL 2024
COVERS SECTIONS 15.5-8, 16.1-8

EXAM KEY

Honor code statement: I will abide strictly by the Georgia Tech honor code at all times. I will not use a calculator. I will not reference any website, application, or other CAS-enabled service. I will not consult with my notes or anyone during this exam. I will not provide aid to anyone else during this exam.

(   ) **Initial here to attest to your integrity.**

Read all instructions carefully before beginning.

- Print your name and GT ID neatly above.
- You have 75 minutes to take the exam.
- You may not use aids of any kind.
- Show your work. Answers without work shown will receive little or no credit.
- The Learning Outcomes for the exam are listed on the back of this front cover.
- The Formula Sheet is on the final page and may be removed. Do not remove any other pages. Do not write any work on the Formula Sheet
- Good luck! Write yourself a message of encouragement on the front page!

Question:	1	2	3	4	5	6	7	8	9	10	11	Total
Points:	1	1	3	3	3	3	3	3	10	10	10	50

Learning Outcomes

- **I3: Polar, Cylindrical, and Spherical Integrals.** Use polar, cylindrical, and spherical coordinates to transform double and triple integrals. Sketch regions based on given polar, cylindrical, and spherical iterated integrals.
- **I4: Triple Integrals.** Set up triple integrals as iterated integrals over any region. Sketch regions based on a given iterated integral. Interpret geometric meanings of triple integrals.
- **I5: Integral Applications.** Use multiple integrals to solve physical problems, such as finding area, average value, volume, mass of a lamina, or solid.
- **I6: Change of Variables.** Use general change of variables to transform double and triple integrals for easier calculation. Choose the most appropriate coordinate system to evaluate a specific integral.
- **V1: Scalar Line Integrals.** Define line integrals of a scalar function over a smooth curve. Evaluate scalar line integrals in two and three dimensions.
- **V2: Vector Line Integrals.** Define line integrals of a vector-valued function or vector field over a smooth curve. Evaluate vector line integrals in two and three dimensions.
- **V3: Conservative Vector Fields.** Test for conservative vector fields. Find potential functions. Evaluate line integrals using the Fundamental Theorem of Line Integrals.
- **V4: Green's Theorem.** Compute two-dimensional curls and divergence of vector fields. Evaluate integrals using Green's theorem. Solve applied problems using Green's Theorem, such as finding area.
- **V5: Surface Integrals.** Define and compute surface integrals for scalar and vector valued functions.
- **V6: Applications of Vector Calculus.** Interpret work, flow, flux, and surface area in terms of line and/or surface integrals, as appropriate.
- **V7: Generalizations of the FTC.** State and apply Stokes' Theorem and the Divergence Theorem to solve problems in three dimensions.

For problems 1-2 choose whether each statement is true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false. Be sure to neatly fill in the bubble corresponding to your answer choice.

1. (1 point) **[V6: Applications of Vector Calculus]** If $\mathbf{F} \cdot \mathbf{T} > 0$ at every point on a curve C, then the net flow of $\mathbf{F}$ along the curve C is positive.

   √ **TRUE** ○ **FALSE**

2. (1 point) **[V3: Conservative Vector Fields]** Let $\mathbf{F}$ be a conservative vector field in $\mathbb{R}^3$. If f and g both satisfy $\nabla f = \nabla g = \mathbf{F}$, then $f = g$.

   ○ **TRUE** √ **FALSE**

3. (3 points) **[V3, V4, V7]** Let $\mathbf{F}(x, y, z) = \langle 4x, -8y, 4z \rangle$ and S be the surface which is the part of the cylinder $x^2 + z^2 = 4$ between $y = 2024$ and $y = 2025$, oriented away from the y-axis. $\mathbf{F}$ is the curl of a vector field $\mathbf{G}$ since $\nabla \cdot \mathbf{F} = 0$. Select all of the integration theorems below that are appropriate to use to compute the flux of $\mathbf{F}$ across S.

   ○ **A)** Fundamental Theorem of Line Integrals
   ○ **B)** Green's Theorem (circulation)
   ○ **C)** Green's Theorem (flux)
   √ **D) Stokes' Theorem**
   ○ **E)** Divergence Theorem

4. (3 points) **[V3, V4, V7]** Let $\mathbf{F}(x, y) = \langle 3x + y, x - 3y \rangle$ and C be the circle $x^2 + y^2 = 25$ oriented clockwise. Select all of the integration theorems below that are appropriate to use to compute the circulation of $\mathbf{F}$ along C.

