MATH 2551-D MIDTERM 1
VERSION A
FALL 2024
COVERS SECTIONS 12.1-12.6, 13.1-13.4, 14.1-2

EXAM KEY

Honor code statement: I will abide strictly by the Georgia Tech honor code at all times. I will not use a calculator. I will not reference any website, application, or other CAS-enabled service. I will not consult with my notes or anyone during this exam. I will not provide aid to anyone else during this exam.

(   ) I attest to my integrity.

Read all instructions carefully before beginning.

- Print your name and GT ID neatly above.
- You have 75 minutes to take the exam.
- You may not use aids of any kind.
- Show your work. Answers without work shown will receive little or no credit.
- Good luck! Write yourself a message of encouragement on the front page!

Question	Points
1	1
2	1
3	2
4	3
5	3
6	10
7	10
8	10
9	10
Total:	50

Learning Outcomes

- **G1: Lines.** Describe lines using the vector equation of a line. Find the intersections of lines. Describe the relationships of lines to each other. Solve problems with lines and planes.
- **G2: Planes.** Describe planes using the general equation of a plane. Find the intersections of planes. Find the equations of planes using a point and a normal vector. Describe the relationships of planes to each other. Solve problems with lines and planes.
- **G3: Tangent Vectors to Curves.** Compute tangent vectors to parametric curves; velocity, speed, and acceleration. Find equations of tangent lines to parametric curves.
- **G4: Arc Length.** Compute the arc length of a curve in two or three dimensions. Use arc length to solve problems.
- **G5: Curvature.** Compute normal vectors and curvature for curves in two and three dimensions. Interpret these objects geometrically and in application.
- **G6: Surfaces.** Graph cylinders from equations. Describe and find traces on surfaces. Identify standard quadric surfaces including: spheres, ellipsoids, elliptic paraboloid, hyperboloids, cones, hyperbolic paraboloid. Sketch quadric surfaces. Graph functions of two variables and determine their domains and ranges. Graph level curves.
- **G7: Limits of Functions.** Calculate the limits of functions of two variables or determine if they do not exist. Apply the Squeeze Theorem for functions of two variables. Define continuity of a function of two or more independent variables.

For problems 1-2 choose whether each statement is true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false. Be sure to neatly fill in the bubble corresponding to your answer choice.

1. (1 point) [**G6: Surfaces**] The contour map for a hill and the contour map for a valley appear the same if the contours are not labeled.

   √ **TRUE** ◯ **FALSE**

2. (1 point) [**G3: Tangent Vectors to Curves**] A particle moving at constant speed must have zero acceleration.

   ◯ **TRUE** √ **FALSE**

3. (2 points) [**G3: Tangent Vectors to Curves**] If $\mathbf{r}(t)$ parameterizes a curve C in space, then $\underline{\mathbf{r}'(0)}$ gives a direction vector for the tangent line to the curve at the point $\mathbf{r}(0)$.

4. (3 points) [**G2: Planes**] Which plane given below is **not** parallel to the plane $x+2y-z=3$?

   ◯ **A)** $x+2y-z=4$
   ◯ **B)** $3x+6y-3z=3$
   √ **C) The plane through the origin which is orthogonal to** $\langle 1,2,1\rangle$
   ◯ **D)** The plane through the origin which is orthogonal to $\langle -1,-2,1\rangle$
   ◯ **E)** The plane containing the points $(3,0,0), (0,3/2,0)$, and $(0,0,-3)$.

5. (3 points) [**G5: Curvature**] Which curve has the greatest curvature at the indicated point?

   ◯ **A)** The line $\ell(t)=\langle t+1, 3t+3, 5t-5\rangle$ at the point $(2,6,0)$.
   √ **B) The circle** $x^2+y^2=1$ **at the point** $(1,0)$
   ◯ **C)** The circle $x^2+y^2=4$ at the point $(\sqrt{2},\sqrt{2})$
   ◯ **D)** A curve with constant curvature $\kappa=\frac{1}{2}$ at any point on the curve.
   ◯ **E)** No single curve above has the greatest curvature at the indicated point.

6. **[G1: Lines], [G2: Planes]** Consider the plane p with equation $-2x + y + 4z = 2$.

(a) (2 points) Give a line ℓ through the point $(1, 1, 1)$ which is orthogonal to p.

Solution: The direction vector of ℓ should be parallel to the normal vector of p for the line to be orthogonal to p. So one equation for this line is

$$\ell(t) = \langle -2, 1, 4\rangle t + \langle 1, 1, 1\rangle, t \in \mathbb{R}.$$

(b) (4 points) Give a different line through the point $(1, 1, 1)$ which is orthogonal to ℓ. Justify why your line is orthogonal to ℓ.

