MATH 2551-K MIDTERM 1
VERSION A
SPRING 2024
COVERS SECTIONS 12.1-12.6, 13.1-13.4, 14.1-14.2

Full name: ________________________________ **GT ID:** ____________________

Honor code statement: I will abide strictly by the Georgia Tech honor code at all times. I will not use a calculator. I will not reference any website, application, or other CAS-enabled service. I will not consult with my notes or anyone during this exam. I will not provide aid to anyone else during this exam.

(   ) I am initialing to the left to attest to my integrity.

Read all instructions carefully before beginning.

- Print your name and GT ID neatly above.
- You have 75 minutes to take the exam.
- You may not use aids of any kind.
- Please show your work.
- Good luck! Write yourself a message of encouragement on the front page!

Question	Points
1	2
2	2
3	2
4	4
5	10
6	10
7	10
8	10
Total:	50

For problems 1-2 choose whether each statement is true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false. Be sure to neatly fill in the bubble corresponding to your answer choice.

1. (2 points) If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in $\mathbb{R}^3$, then $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$.

   √ **TRUE** ○ **FALSE**

2. (2 points) The domain of the function $f(x, y) = \sqrt{y-3} + xe^x$ is $[3, \infty)$.

   ○ **TRUE** √ **FALSE**

3. (2 points) A hummingbird with velocity vector $\mathbf{r}'(t)$ starts flying from $(2, 1, -3)$ at time $t = 0$ and flies around for 5 seconds. Where is the hummingbird located at time $t = 5$ if $\int_0^5 \mathbf{r}'(t)\,dt = 0$? You do not need to show your work on this problem.

Solution: The given integral is the hummingbird's displacement over this time interval, so its final position is the same as its initial position: $(2, 1, -3)$.

4. (4 points) Find the value of the limit below or show that it does not exist. To receive full credit, your answer must show work supporting your answer and if you show it does not exist, mention the test that you are using.

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$$

Solution: This limit does not exist. To see this, we can compute the limit along a straight line $y = mx$ through the origin. Along such a line, points have the form (x, mx), so the limit becomes

$$\lim_{(x,mx)\to(0,0)} f(x, y) = \lim_{x\to 0} \frac{x(mx)}{x^2 + (mx)^2} = \lim_{x\to 0} \frac{mx^2}{(1 + m^2)x^2} = \frac{m}{1 + m^2}.$$

Since this expression depends on the slope m, we will get different values for the limit along different lines through the origin and thus by the **Two-Path Test** the limit does not exist.

5. In this problem, you will work with the vector-valued function $\mathbf{r}(t) = \langle \sqrt{2}t, e^t, e^{-t} \rangle$, defined for $t \in \mathbb{R}$.

   (a) (2 points) Find the two values of t where $\mathbf{r}(t)$ is $\langle 0, 1, 1 \rangle$ and $2\sqrt{2}\mathbf{i} + e^2\mathbf{j} + \frac{1}{e^2}\mathbf{k}$, respectively.

   Solution: We have $\mathbf{r}(0) = \langle 0, 1, 1 \rangle$ and $\mathbf{r}(2) = 2\sqrt{2}\mathbf{i} + e^2\mathbf{j} + \frac{1}{e^2}\mathbf{k}$.

   (b) (8 points) Find the length of the curve $\mathbf{r}(t)$ between $(0, 1, 1)$ and $(2\sqrt{2}, e^2, \frac{1}{e^2})$. Show all of your work and fully simplify your final answer.

   Solution: By part (a), this is $\int_0^2 |\mathbf{r}'(t)|\, dt$.

$$
\begin{aligned}
L(t) &= \int_0^2 |\mathbf{r}'(t)|\, dt \\
&= \int_0^2 |\langle \sqrt{2}, e^t, e^{-t} \rangle|\, dt \\
&= \int_0^2 \sqrt{2 + e^{2t} + e^{-2t}}\, dt \\
&= \int_0^2 \sqrt{(e^t + e^{-t})^2}\, dt \\
&= \int_0^2 e^t + e^{-t}\, dt \\
&= e^t - e^{-t}|_0^2 \\
&= e^2 - e^{-2} - (1 - 1) \\
&= e^2 - \frac{1}{e^2}.
\end{aligned}
$$

6. In this problem, you will work with the spiral plane curve

$$\mathbf{r}(t) = \langle \cos(t) + t\sin(t), \sin(t) - t\cos(t) \rangle$$

for $t > 0$.

(a) (5 points) Compute the unit tangent vector $\mathbf{T}(t)$.

