MATH 2551-K MIDTERM 3
VERSION A
SPRING 2024
COVERS SECTIONS 15.5-15.8, 16.1-16.8

Full name: ______________________________ **GT ID:** ________________

Honor code statement: I will abide strictly by the Georgia Tech honor code at all times. I will not use a calculator. I will not reference any website, application, or other CAS-enabled service. I will not consult with my notes or anyone during this exam. I will not provide aid to anyone else during this exam.

(   ) I attest to my integrity.

Read all instructions carefully before beginning.

- Print your name and GT ID neatly above.
- You have 75 minutes to take the exam.
- You may not use aids of any kind.
- Please show your work. Answers without work shown will receive little or no credit.
- Good luck! Write yourself a message of encouragement on the front page!

Question	Points
1	2
2	2
3	2
4	6
5	8
6	10
7	10
8	10
Total:	50

For problems 1-3 choose whether each statement is true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false. Be sure to neatly fill in the bubble corresponding to your answer choice.

1. (2 points) If $\mathbf{F}$ is the curl of another vector field $\mathbf{G}$, then the flux of $\mathbf{F}$ out of any closed surface is zero.

   ○ **TRUE** ○ **FALSE**

2. (2 points) If f has continuous partial derivatives on $\mathbb{R}^3$ and C is any circle, then $\int_C \nabla f \cdot d\mathbf{r} = 0$.

   ○ **TRUE** ○ **FALSE**

3. (2 points) Every curl-free vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is conservative (i.e. if $\operatorname{curl} \mathbf{F}$ is the zero vector then $\mathbf{F} = \nabla f$ for some f).

   ○ **TRUE** ○ **FALSE**

4. (6 points) Match the volume integrals below to the regions described by filling in the blanks below. You will not use all of the integrals.

   (A) A cylinder of radius 2 and height $\pi/4$.

   (B) A cube with side lengths $2, \pi/4$ and 2π.

   (C) The smaller region between a sphere of radius 2 and a cone making an angle of $\pi/4$ with the positive z-axis.

   (I) $\int_0^{2\pi} \int_0^{\pi/4} \int_0^2 dy\ dx\ dz$

   (II) $\int_0^{2\pi} \int_0^{\pi/4} \int_0^2 r\ dr\ dz\ d\theta.$

   (III) $\int_0^{2\pi} \int_0^{\pi/4} \int_0^2 \rho^2 \sin(\varphi)\, d\rho\, d\theta\, d\varphi.$

   (IV) $\int_0^{2\pi} \int_0^{\pi/4} \int_0^2 r\ dz\ dr\ d\theta.$

   (V) $\int_0^{2\pi} \int_0^{\pi/4} \int_0^2 \rho^2 \sin(\varphi)\, d\rho\, d\varphi\, d\theta.$

(A) corresponds to __________
(B) corresponds to __________
(C) corresponds to __________

5. (8 points) Let $f(x, y, z) = 8xz$. Set up but do not evaluate an iterated integral in Cartesian coordinates using any order of integration for $\iiint_D f(x, y, z)\, dV$, where D is the region

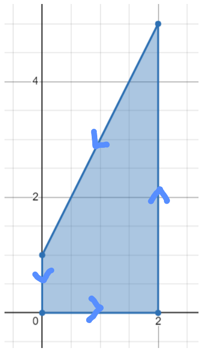
$$y^2 \le z \le 8 - 2x^2 - y^2, \qquad x \ge 0$$

A complete answer will include a sketch of the shadow of the region in an appropriate coordinate plane for your chosen direction of integration.

Explain in 1-2 sentences why you chose the order of integration that you did.

6. Let C be the boundary curve of the trapezoid with vertices $(0,0), (2,0), (0,1)$, and $(2,5)$, oriented counterclockwise.

(a) (2 points) Sketch the curve C.



(b) (8 points) Compute the circulation of a fluid with velocity field

$$\mathbf{F}(x,y) = \langle e^{x^2+3x}\cos(7x), \sin(y) + \ln(2x+1)\rangle$$

around C.

7. In this problem, you will compute the flux of the vector field $\mathbf{F} = z\langle x, y, 1\rangle$ across the surface S consisting of the portion of the paraboloid $z = 4 - x^2 - y^2$ with $z \geq 0$, oriented with normal vectors away from the origin.

   (a) (3 points) Find a parameterization of S.

   Hint: Consider what will make your integration in part (b) simplest.

   (b) (7 points) Write and evaluate an integral expression to compute the flux of $\mathbf{F}$ across S. Fully simplify your answer.

