MATH 2551-D MIDTERM 2
VERSION A
FALL 2024
COVERS SECTIONS 14.3-14.8, 15.1-15.4

EXAM KEY

Honor code statement: I will abide strictly by the Georgia Tech honor code at all times. I will not use a calculator. I will not reference any website, application, or other CAS-enabled service. I will not consult with my notes or anyone during this exam. I will not provide aid to anyone else during this exam.

(   ) **Initial here to attest to your integrity.**

Read all instructions carefully before beginning.

- Print your name and GT ID neatly above.
- You have 75 minutes to take the exam.
- You may not use aids of any kind.
- Show your work. Answers without work shown will receive little or no credit.
- The Learning Outcomes for the exam are listed on the back of this front cover. The Formula Sheet is on the final page and may be removed.
- Good luck! Write yourself a message of encouragement on the front page!

Question	Points
1	1
2	1
3	4
4	8
5	6
6	10
7	10
8	10
Total:	50

Learning Outcomes

- **D1: Computing Derivatives.** Compute partial derivatives, total derivatives, directional derivatives, and gradients.
- **D2: Chain Rule.** Apply the Chain Rule for multivariable functions to compute derivatives of composite functions.
- **D3: Gradients.** Interpret the meaning of the gradient and directional derivative of a function at a given point in a specified direction, including in the context of a graph or a contour plot.
- **D4: Tangent Planes and Linear Approximations.** Find equations for tangent planes to surfaces and linear approximations of functions at a given point and use these to solve problems.
- **D5: Optimization.** Find local maxima and minima by using derivative tests for functions of two or more independent variables. Analyze and locate critical points. Find absolute maxima and minima on closed bounded sets. Use maxima and minima to solve application problems.
- **D6: Constrained Optimization.** Define and compute Lagrange multipliers in two and three variables. Use the method of Lagrange multipliers to maximize and minimize functions subject to constraints. Use maxima and minima to solve application problems.
- **I1: Double Integrals.** Set up double integrals as iterated integrals over any region. Sketch regions based on a given iterated integral. Interpret geometric meanings of double integrals.
- **I2: Iterated Integrals.** Compute iterated integrals of two and three variable functions. Apply Fubini's Theorem to change the order of integration of an iterated integral.
- **I3: Polar, Cylindrical, and Spherical Integrals.** Use polar, cylindrical, and spherical coordinates to transform double and triple integrals. Sketch regions based on given polar, cylindrical, and spherical iterated integrals.

For problems 1-2 choose whether each statement is true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false. Be sure to neatly fill in the bubble corresponding to your answer choice.

1. (1 point) **[D1: Computing Derivatives]** There exists a continuous function $f(x, y)$ such that $f_x = -3x^3 + 4y$ and $f_y = x^2 + 5x$.

   ○ **TRUE** √ **FALSE**

2. (1 point) **[D4: Tangent Planes and Linear Approximations]** A function $f(y, z)$ is differentiable at a point (b, c) if the surface $x = f(y, z)$ has a unique tangent plane at the point $(f(b, c), b, c)$.

   √ **TRUE** ○ **FALSE**

3. (4 points) **[D2: Chain Rule]** Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}^3$ are differentiable functions such that the composition $h = g \circ f$ is well-defined. Select all true statements below.

   ○ **A)** Dh is a single-variable derivative given by the product

   $$Df(g(t))Dg(t).$$

   √ **B) Dh is a multi-variable derivative given by the product**

   $$Dg(f(x, y, z))Df(x, y, z).$$

   √ **C) The value of Dh at a given point in its domain is independent of the value of g at the corresponding point in its domain.**

   ○ **D)** The value of Dh at a given point in its domain is independent of the value of f at the corresponding point in its domain

   ○ **E)** None of the choices above are true.