Solution: A line which is orthogonal to ℓ should have a direction vector which is orthogonal to $\langle -2, 1, 4\rangle$. There is a two-dimensional subspace of such vectors, spanned by $\langle 1, 2, 0\rangle$ and $\langle 0, 4, -1\rangle$. So there are many correct answers, e.g.

$$\langle 1, 2, 0\rangle t + \langle 1, 1, 1\rangle.$$

(c) (4 points) Give a third line which is skew to ℓ (it is not parallel to ℓ and does not intersect ℓ. Justify why your line is skew to ℓ.

Solution: We can produce a line skew to ℓ by taking the line from (b) (which is not parallel to ℓ) and shifting it in any direction which is different from its own direction and the direction of ℓ (this preserves the direction and forces it to no longer intersect ℓ). So, for instance, we could shift by subtracting $\langle 1, 1, 1\rangle$ to produce the new line

$$L(s) = \langle 1, 2, 0\rangle s.$$

We can then check this line does not intersect ℓ by setting components equal and trying to solve.

$$\begin{aligned} 1 - 2t &= s \\ 1 + t &= 2s \\ 1 + 4t &= 0 \end{aligned}$$

The final equation gives $t = -1/4$, which causes the first equation to become $s = 3/2$ and the second to become $s = -3/8$. This is a contradiction, so the lines do not intersect.

7. [**G4: Arc Length**] Consider the curve C which is the part of a helix between $(2,0,0)$ and $(2,0,4\pi)$ parameterized by

$$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), t\rangle, \quad 0 \le t \le 4\pi.$$

(a) (7 points) Compute the arc length function for C based at $(0,2,\pi/2)$.

Solution: The value of t corresponding to $(0,2,\pi/2)$ is $t=\pi/2$. So we have

$$\begin{aligned} s(t) &= \int_{t_0}^{t} \|\mathbf{r}'(T)\|\, dT \\ &= \int_{\pi/2}^{t} \|\langle -2\sin(T), 2\cos(T), 1\rangle\|\, dT \\ &= \int_{\pi/2}^{t} \sqrt{4\sin^2(T) + 4\cos^2(T) + 1}\, dT \\ &= \int_{\pi/2}^{t} \sqrt{5}\, dT \\ &= \sqrt{5}\left(t - \frac{\pi}{2}\right) \end{aligned}$$

(b) (3 points) Use your arc length function to compute the length of the section of the curve between $(-2,0,\pi)$ and $(0,2,5\pi/2)$. Fully simplify your answer.

Solution: These points correspond to $t_1 = \pi$ and $t_2 = 5\pi/2$ respectively. So the length of this part of the curve is

$$L = s(t_2) - s(t_1) = s(5\pi/2) - s(\pi) = \sqrt{5}(\frac{5\pi}{2} - \frac{\pi}{2}) - \sqrt{5}(\pi - \frac{\pi}{2}) = \frac{3\pi\sqrt{5}}{2}$$

8. **[G6: Surfaces]**

(a) (6 points) In this part, you will consider the surface with equation $y - x^2 - z^2 = 1$. Fill in each blank with the appropriate letter.

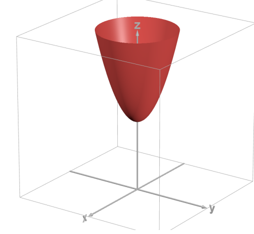
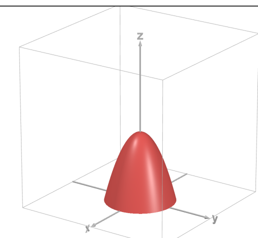
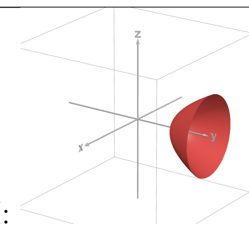
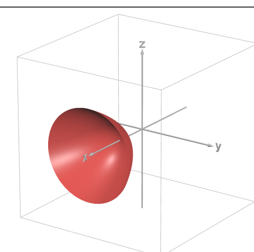
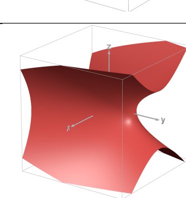
i. The cross sections in the planes $x = k$ are **C: Parabolas**

ii. The cross sections in the planes $y = k$ are **A: Ellipses/Circles**

iii. The cross sections in the planes $z = k$ are **C: Parabolas**

iv. The quadric surface is a **F: Elliptical Paraboloid**

v. The surface looks most like graph **M**

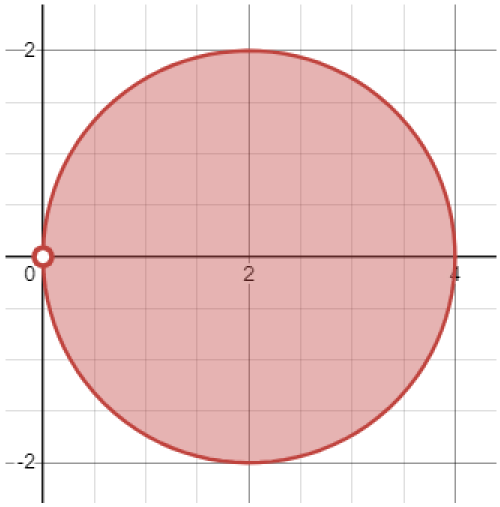
Cross Section Choices	**Quadric Surface Choices**	**Graph Choices**
A: Ellipses/Circles	E: Ellipsoid	K: 
B: Lines	F: Elliptical Paraboloid	L: 
C: Parabolas	G: Hyperbolic Paraboloid	M: 
D: Hyperbolas	H: Hyperboloid of 1 or 2 Sheets	N: 
	I: Cone	O: 