8. (a) (6 points) Let C be the curve composed of the line segment from $(0,0)$ to $(1,1)$ and the line segment from $(1,1)$ to $(1,0)$. Write an integral expression for the average value of $f(x,y) = x + y^2$ on this curve. Do not evaluate your integral expression.

   Note that the length of this curve is $1 + \sqrt{2}$ by geometry.

   (b) (4 points) Find the flux of the vector field $\mathbf{F}(x,y,z) = \langle 5y, 6x, 2z \rangle$ out of the unit sphere $x^2 + y^2 + z^2 = 1$.

   Hint: This surface is closed and the volume of this sphere is $\frac{4}{3}\pi$.

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FORMULA SHEET

- Trig identities: $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$, $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$
- Volume$(D) = \iiint_D dV$, $\quad f_{avg} = \dfrac{\iiint_D f(x,y,z)dV}{\text{volume of } D}$ or $\dfrac{\int_C f(x,y,z)\ ds}{\text{length of } C}$,
  Mass: $M = \iiint_D \delta\ dV$
- Cylindrical coordinates: $x = r\cos(\theta)$, $\quad y = r\sin(\theta)$, $z = z$, $dV = r\ dz\ dr\ d\theta$
- Spherical coordinates: $x = \rho\sin(\phi)\cos(\theta)$, $y = \rho\sin(\phi)\sin(\theta)$, $z = \rho\cos(\phi)$,
  $dV = \rho^2\sin(\phi)\ d\rho\ d\phi\ d\theta$
- Substitution for double integrals: If R is the image of G under a coordinate transformation $\mathbf{T}(u,v) = \langle x(u,v), y(u,v)\rangle$ then

$$\iint_R f(x,y)\ dx\ dy = \iint_G f(\mathbf{T}(u,v))|\det D\mathbf{T}(u,v)|\ du\ dv.$$

- Scalar line integral: $\int_C f(x,y,z)\ ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|dt$
- Tangential vector line integral: $\int_C \mathbf{F}\cdot\mathbf{T}\ ds = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\ dt$
- Normal vector line integral: $\int_C \mathbf{F}(x,y)\cdot\mathbf{n}\ ds = \int_C P\ dy - Q\ dx = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\langle y'(t), -x'(t)\rangle\ dt$.
- Fundamental Theorem of Line Integrals: $\int_C \nabla f\cdot d\mathbf{r} = f(B) - f(A)$ if C is a path from A to B
- Curl (Mixed Partials) Test: $\mathbf{F} = \nabla f$ if curl $\mathbf{F} = \mathbf{0} \Leftrightarrow P_z = R_x, Q_z = R_y$, and $Q_x = P_y$.
- $\nabla = \left\langle \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z}\right\rangle$ $\qquad$ div $\mathbf{F} = \nabla\cdot\mathbf{F}$ $\qquad$ curl $\mathbf{F} = \nabla\times\mathbf{F}$
- Green's Theorem: If C is a simple closed curve with positive orientation and R is the simply-connected region it encloses, then

$$\int_C \mathbf{F}\cdot\mathbf{T}\ ds = \iint_R (\nabla\times\mathbf{F})\cdot\mathbf{k}\ dA \qquad \int_C \mathbf{F}\cdot\mathbf{n}\ ds = \iint_R (\nabla\cdot\mathbf{F})\ dA.$$

- Surface Area$=\iint_S 1\ d\sigma$
- Scalar surface integral: $\iint_S f(x,y,z)\ d\sigma = \iint_R f(\mathbf{r}(u,v))\ |\mathbf{r}_u\times\mathbf{r}_v|dA$
- Flux surface integral: $\iint_S \mathbf{F}\cdot\mathbf{n}\ d\sigma = \iint_S \mathbf{F}\cdot d\boldsymbol{\sigma} = \iint_R \mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u\times\mathbf{r}_v)\ dA$
- Stokes' Theorem: If S is a piecewise smooth oriented surface bounded by a piecewise smooth curve C and $\mathbf{F}$ is a vector field whose components have continuous partial derivatives on an open region containing S, then

$$\int_C \mathbf{F}\cdot\mathbf{T}\ ds = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\ d\sigma.$$

- Divergence Theorem: If S is a piecewise smooth closed oriented surface enclosing a volume D and $\mathbf{F}$ is a vector field whose components have continuous partial derivatives on D, then

$$\iint_S \mathbf{F}\cdot\mathbf{n}\ d\sigma = \iiint_D \nabla\cdot\mathbf{F}\ dV.$$

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