MATH 2551-K MIDTERM 2
VERSION A
SPRING 2024
COVERS SECTIONS 14.3-14.8, 15.1-15.4

Full name: ______________________ **GT ID:** ______________

Honor code statement: I will abide strictly by the Georgia Tech honor code at all times. I will not use a calculator. I will not reference any website, application, or other CAS-enabled service. I will not consult with my notes or anyone during this exam. I will not provide aid to anyone else during this exam.

(   ) I attest to my integrity.

Read all instructions carefully before beginning.

- Print your name and GT ID neatly above.
- You have 75 minutes to take the exam.
- You may not use aids of any kind.
- Show your work. Answers without work shown will receive little or no credit.
- Good luck! Write yourself a message of encouragement on the front page!

Question	Points
1	2
2	2
3	2
4	4
5	4
6	10
7	10
8	6
9	10
Total:	50

For problems 1-2 choose whether each statement is true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false. Be sure to neatly fill in the bubble corresponding to your answer choice.

1. (2 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$. If f and all of its first and second partial derivatives are continuous, then $f_{xy} = f_{yx}$.

   $\surd$ **TRUE** ○ **FALSE**

2. (2 points) Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is differentiable at (a, b, c) with $f(a, b, c) = d$. Then the equation of the linearization $L(x, y, z)$ of f at the point (a, b, c) is also the equation of the plane tangent to the level surface $f(x, y, z) = d$ at the point (a, b, c).

   ○ **TRUE** $\surd$ **FALSE**

3. (2 points) The area of a small town is less than 3 square kilometers. If the average population density is 140 people per square kilometer, explain why the total population of the town cannot possibly be 500.

   Your answer does not need to include a derivative or integral computation, but should relate to at least one idea we discussed in this unit. Your answer should include at least one complete sentence.

   Solution: We know that the total population is the average population density times the area. Since this quantity is less than $140 \times 3 = 420$, the total population cannot be 500.

4. (4 points) Suppose $f_x = 2x$ and $f_y = 3y^2 - 3$ with critical points $(0, -1)$ and $(0, 1)$. Classify each critical point.

Solution: We have $Hf(x, y) = \begin{bmatrix} 2 & 0 \\ 0 & 6y \end{bmatrix}$.

At $(0, -1)$, $\det Hf(0, -1) = \begin{vmatrix} 2 & 0 \\ 0 & -6 \end{vmatrix} = -12 < 0$, so by the Second Derivative Test f has a saddle point at $(0, -1)$.

At $(0, 1)$, $\det Hf(0, 1) = \begin{vmatrix} 2 & 0 \\ 0 & 6 \end{vmatrix} = 12 > 0$ and $f_{xx}(0, 1) = 2 > 0$, so by the Second Derivative Test f has a local minimum at $(0, 1)$.

5. (4 points) Convert the integral $\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} x^2 + y^2 \, dy \, dx$ to an equivalent integral in polar coordinates.

Solution: We have $-\sqrt{9 - x^2} \leq y \leq \sqrt{9 - x^2}$ and $0 \leq x \leq 3$, so this region of integration is the right half of a circle of radius 3 centered at the origin.

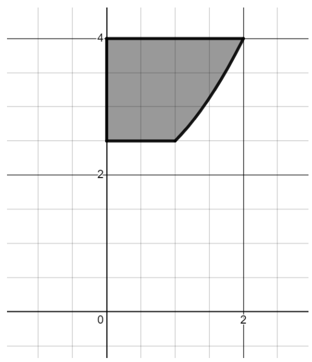
Thus the integral becomes

$$\int_{-\pi/2}^{\pi/2} \int_0^3 (r^2) \, r \, dr \, d\theta.$$

6. In this problem, you will compute the area of the region R contained between the curves $y = 5/2$, $y = 4$, $x = 0$, and $y = \frac{1}{2}x^2 + 2$.

   (a) (2 points) Sketch the region R. Label your axes and each curve.

   Solution:



   (b) (2 points) Is R horizontally simple, vertically simple, both, or neither?

   Solution: R is only horizontally simple.

   (c) (6 points) Write a double iterated integral to find the area of R, then evaluate it.

   Solution: The simpler order of integration is $dx\ dy$, resulting in

$$\int_{5/2}^{4}\int_{0}^{\sqrt{2y-4}} 1\ dx\ dy.$$

   The other order requires splitting the integral at $x = 1$:

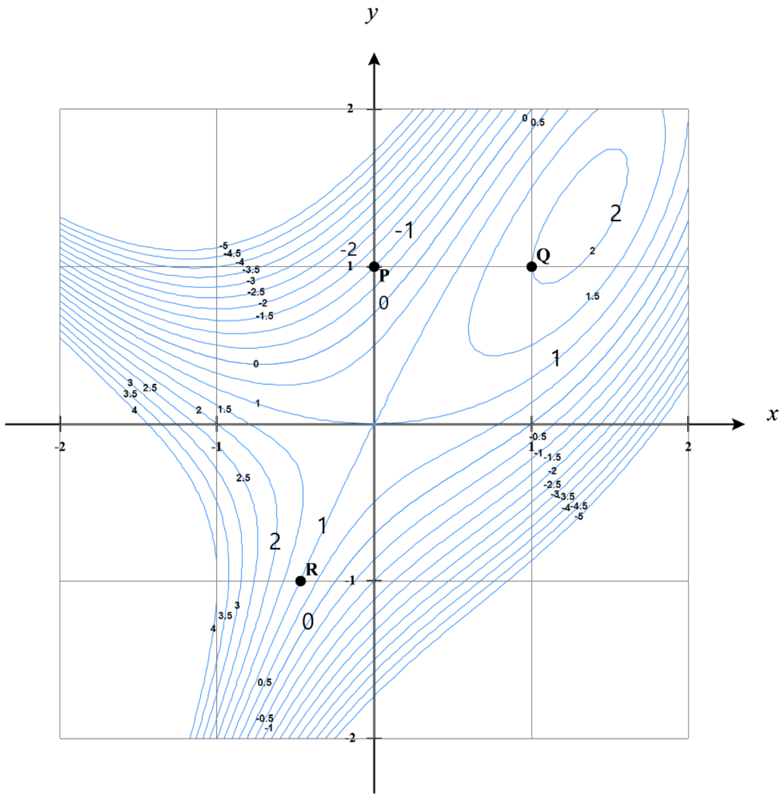
$$\int_{0}^{1}\int_{5/2}^{4} 1\ dy\ dx + \int_{1}^{2}\int_{\frac{1}{2}x^2+2}^{4} 1\ dy\ dx.$$

   Evaluating the first integral by letting $u = 2y - 4$, we have

$$\begin{aligned}\int_{5/2}^{4}\int_{0}^{\sqrt{2y-4}} 1\ dx\ dy &= \int_{5/2}^{4} \sqrt{2y-4}\ dx\ dy \\ &= \int_{1}^{4} \frac{1}{2}\sqrt{u}\ du \\ &= \frac{1}{3}u^{3/2}|_1^4 \\ &= \frac{7}{3}\end{aligned}$$

7. In this problem, you will work with the function $g : \mathbb{R}^2 \to \mathbb{R}$, whose contour plot is given below.

$$g(x, y) = -x^3 + 4xy - 2y^2 + 1.$$



In parts (a)-(c), write the sign $(+, -, 0)$ of the directional derivative at the given point in the given direction in the answer box provided. You do not need to show work for these parts.

(a) (2 points) At the point P in the direction $\langle 1, -1\rangle$: $\boxed{+}$

(b) (2 points) At the point Q in the direction $\langle -1, 0\rangle$: $\boxed{-}$

(c) (2 points) At the point R in a direction orthogonal to $\nabla g(R)$: $\boxed{0}$

(d) (4 points) Let $\mathbf{u} = \langle 4/5, 3/5\rangle$. Compute $D_{\mathbf{u}}g(1, 0)$.

Solution: We have $\nabla g(x, y) = \langle -3x^2 + 4y, 4x - 4y\rangle$, so $\nabla g(1, 0) = \langle -3, 4\rangle$. Then

$$D_{\mathbf{u}}g(1, 0) = \nabla g(1, 0) \cdot \mathbf{u} = \langle -3, 4\rangle \cdot \langle 4/5, 3/5\rangle = -12/5 + 12/5 = 0.$$

8. (6 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function $f(x, y) = x^2y + xy^2$ and $\mathbf{r}(s, t) : \mathbb{R}^2 \to \mathbb{R}^2$ be the function

$$\mathbf{r}(s,t) = \begin{bmatrix} 3s+t \\ 5st \end{bmatrix}.$$

Use the Chain Rule to compute the derivative (either the total derivative or all relevant partial derivatives) of the composite function $g : \mathbb{R}^2 \to \mathbb{R}$ where $g(s, t) = f(\mathbf{r}(s, t))$ at the point $(s, t) = (0, 1)$.

To receive full credit, your answer must either include total derivatives or a tree/chain diagram showing the relationships among the variables.

Solution: To apply the Chain Rule, we need to compute the total derivatives of both functions.

$$Df = \begin{bmatrix} 2xy + y^2 & x^2 + 2xy \end{bmatrix} \qquad D\mathbf{r} = \begin{bmatrix} 3 & 1 \\ 5t & 5s \end{bmatrix}$$

Now we evaluate Df at $\mathbf{r}(0, 1)$ and $D\mathbf{r}$ at $(0, 1)$.

$$\mathbf{r}(0,1) = \langle 1, 0 \rangle, \qquad Df(\mathbf{r}(1,0)) = \begin{bmatrix} 0 & 1 \end{bmatrix} \qquad D\mathbf{r}(0,1) = \begin{bmatrix} 3 & 1 \\ 5 & 0 \end{bmatrix}$$

Thus

$$Dg(0,1) = Df(\mathbf{r}(0,1))D\mathbf{r}(0,1) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 0 \end{bmatrix},$$

i.e. $g_s(0, 1) = 5$ and $g_t(0, 1) = 0$.

