MATH 2551-K MIDTERM 3
VERSION A
SPRING 2024
COVERS SECTIONS 15.5-15.8, 16.1-16.8

EXAM KEY

Honor code statement: I will abide strictly by the Georgia Tech honor code at all times. I will not use a calculator. I will not reference any website, application, or other CAS-enabled service. I will not consult with my notes or anyone during this exam. I will not provide aid to anyone else during this exam.

(   ) I attest to my integrity.

Read all instructions carefully before beginning.

- Print your name and GT ID neatly above.
- You have 75 minutes to take the exam.
- You may not use aids of any kind.
- Please show your work. Answers without work shown will receive little or no credit.
- Good luck! Write yourself a message of encouragement on the front page!

Question	Points
1	2
2	2
3	2
4	6
5	8
6	10
7	10
8	10
Total:	50

For problems 1-3 choose whether each statement is true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false. Be sure to neatly fill in the bubble corresponding to your answer choice.

1. (2 points) If **F** is the curl of another vector field **G**, then the flux of **F** out of any closed surface is zero.

   √ **TRUE** ◯ **FALSE**

2. (2 points) If f has continuous partial derivatives on $\mathbb{R}^3$ and C is any circle, then $\int_C \nabla f \cdot d\mathbf{r} = 0$.

   √ **TRUE** ◯ **FALSE**

3. (2 points) Every curl-free vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is conservative (i.e. if $\operatorname{curl} \mathbf{F}$ is the zero vector then $\mathbf{F} = \nabla f$ for some f).

   √ **TRUE** ◯ **FALSE**

4. (6 points) Match the volume integrals below to the regions described by filling in the blanks below. You will not use all of the integrals.

   (A) A cylinder of radius 2 and height $\pi/4$.

   (B) A cube with side lengths $2, \pi/4$ and 2π.

   (C) The smaller region between a sphere of radius 2 and a cone making an angle of $\pi/4$ with the positive z-axis.

   (I) $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^2 dy\ dx\ dz$

   (II) $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^2 r\ dr\ dz\ d\theta.$

   (III) $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^2 \rho^2\sin(\varphi)\ d\rho\ d\theta\ d\varphi.$

   (IV) $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^2 r\ dz\ dr\ d\theta.$

   (V) $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^2 \rho^2\sin(\varphi)\ d\rho\ d\varphi\ d\theta.$

(A) corresponds to (II)
(B) corresponds to (I)
(C) corresponds to (V)

5. (8 points) Let $f(x, y, z) = 8xz$. Set up but do not evaluate an iterated integral in Cartesian coordinates using any order of integration for $\iiint_D f(x, y, z)\, dV$, where D is the region

$$y^2 \leq z \leq 8 - 2x^2 - y^2, \qquad x \geq 0$$

A complete answer will include a sketch of the shadow of the region in an appropriate coordinate plane for your chosen direction of integration.

Explain in 1-2 sentences why you chose the order of integration that you did.

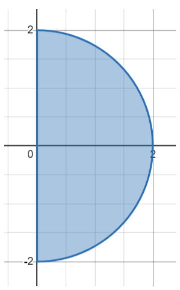
Solution: Use the order $dz\ dx\ dy$. The shadow of this region in the xy-plane is given by

$$y^2 \leq 8 - 2x^2 - y^2, \qquad x \geq 0.$$

Simplifying the first inequality yields $2x^2 + 2y^2 \leq 8$, so our shadow is

$$x^2 + y^2 \leq 4, x \geq 0,$$

which is the right half of a circle of radius 2 centered at the origin.



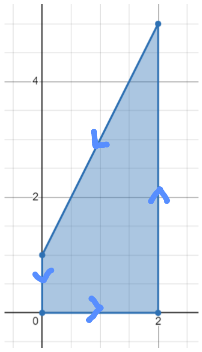
Therefore our triple iterated integral is

$$\int_{-2}^{2} \int_{0}^{\sqrt{4-y^2}} \int_{y^2}^{8-2x^2-y^2} 8xz\ dz\ dx\ dy.$$

We chose this order because both z bounds are given already, so starting with z makes sense. Then we could have chosen either order for x and y, but the bounds on x are slightly nicer (no negative square root) so we use $dx\ dy$.

6. Let C be the boundary curve of the trapezoid with vertices $(0,0),(2,0),(0,1)$, and $(2,5)$, oriented counterclockwise.

(a) (2 points) Sketch the curve C.



(b) (8 points) Compute the circulation of a fluid with velocity field

$$\mathbf{F}(x,y) = \langle e^{x^2+3x}\cos(7x), \sin(y) + \ln(2x+1)\rangle$$

around C.