   √ **A) Fundamental Theorem of Line Integrals**
   √ **B) Green's Theorem (circulation)**
   ○ **C)** Green's Theorem (flux)
   ○ **D)** Stokes' Theorem
   ○ **E)** Divergence Theorem

5. (3 points) **[I6: Change of Variables]** Suppose we wish to evaluate the integral $\iint_R f(x,y)\ dA$ using the change of variables

$$u = 3x \qquad v = y^5.$$

Which of the following is the new area element dA in terms of du and dv?

○ **A)** $dA = 15v^{4/5}\ du\ dv$

○ **B)** $dA = 15v^4\ du\ dv$

√ **C)** $dA = \dfrac{1}{15}v^{-4/5}\ du\ dv$

○ **D)** $dA =\ du\ dv$

○ **E)** $dA = \dfrac{5}{3}v^{-4/5}\ du\ dv$

6. (3 points) **[V3: Conservative Vector Fields]** If possible, find a potential function for the vector field

$$\mathbf{F}(x,y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle.$$

√ **A)** $f(x,y) = \arctan(y/x)$

○ **B)** $f(x,y) = \arctan(y/x) + \arctan(x/y)$

○ **C)** $f(x,y) = -\dfrac{1}{2}\ln(x^2+y^2)$

○ **D)** $f(x,y) = \dfrac{1}{2}\ln(x^2+y^2)$

○ **E)** This vector field is not conservative.

7. (3 points) **[I3: Polar, Cylindrical, and Spherical Integrals]** Which integral below computes the volume of the region inside the sphere $x^2 + y^2 + z^2 = 4$ and outside the cylinder $x^2 + y^2 = 2$?

◯ **A)** $\int_0^{2\pi} \int_{\sqrt{2}}^{2} \int_0^{\sqrt{4-r^2}} dz\ dr\ d\theta$

◯ **B)** $\int_0^{\pi} \int_0^{2} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\ dz\ dr\ d\theta$

◯ **C)** $\int_0^{2\pi} \int_0^{\pi/4} \int_{\sqrt{2}\csc(\phi)}^{2} \rho^2 \sin(\phi)\ d\rho\ d\phi\ d\theta$

◯ **D)** $\int_0^{2\pi} \int_{\pi/4}^{3\pi/4} \int_0^{2} \rho^2 \sin(\phi)\ d\rho\ d\phi\ d\theta$

√ **E)** $\int_0^{2\pi} \int_{\pi/4}^{3\pi/4} \int_{\sqrt{2}\csc(\phi)}^{2} \rho^2 \sin(\phi)\ d\rho\ d\phi\ d\theta$

8. (3 points) **[V5: Surface Integrals]** Select all of the parameterizations $\mathbf{r}(s,t)$ below corresponding to the surface which is the part of the elliptical paraboloid $x = y^2 + 4z^2$ with $0 \le x \le 4$.

◯ **A)** $\mathbf{r}(s,t) = \langle s, t, s^2 + 4t^2 \rangle$, $0 \le s^2 + 4t^2 \le 4$

√ **B)** $\mathbf{r}(s,t) = \langle s^2 + 4t^2, s, t \rangle$, $0 \le s^2 + 4t^2 \le 4$

◯ **C)** $\mathbf{r}(s,t) = \langle s\cos(t), s\sin(t), s^2\cos^2(t) + 4s^2\sin^2(t) \rangle$, $0 \le s \le 4, 0 \le t \le 2\pi$

√ **D)** $\mathbf{r}(s,t) = \langle s, \sqrt{s}\cos(t), \frac{\sqrt{s}}{2}\sin(t) \rangle$, $0 \le s \le 4, 0 \le t \le 2\pi$

√ **E)** $\mathbf{r}(s,t) = \langle 4t^2, 2t\cos(s), t\sin(s) \rangle$, $0 \le s \le 2\pi, 0 \le t \le 1$

9. (10 points) **[I4: Triple Integrals, I5: Integral Applications]** Set up but do not compute an integral to compute the mass of the solid D that has mass density function $\delta(x, y, z) = 4 + \cos(x)\sin(y) + z/10$ and occupies the region

$$y \leq 8 - x^2, \qquad x^2 + 2z^2 \leq y, \qquad x \geq 0.$$

A complete solution will include a sketch of the shadow of D in the plane of the two variables that you do not integrate first.