(b) (4 points) Find and sketch the domain of the function $f(x, y) = \dfrac{\sqrt{4-(x-2)^2-y^2}}{x^2e^y}$. Be sure to clearly indicate which points on the boundary are included or excluded.

Solution: The domain is all points (x, y) such that

$$4-(x-2)^2-y^2 \geq 0 \quad \text{and} \quad x^2e^y \neq 0.$$

The former gives the disk $(x-2)^2+y^2 \leq 4$ of radius 2 centered at $(2, 0)$, and the latter excludes the line $x = 0$. The only point in this disk with $x = 0$ is the origin, so we get the sketch below.



9. **[G7: Limits of Functions]**

(a) (4 points) Compute

$$\lim_{(x,y)\to(1,2)} \frac{2x-y}{2x^2+xy-y^2}$$

or show that this limit does not exist.

Solution: Evaluating the limit directly gives the indeterminate form 0/0, but we can simplify.

$$\begin{aligned}\lim_{(x,y)\to(1,2)} \frac{2x-y}{2x^2+xy-y^2} &= \lim_{(x,y)\to(1,2)} \frac{2x-y}{(2x-y)(x+y)} \\ &= \lim_{(x,y)\to(1,2)} \frac{1}{x+y} \\ &= \frac{1}{3}\end{aligned}$$

(b) (2 points) Find a value C so that the function $h(x, y) = \dfrac{3x^2}{x^2 + y^2}$ satisfies $|h(x, y)| \leq C$.

Solution: We have $0 \leq x^2 \leq x^2 + y^2$, so

$$0 \leq \frac{x^2}{x^2 + y^2} \leq 1$$

and therefore we can let $C = 3$.

(c) (4 points) Use the Squeeze Theorem to show that

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y^3}{x^2y^2 + y^4} = 0.$$

Solution: We can see that

$$\frac{3x^2y^3}{x^2y^2 + y^4} = \frac{y^3}{y^2} \cdot \frac{3x^2}{x^2 + y^2},$$

so we let $g(x, y) = \dfrac{y^3}{y^2}$ and $h(x, y) = \dfrac{3x^2}{x^2 + y^2}$ in the statement of the Squeeze Theorem. Since

$$\lim_{(x,y)\to(0,0)} \frac{y^3}{y^2} = \lim_{y\to 0} y = 0$$

and we showed in (b) that $\left|\dfrac{3x^2}{x^2 + y^2}\right| \leq 3$, we have by the Squeeze Theorem that

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y^3}{x^2y^2 + y^4} = 0.$$

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FORMULA SHEET

- Dot product: $\langle u_1, u_2, u_3\rangle \cdot \langle v_1, v_2, v_3\rangle = u_1v_1 + u_2v_2 + u_3v_3$
- Dot product magnitudes: $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos(\theta)$
- Cross product: $\langle u_1, u_2, u_3\rangle \times \langle v_1, v_2, v_3\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$
- Cross product magnitude: $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\||\sin(\theta)|$
- Arc length: $L = \int_a^b \|\mathbf{r}'(t)\|\ dt$
- Arc length function: $s(t) = \int_{t_0}^t \|\mathbf{r}'(T)\|\ dT$
- Unit tangent vector: $\mathbf{T} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{d\mathbf{r}}{ds}$
- Curvature: $\kappa = \left\|\dfrac{d\mathbf{T}}{ds}\right\| = \dfrac{1}{\|\mathbf{v}\|}\left\|\dfrac{d\mathbf{T}}{dt}\right\| = \dfrac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3}$
- Principal unit normal vector: $\mathbf{N} = \dfrac{1}{\kappa}\dfrac{d\mathbf{T}}{ds} = \dfrac{d\mathbf{T}/dt}{\|d\mathbf{T}/dt\|}$
- Two-Path Test: If there exist two different paths C_1 and C_2 through the point (a, b) along which the limit of $f(x, y)$ takes on different values, then

$$\lim_{(x,y)\to(a,b)} f(x, y)$$

  does not exist.
- Squeeze theorem: If $f(x, y) = g(x, y)h(x, y)$, where

$$\lim_{(x,y)\to(a,b)} g(x, y) = 0 \qquad \text{and} \qquad |h(x, y)| \leq C \text{ near } (a, b)$$

  for some constant $C > 0$, then

$$\lim_{(x,y)\to(a,b)} f(x, y) = 0.$$