4. (8 points) [**I3: Polar, Cylindrical, and Spherical Integrals**] Use polar coordinates to find

$$\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \frac{-6}{x^2+y^2+1}\, dy\, dx.$$

Solution: In polar coordinates, the region

$$0 \leq x \leq 2, \quad -\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2}$$

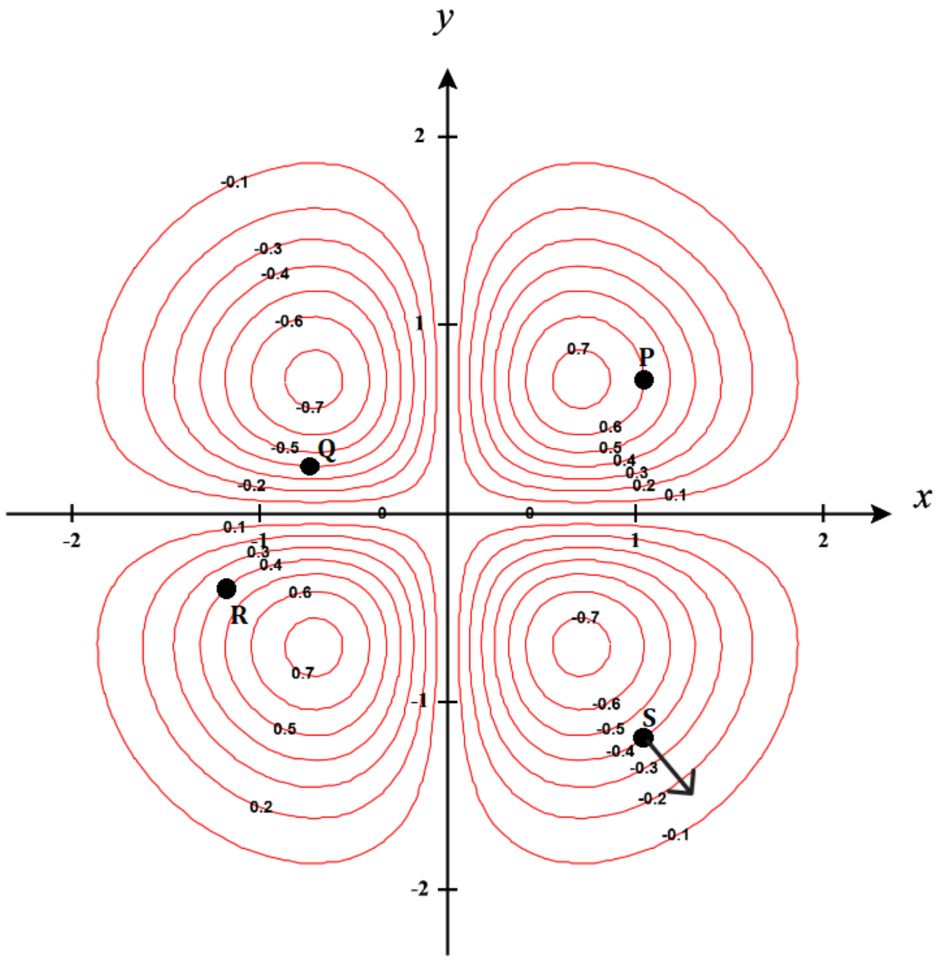
becomes

$$0 \leq r \leq 2, -\pi/2 \leq \theta \leq \pi/2,$$

since this is the right half of the disk of radius 2 centered at the origin. So we have

$$\begin{aligned} \int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \frac{-6}{x^2+y^2+1}\, dy\, dx &= \int_{-\pi/2}^{\pi/2} \int_0^2 \frac{-6}{r^2+1} r\, dr\, d\theta \\ &= \int_{-\pi/2}^{\pi/2} -3\ln(r^2+1)\Big|_0^2\, d\theta \\ &= \int_{-\pi/2}^{\pi/2} -3\ln(5)\, d\theta \\ &= -3\ln(5)\theta\Big|_{-\pi/2}^{\pi/2} \\ &= -3\pi\ln(5) \end{aligned}$$

5. [**D3: Gradients**] In this problem, you will work with the function $f : \mathbb{R}^2 \to \mathbb{R}$ whose contour plot near the origin is shown below.



(a) (1 point) Determine the sign $(+, -, 0)$ of the directional derivative at the point P in the direction towards the point $(2, 0)$. $-$

(b) (1 point) Determine the sign $(+, -, 0)$ of the directional derivative at the point Q in the x-direction. 0

(c) (1 point) Determine the sign $(+, -, 0)$ of the directional derivative at the point R in the direction of $\nabla f(R)$. $+$

(d) (2 points) Draw a vector parallel to the gradient vector of f at the point S.