9. (10 points) Determine the largest value of the function $f(x, y) = e^{xy}$ such that $x^3+y^3 = 16$.

Solution: We can answer this problem using the method of Lagrange multipliers. Our objective function is $f(x, y) = e^{xy}$ and our constraint is $g(x, y) = x^3 + y^3 = 16$. We have $\nabla f = \langle ye^{xy}, xe^{xy}\rangle$ and $\nabla g = \langle 3x^2, 3y^2\rangle$. Equating $\nabla f = \lambda\nabla g$, we get the system of equations

$$\begin{cases} ye^{xy} = 3x^2\lambda \\ xe^{xy} = 3y^2\lambda \\ x^3 + y^3 = 16. \end{cases}$$

The easiest variable to isolate is λ. Doing so in the first equation yields two cases: either $x = 0$ or $\lambda = \dfrac{ye^{xy}}{3x^2}$.

<u>Case 1: $x = 0$</u>. From equation 1, we also have $ye^0 = 0$, i.e. $y = 0$. But then equation 3 is false: $0^3 + 0^3 \neq 16$. So this is impossible.

<u>Case 2: $\lambda = \dfrac{ye^{xy}}{3x^2}$</u>. Substitution into equation 2 gives

$$xe^{xy} = 3y^2\frac{ye^{xy}}{3x^2}.$$

Simplifying, dividing by e^{xy} (always nonzero), and multiplying by x^2 to clear the fractions gives $x^3 = y^3$ or $x = y$. So equation 3 becomes $2x^3 = 16$, i.e. $x^3 = 8$, i.e. $x = 2$. Then $y = 2$ also and this is the only solution point.

The largest value of $f(x, y)$ subject to this constraint is therefore $f(2, 2) = e^4$.

Bonus: (2 points) Provide a complete mathematical justification for why $f(x, y)$ does not attain a minimum value in the problem above.

Solution: f cannot attain a minimum value subject to this constraint because as $x \to -\infty$ along the constraint, $y \to \infty$ and $xy \to -\infty$, so f approaches (but does not reach) 0. Hence there is no minimum value.

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FORMULA SHEET

- Total Derivative: For $\mathbf{f}(x_1, \ldots, x_n) = \langle f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n) \rangle$

$$D\mathbf{f} = \begin{bmatrix} (f_1)_{x_1} & (f_1)_{x_2} & \cdots & (f_1)_{x_n} \\ (f_2)_{x_1} & (f_2)_{x_2} & \cdots & (f_2)_{x_n} \\ \vdots & \ddots & \cdots & \vdots \\ (f_m)_{x_1} & (f_m)_{x_2} & \cdots & (f_m)_{x_n} \end{bmatrix}$$

- Linearization: Near $\mathbf{a}$, $f(\mathbf{x}) \approx L(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a})$

- Chain Rule: If $h = g(f(\mathbf{x}))$ then $Dh(\mathbf{x}) = Dg(f(\mathbf{x}))Df(\mathbf{x})$

- Implicit Differentiation: If z is implicitly given in terms of x and y by $F(x, y, z) = c$, then $\dfrac{\partial z}{\partial x} = \dfrac{-F_x}{F_z}$ and $\dfrac{\partial z}{\partial y} = \dfrac{-F_y}{F_z}$.

- Directional Derivative: If $\mathbf{u}$ is a unit vector, $D_{\mathbf{u}}f(P) = Df(P)\mathbf{u} = \nabla f(P) \cdot \mathbf{u}$

- The tangent line to a level curve of $f(x, y)$ at (a, b) is $0 = \nabla f(a, b) \cdot \langle x - a, y - b \rangle$

- The tangent plane to a level surface of $f(x, y, z)$ at (a, b, c) is

$$0 = \nabla f(a, b, c) \cdot \langle x - a, y - b, z - c \rangle.$$

- Hessian Matrix: For $f(x, y)$, $Hf(x, y) = \begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix}$

- Second Derivative Test: If (a, b) is a critical point of $f(x, y)$ then
  1. If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) < 0$ then f has a local maximum at (a, b)
  2. If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) > 0$ then f has a local minimum at (a, b)
  3. If $\det(Hf(a, b)) < 0$ then f has a saddle point at (a, b)
  4. If $\det(Hf(a, b)) = 0$ the test is inconclusive

- Area/volume: $\text{area}(R) = \iint_R dA$

- Trig identities: $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$, $\quad \cos^2(x) = \frac{1}{2}(1 + \cos(2x))$

- Average value: $f_{avg} = \dfrac{\iint_R f(x, y)dA}{\text{area of } R}$

- Polar coordinates: $x = r\cos(\theta)$, $\quad y = r\sin(\theta)$, $\quad dA = r\ dr\ d\theta$

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