Solution: The given constraints suggest integrating with respect to y first since we have

$$x^2 + 2z^2 \leq y \leq 8 - x^2.$$

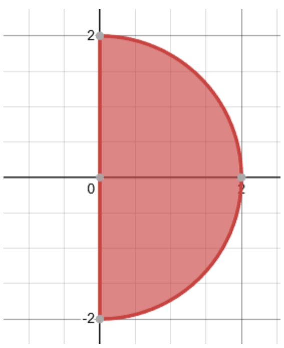
The shadow of this solid in the xz-plane is thus the region with

$$x^2 + 2z^2 \leq 8 - x^2 \text{ and } x \geq 0$$

i.e. the region

$$x^2 + z^2 \leq 4 \text{ and } x \geq 0.$$

A sketch of this region is given below; it is the right half of the disk of radius 2 centered at the origin in the xz-plane.



Therefore an integral to compute this mass is

$$\iiint_D \delta(x, y, z)\, dV = \int_{-2}^{2} \int_{0}^{\sqrt{4-z^2}} \int_{x^2+2z^2}^{8-x^2} 4 + \cos(x)\sin(y) + z/10 \; dy\, dx\, dz$$

$$\text{or } \int_{0}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+2z^2}^{8-x^2} 4 + \cos(x)\sin(y) + z/10 \; dy\, dz\, dx$$

$$\text{or } \int_{0}^{2} \int_{x^2}^{8-x^2} \int_{-\sqrt{(y-x^2)/2}}^{\sqrt{(y-x^2)/2}} 4 + \cos(x)\sin(y) + z/10 \; dz\, dy\, dx$$

10. (10 points) **[V5: Surface Integrals, V6: Applications of Vector Calculus, V7: Generalizations of the FTC.]** Compute the flux of the vector field

$$\mathbf{F}(x, y, z) = \langle xy^2 + \frac{1}{3}x^3, yx^2 + e^{x\cos(z)}, y^2z + \sin(\arctan(x))\rangle$$

out of the closed surface S which is the portion of the paraboloid $z = x^2 + y^2$ from $z = 0$ to $z = 4$ together with the disk $x^2 + y^2 \leq 4$ in the plane $z = 4$. Simplify your final answer.

Hint: Cylindrical coordinates may be useful.

Solution: The surface is closed so we can apply the Divergence Theorem. Therefore the flux is

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma &= \iiint_D \nabla \cdot \mathbf{F}\, dV \\
&= \iiint_D (y^2 + x^2) + (x^2) + (y^2)\, dV \\
&= \int_0^{2\pi} \int_0^2 \int_{r^2}^4 2r^3\, dz\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^2 8r^3 - 2r^5\, dr\, d\theta \\
&= \int_0^{2\pi} 2r^4 - \frac{r^6}{3}\Big|_0^2\, d\theta \\
&= \int_0^{2\pi} 32 - \frac{64}{3}\, d\theta \\
&= \int_0^{2\pi} \frac{32}{3}\, d\theta \\
&= \frac{64\pi}{3}.
\end{aligned}$$

11. **[V2: Vector Line Integrals, V3: Conservative Vector Fields, V4: Green's Theorem]** Consider the vector field $\mathbf{F} = \langle 2y, x\rangle$ and the curve C which is the circle $x^2+y^2 = 25$, oriented counterclockwise with outward normal vector.

   (a) (4 points) Compute $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$.

   Solution: We can apply the circulation form of Green's Theorem:

$$\begin{aligned}\int_C \mathbf{F}\cdot\mathbf{T}\,ds &= \iint_R (\nabla\times\mathbf{F})\cdot\mathbf{k}\,dA\\ &= \iint_R (1-2)\,dA\\ &= -\text{area}(R)\\ &= -25\pi.\end{aligned}$$

   (b) (4 points) Compute $\int_C \mathbf{F}\cdot\mathbf{n}\,ds$.

   Solution: We can apply the flux form of Green's Theorem:

$$\begin{aligned}\int_C \mathbf{F}\cdot\mathbf{n}\,ds &= \iint_R (\nabla\cdot\mathbf{F})\,dA\\ &= \iint_R (0+0)\,dA\\ &= 0\end{aligned}$$

   (c) (2 points) Based on your work above, what can you conclude about whether $\mathbf{F}$ is conservative? Justify your answer.