Solution: We need to compute $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$. This is easiest to do using Green's Theorem, since this curve is simple and closed with positive orientation. The region is vertically simple, bounded by $0 \le y \le 2x+1$ and $0 \le x \le 2$.

$$\begin{aligned}
\int_C \mathbf{F}\cdot\mathbf{T}\,ds &= \iint_R (\nabla\times\mathbf{F})\cdot\mathbf{k}\,dA \\
&= \int_0^2\int_0^{2x+1} \frac{\partial}{\partial x}(\sin(y)+\ln(2x+1)) - \frac{\partial}{\partial y}(e^{x^2+3x}\cos(7x))\,dy\,dx \\
&= \int_0^2\int_0^{2x+1} 0 + \frac{2}{2x+1} - 0\,dy\,dx \\
&= \int_0^2 \frac{2y}{2x+1}\Big|_0^{2x+1}\,dx \\
&= \int_0^2 2\,dx \\
&= 4.
\end{aligned}$$

7. In this problem, you will compute the flux of the vector field $\mathbf{F} = z\langle x, y, 1\rangle$ across the surface S consisting of the portion of the paraboloid $z = 4 - x^2 - y^2$ with $z \geq 0$, oriented with normal vectors away from the origin.

   (a) (3 points) Find a parameterization of S.

   Hint: Consider what will make your integration in part (b) simplest.

   Solution: Typically these integrals are nicest in a parameterization based on cylindrical coordinates. For this surface, that gives

   $$\mathbf{r}(r,\theta) = \langle r\cos(\theta), r\sin(\theta), 4 - r^2\rangle, 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi.$$

   There are of course many other parameterizations that will work.

   (b) (7 points) Write and evaluate an integral expression to compute the flux of $\mathbf{F}$ across S. Fully simplify your answer.

   Solution: We need to compute a normal vector via the cross products of our partial derivatives. We have $\mathbf{r}_r = \langle \cos(\theta), \sin(\theta), -2r\rangle$ and $\mathbf{r}_\theta = \langle -r\sin(\theta), r\cos(\theta), 0\rangle$. So $\mathbf{r}_r \times \mathbf{r}_\theta = \langle 2r^2\cos(\theta), 2r^2\sin(\theta), r\cos^2(\theta) + r\sin^2(\theta)\rangle = \langle 2r^2\cos(\theta), 2r^2\sin(\theta), r\rangle$. We also have $\mathbf{F}(\mathbf{r}(r,\theta)) = (4 - r^2)\langle r\cos(\theta), r\sin(\theta), 1\rangle$.

   $$\begin{aligned}
   \text{flux} &= \iint_S \mathbf{F}\cdot\mathbf{n}\, d\sigma \\
   &= \iint_R \mathbf{F}(\mathbf{r}(r,\theta))\cdot(\mathbf{r}_r\times\mathbf{r}_\theta)\, dA \\
   &= \int_0^{2\pi}\int_0^2 (4-r^2)\langle r\cos(\theta), r\sin(\theta), 1\rangle \cdot \langle 2r^2\cos(\theta), 2r^2\sin(\theta), r\rangle\, dr\, d\theta \\
   &= \int_0^{2\pi}\int_0^2 (4-r^2)(2r^3\cos^2(\theta) + 2r^3\sin^2(\theta) + r)\, dr\, d\theta \\
   &= \int_0^{2\pi}\int_0^2 (4-r^2)(2r^3 + r)\, dr\, d\theta \\
   &= \int_0^{2\pi}\int_0^2 8r^3 + 4r - 2r^5 - r^3\, dr\, d\theta \\
   &= \int_0^{2\pi} 2r^4 + 2r^2 - \frac{1}{3}r^6 - \frac{1}{4}r^4\Big|_0^2\, d\theta \\
   &= \int_0^{2\pi} 32 + 8 - \frac{64}{3} - 4\, d\theta \\
   &= \int_0^{2\pi} \frac{44}{3}\, d\theta \\
   &= \frac{88\pi}{3}
   \end{aligned}$$

8. (a) (6 points) Let C be the curve composed of the line segment from $(0,0)$ to $(1,1)$ and the line segment from $(1,1)$ to $(1,0)$. Write an integral expression for the average value of $f(x,y) = x + y^2$ on this curve. Do not evaluate your integral expression.

Note that the length of this curve is $1+\sqrt{2}$ by geometry.

Solution: The average value of f on C is $\int_C f\ ds$ divided by the length of C. We separate C into the two given segments and parameterize them separately as

$$\mathbf{r}_1(t) = \langle t, t\rangle, \qquad \mathbf{r}_2(t) = \langle 1, 1-t\rangle,$$

both with domain $0 \le t \le 1$.
Then $\mathbf{r}_1'(t) = \langle 1, 1\rangle$, $\mathbf{r}_2'(t) = \langle 0, -1\rangle$, $f(\mathbf{r}_1(t)) = t + t^2$, and $f(\mathbf{r}_2(t)) = 1 + (1-t)^2$. So we have

$$\begin{aligned} f_{avg} &= \frac{\int_{C_1} f(\mathbf{r}_1(t))|\mathbf{r}_1'(t)|\ ds + \int_{C_2} f(\mathbf{r}_2(t))|\mathbf{r}_2'(t)|\ ds}{|C|} \\ &= \frac{\int_0^1 (t+t^2)\sqrt{2}\ dt + \int_0^1 (1+(1-t)^2)(1)\ dt}{1+\sqrt{2}} \end{aligned}$$

(b) (4 points) Find the flux of the vector field $\mathbf{F}(x,y,z) = \langle 5y, 6x, 2z\rangle$ out of the unit sphere $x^2+y^2+z^2 = 1$.