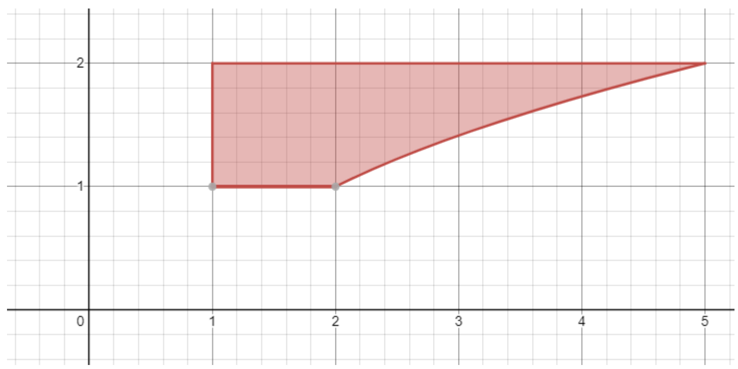
(e) (1 point) This function has a critical point at the center of the concentric contours near S. Based on the plot and the gradient at S, classify this critical point.

Solution: This critical point is a minimum since the gradient nearby points away from the critical point.

6. **[I1: Double Integrals & I2: Iterated Integrals]** In this problem you will work with the region R bounded by $x = 1 + y^2, x = 1, y = 1$, and $y = 2$ to set up a double integral

$$I = \iint_R f(x, y)\, dA.$$

(a) (3 points) Sketch this region. Clearly label the region R and find all intersection points.



Solution:

The horizontal lines intersect the parabola at $(2, 1)$ and $(5, 2)$.

(b) (2 points) Determine whether the region is horizontally simple, vertically simple, both, or neither. Explain how you determined this.

Solution: The region is only horizontally simple: arrows drawn from left to right always enter the region along the vertical line and leave along the parabola, but arrows drawn from bottom to top change behavior at $x = 2$ where they shift from entering along the horizontal line to entering along the parabola.

(c) (1 point) Based on your answer to (b), which order of integration will produce a simpler iterated integral?

Solution: Since the region is only horizontally simple, we should use the $dx\ dy$ order of integration.

(d) (4 points) Give an iterated integral expression (which may involve multiple integrals) for I using the order you chose in (c).

Solution:

$$I = \int_1^2 \int_1^{1+y^2} f(x, y)\ dx\ dy.$$

7. (10 points) **[D1: Computing Derivatives & D5: Optimization]** Find and classify all of the critical points of the function

$$f(x,y) = y^2 + x^2y + 2x^2 + 2.$$

Solution: To find the critical points, we look for points where $Df = \begin{bmatrix} 0 & 0 \end{bmatrix}$ or where Df is undefined. In this case, Df exists at all points in the domain of f since f is a polynomial, so we just solve

$$\begin{bmatrix} 2xy + 4x & 2y + x^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

From the first entry we have $2x(y+2) = 0$, so either $x = 0$ or $y = -2$. Substituting $x = 0$ into the second equation gives $2y = 0$, so $y = 0$ and we have $(0,0)$ as a critical point. Substituting $y = -2$ into the second equation gives $-4 + x^2 = 0$, so $x = \pm 2$ and we get $(2,-2)$ and $(-2,-2)$ as two more critical points.

Now we compute

$$Hf(x,y) = \begin{bmatrix} 2y+4 & 2x \\ 2x & 2 \end{bmatrix}.$$

At $(0,0)$ we have

$$\det Hf(0,0) = \begin{vmatrix} 4 & 0 \\ 0 & 2 \end{vmatrix} = 8 > 0$$

and $f_{xx} > 0$ so f has a local minimum at $(0,0)$ by the Second Derivative Test. At $(2,-2)$ we have

$$\det Hf(2,-2) = \begin{vmatrix} 0 & 4 \\ 4 & 2 \end{vmatrix} = -16 < 0$$

so f has a saddle point at $(2,-2)$ by the Second Derivative Test. At $(-2,-2)$ we have

$$\det Hf(2,-2) = \begin{vmatrix} 0 & -4 \\ -4 & 2 \end{vmatrix} = -16 < 0$$

so f also has a saddle point at $(-2,-2)$ by the Second Derivative Test.

8. **[D1: Computing Derivatives & D6: Constrained Optimization]** In this problem, you will determine the extreme values of the function $f(x, y) = y^2 - x^2$ on the elliptical region $E : x^2 + 4y^2 \leq 4$.

(a) (2 points) Find any possible extreme values of f in the interior of the region E.