   Solution: We know that $\mathbf{F}$ is not conservative because its circulation around the closed curve C is nonzero.

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FORMULA SHEET

Rules for Triple Integrals for the Sketching Impaired

1: Choose a variable appearing exactly twice for the next integral.

2: After setting up an integral, cross out any constraints involving the variable just used.

3: Create a new constraint by setting the lower limit of the preceding integral less than the upper limit.

4: A squared variable counts twice.

5: The argument of a square root must be nonnegative.

- Trig identities: $\sin^2(x) = \frac{1}{2}(1-\cos(2x))$, $\cos^2(x) = \frac{1}{2}(1+\cos(2x))$
- Volume$(D) = \iiint_D dV$, $\quad f_{avg} = \dfrac{\iiint_D f(x,y,z)dV}{\text{volume of } D}$, Mass: $M = \iiint_D \delta \, dV$
- Coordinate Transforms:
  - Cylindrical: $x = r\cos(\theta)$, $\quad y = r\sin(\theta)$, $z = z$, $dV = r\ dz\ dr\ d\theta$
  - Spherical: $x = \rho\sin(\phi)\cos(\theta)$, $\quad y = \rho\sin(\phi)\sin(\theta)$, $\quad z = \rho\cos(\phi)$, $dV = \rho^2\sin(\phi)\ d\rho\ d\phi\ d\theta$
- Substitution Theorem: If R is the image of G under a coordinate transformation $\mathbf{T}(u,v) = \langle x(u,v), y(u,v)\rangle$ then

$$\iint_R f(x,y)\ dx\ dy = \iint_G f(\mathbf{T}(u,v))|\det D\mathbf{T}(u,v)|\ du\ dv.$$

- Line integrals:
  - $\int_C f(x,y,z)\ ds = \int_a^b f(\mathbf{r}(t))\|\mathbf{r}'(t)\|dt$
  - $\int_C \mathbf{F}\cdot\mathbf{T}\ ds = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\ dt$
  - $\int_C \mathbf{F}\cdot\mathbf{n}\ ds = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\langle y'(t), -x'(t)\rangle\ dt.$
- Fundamental Theorem of Line Integrals: $\int_C \nabla f\cdot d\mathbf{r} = f(B) - f(A)$ if C is a path from A to B
- Curl (Mixed Partials) Test: On a simply-connected region $\mathbf{F} = \nabla f \Leftrightarrow \text{curl } \mathbf{F} = \mathbf{0}$
- $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle \qquad \text{div } \mathbf{F} = \nabla\cdot\mathbf{F} \qquad \text{curl } \mathbf{F} = \nabla\times\mathbf{F}$
- Green's Theorem: If C is a simple closed curve with positive orientation and R is the simply-connected region it encloses, then

$$\int_C \mathbf{F}\cdot\mathbf{T}\ ds = \iint_R (\nabla\times\mathbf{F})\cdot\mathbf{k}\ dA \qquad \int_C \mathbf{F}\cdot\mathbf{n}\ ds = \iint_R (\nabla\cdot\mathbf{F})\ dA.$$

FORMULA SHEET CONTINUED

- Surface integrals:
  - $\iint_S f(x,y,z)\ d\sigma = \iint_R f(\mathbf{r}(u,v))\ \|\mathbf{r}_u \times \mathbf{r}_v\| dA$
  - $\iint_S \mathbf{F} \cdot \mathbf{n}\ d\sigma = \iint_S \mathbf{F} \cdot d\boldsymbol{\sigma} = \iint_R \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\ dA$

- Stokes' Theorem: If S is a piecewise smooth oriented surface bounded by a piecewise smooth curve C and $\mathbf{F}$ is a vector field whose components have continuous partial derivatives on an open region containing S, then

$$\int_C \mathbf{F} \cdot \mathbf{T}\ ds = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\ d\sigma.$$

- Divergence Theorem: If S is a piecewise smooth closed oriented surface enclosing a volume D and $\mathbf{F}$ is a vector field whose components have continuous partial derivatives on D, then

$$\iint_S \mathbf{F} \cdot \mathbf{n}\ d\sigma = \iiint_D \nabla \cdot \mathbf{F}\ dV.$$