Hint: This surface is closed and the volume of this sphere is $\frac{4}{3}\pi$.

Solution: By the Divergence Theorem, we have

$$\begin{aligned} \iint_S \mathbf{F}\cdot\mathbf{n}\ d\sigma &= \iiint_D \nabla\cdot\mathbf{F}\ dV \\ &= \iiint_D 0+0+2\ dV \\ &= 2\cdot \text{volume of } D \\ &= \frac{8\pi}{3} \end{aligned}$$

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FORMULA SHEET

- Trig identities: $\sin^2(x) = \frac{1}{2}(1-\cos(2x))$, $\cos^2(x) = \frac{1}{2}(1+\cos(2x))$
- Volume$(D) = \iiint_D dV$, $\quad f_{avg} = \dfrac{\iiint_D f(x,y,z)dV}{\text{volume of } D}$ or $\dfrac{\int_C f(x,y,z)\ ds}{\text{length of } C}$,
  Mass: $M = \iiint_D \delta\ dV$
- Cylindrical coordinates: $x = r\cos(\theta)$, $\quad y = r\sin(\theta)$, $z = z$, $dV = r\ dz\ dr\ d\theta$
- Spherical coordinates: $x = \rho\sin(\phi)\cos(\theta)$, $y = \rho\sin(\phi)\sin(\theta)$, $z = \rho\cos(\phi)$, $dV = \rho^2\sin(\phi)\ d\rho\ d\phi\ d\theta$
- Substitution for double integrals: If R is the image of G under a coordinate transformation $\mathbf{T}(u,v) = \langle x(u,v), y(u,v)\rangle$ then

$$\iint_R f(x,y)\ dx\ dy = \iint_G f(\mathbf{T}(u,v))|\det D\mathbf{T}(u,v)|\ du\ dv.$$

- Scalar line integral: $\int_C f(x,y,z)\ ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|dt$
- Tangential vector line integral: $\int_C \mathbf{F}\cdot\mathbf{T}\ ds = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\ dt$
- Normal vector line integral: $\int_C \mathbf{F}(x,y)\cdot\mathbf{n}\ ds = \int_C P\ dy - Q\ dx = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\langle y'(t), -x'(t)\rangle\ dt$.
- Fundamental Theorem of Line Integrals: $\int_C \nabla f\cdot d\mathbf{r} = f(B) - f(A)$ if C is a path from A to B
- Curl (Mixed Partials) Test: $\mathbf{F} = \nabla f$ if curl $\mathbf{F} = \mathbf{0} \Leftrightarrow P_z = R_x, Q_z = R_y$, and $Q_x = P_y$.
- $\nabla = \left\langle \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z}\right\rangle$ $\qquad$ div $\mathbf{F} = \nabla\cdot\mathbf{F}$ $\qquad$ curl $\mathbf{F} = \nabla\times\mathbf{F}$
- Green's Theorem: If C is a simple closed curve with positive orientation and R is the simply-connected region it encloses, then

$$\int_C \mathbf{F}\cdot\mathbf{T}\ ds = \iint_R (\nabla\times\mathbf{F})\cdot\mathbf{k}\ dA \qquad \int_C \mathbf{F}\cdot\mathbf{n}\ ds = \iint_R (\nabla\cdot\mathbf{F})\ dA.$$

- Surface Area$=\iint_S 1\ d\sigma$
- Scalar surface integral: $\iint_S f(x,y,z)\ d\sigma = \iint_R f(\mathbf{r}(u,v))\ |\mathbf{r}_u\times\mathbf{r}_v|dA$
- Flux surface integral: $\iint_S \mathbf{F}\cdot\mathbf{n}\ d\sigma = \iint_S \mathbf{F}\cdot d\boldsymbol{\sigma} = \iint_R \mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u\times\mathbf{r}_v)\ dA$
- Stokes' Theorem: If S is a piecewise smooth oriented surface bounded by a piecewise smooth curve C and $\mathbf{F}$ is a vector field whose components have continuous partial derivatives on an open region containing S, then

$$\int_C \mathbf{F}\cdot\mathbf{T}\ ds = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\ d\sigma.$$

- Divergence Theorem: If S is a piecewise smooth closed oriented surface enclosing a volume D and $\mathbf{F}$ is a vector field whose components have continuous partial derivatives on D, then

$$\iint_S \mathbf{F}\cdot\mathbf{n}\ d\sigma = \iiint_D \nabla\cdot\mathbf{F}\ dV.$$

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