> **Solution:** We solve $Df = \begin{bmatrix} 0 & 0 \end{bmatrix}$ to get the system $-2x = 0, 2y = 0$. This has a single solution $(0, 0)$ which lies inside E. We have $f(0, 0) = 0$, so 0 is a possible extreme value in the interior of E.

(b) (6 points) Use the method of Lagrange multipliers to find all possible extreme values of f on the boundary ellipse of E.

> **Solution:** Our objective function is $f(x, y) = y^2 - x^2$ and our constraint function is $g(x, y) = x^2 + 4y^2 = 4$. This results in the system
>
> $$\begin{cases} -2x = 2\lambda x \\ 2y = 8\lambda y \\ 4 = x^2 + 4y^2 \end{cases}$$
>
> Solving the first equation yields either $x = 0$ or $\lambda = -1$. Substituting $x = 0$ into the third equation gives $y = \pm 1$, so two candidate points are $(0, \pm 1)$. Substituting $\lambda = -1$ into the second equation gives $y = 0$ and substituting this into the third equation gives $x = \pm 2$. So we have two more candidate points $(\pm 2, 0)$.
>
> Finally, the value of f at $(0, \pm 1)$ is 1 and the value of f at $(\pm 2, 0)$ is -4, so the mininum of f on the boundary is -4 and the maximum of f is 1.

(c) (2 points) Determine the overall extreme values of f on the region E.

Solution: Comparing our answers from (a) and (b) we see that the overall maximum value of f on E is 1 and the overall minimum value is -4.

Bonus: (2 points) Explain what properties of the function f and the region E allow the method in the problem above to correctly find the extreme values.

Solution: Here we used the fact that f is continuous and E is closed and bounded to guarantee that our method worked, via the Extreme Value Theorem.

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FORMULA SHEET

- Total Derivative: For $\mathbf{f}(x_1, \ldots, x_n) = \langle f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n) \rangle$

$$D\mathbf{f} = \begin{bmatrix} (f_1)_{x_1} & (f_1)_{x_2} & \cdots & (f_1)_{x_n} \\ (f_2)_{x_1} & (f_2)_{x_2} & \cdots & (f_2)_{x_n} \\ \vdots & \ddots & \cdots & \vdots \\ (f_m)_{x_1} & (f_m)_{x_2} & \cdots & (f_m)_{x_n} \end{bmatrix}$$

- Linearization: Near $\mathbf{a}$, $f(\mathbf{x}) \approx L(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a})$

- Chain Rule: If $h = g(f(\mathbf{x}))$ then $Dh(\mathbf{x}) = Dg(f(\mathbf{x}))Df(\mathbf{x})$

- Implicit Differentiation: If z is implicitly given in terms of x and y by $F(x, y, z) = c$, then $\dfrac{\partial z}{\partial x} = \dfrac{-F_x}{F_z}$ and $\dfrac{\partial z}{\partial y} = \dfrac{-F_y}{F_z}$.

- Directional Derivative: If $\mathbf{u}$ is a unit vector, $D_{\mathbf{u}}f(P) = Df(P)\mathbf{u} = \nabla f(P) \cdot \mathbf{u}$

- The tangent plane to a level surface of $f(x, y, z)$ at (a, b, c) is

$$0 = \nabla f(a, b, c) \cdot \langle x - a, y - b, z - c \rangle.$$

- Hessian Matrix: For $f(x, y)$, $Hf(x, y) = \begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix}$

- Second Derivative Test: If (a, b) is a critical point of $f(x, y)$ then
  1. If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) < 0$ then f has a local maximum at (a, b)
  2. If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) > 0$ then f has a local minimum at (a, b)
  3. If $\det(Hf(a, b)) < 0$ then f has a saddle point at (a, b)
  4. If $\det(Hf(a, b)) = 0$ the test is inconclusive

- Area/volume: $\text{area}(R) = \iint_R dA$

- Trig identities: $\sin^2(x) = \frac{1}{2}(1 - \cos(2x)), \quad \cos^2(x) = \frac{1}{2}(1 + \cos(2x))$

- Average value: $f_{avg} = \dfrac{\iint_R f(x, y)dA}{\text{area of } R}$

- Polar coordinates: $x = r\cos(\theta), \quad y = r\sin(\theta), \quad dA = r\ dr\ d\theta$

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