Solution: We have $\mathbf{r}'(t) = \langle -\sin(t) + \sin(t) + t\cos(t), \cos(t) - \cos(t) + t\sin(t) \rangle = t\langle \cos(t), \sin(t) \rangle$. Thus $|\mathbf{r}'(t)| = t\sqrt{\cos^2(t) + \sin^2(t)} = t$ and we have

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \langle \cos(t), \sin(t) \rangle.$$

(b) (3 points) Compute the principal unit normal vector $\mathbf{N}(t)$.

Solution: We have $\mathbf{T}'(t) = \langle -\sin(t), \cos(t) \rangle$ and $|\mathbf{T}'(t)| = \sqrt{\sin^2(t) + \cos^2(t)} = 1$, so

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \langle -\sin(t), \cos(t) \rangle.$$

(c) (2 points) Compute the curvature $\kappa(t)$.

Solution: Using our calculation in (b), we have

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{1}{t}.$$

7. Let L_1 be the line parameterized by $\mathbf{r}_1(t) = \langle 1, 2, 3\rangle t + \langle 1, -4, -4\rangle$ and L_2 be the line parameterized by $\mathbf{r}_2(p) = \langle 1, 1, 1\rangle + p\langle 2, -1, 1\rangle$.

   (a) (4 points) Show that these lines are not skew (i.e. they are either parallel or intersect).

   Solution: The lines are not parallel because their direction vectors $\langle 1, 2, 3\rangle$ and $\langle 2, -1, 1\rangle$ are not scalar multiples of each other.

   To see that they intersect, we first set up the linear system resulting from equating components:

$$\begin{aligned} t + 1 &= 2p + 1 \\ 2t - 4 &= 1 - p \\ 3t - 4 &= p + 1 \end{aligned}$$

   From the first equation, we have $t = 2p$, and substituting this into the other equations gives $p = 1$ for both. So the system is consistent and there is a point where the lines meet, i.e. when $p = 1$ and $t = 2$, which result in the common point $(3, 0, 2)$.

   (b) (5 points) Find an equation for the unique plane which contains both lines. Give your answer in the form $Ax + By + Cz = D$.

   Solution: A plane containing both lines will have a normal vector orthogonal to both direction vectors. So we compute $\mathbf{n} = \langle 1, 2, 3\rangle \times \langle 2, -1, 1\rangle = (2 + 3)\mathbf{i} - (1 - 6)\mathbf{j} + (-1 - 4)\mathbf{k} = \langle 5, 5, -5\rangle$.
   The plane also contains all points on both lines, so in particular the point $(1, 1, 1)$. Thus an equation for the plane is

$$5(x - 1) + 5(y - 1) - 5(z - 1) = 0.$$

   Rewriting in the requested form we get

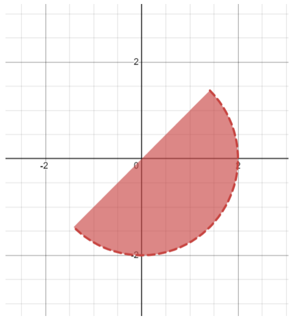
$$5x + 5y - 5z = 5 \text{ or } x + y - z = 1.$$

   (c) (1 point) Why is part (a) necessary for the problem? In other words, why would part (b) be impossible if the lines were skew?

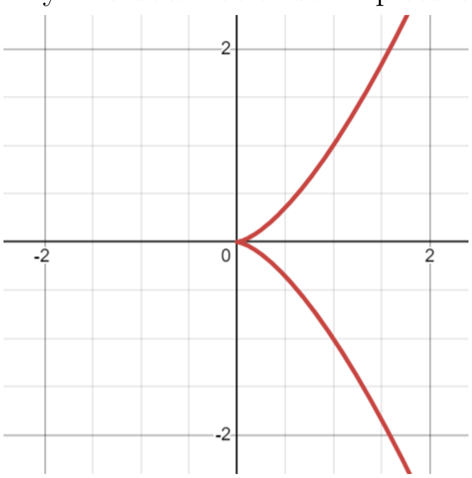
   Solution: Within any plane, any two lines fully contained in that plane are either parallel or intersect. Therefore, if two lines are neither parallel nor intersect, they cannot be contained in the same plane.

8. (a) (6 points) Let $f(x,y) = \dfrac{\ln(x-y)}{\sqrt{4-x^2-y^2}}$. Graph the domain of f on the provided axes below. Be sure to show all your work in finding the domain. Indicate whether or not each part of the boundary of the domain is included.

Solution: We need $x - y > 0$ and $4 - x^2 - y^2 > 0$ so that all functions are well-defined and we do not divide by zero. Thus the domain is all points (x, y) which are both to the right/below $y = x$ and inside the circle $x^2 + y^2 = 4$. No boundary points are included.



(b) (4 points) Compute the tangent line for the curve $\mathbf{r}(t) = \langle t^2, t^3 \rangle$ at $t = 0$ or explain why this does not exist. A picture of the curve is below.



Solution: The tangent line at the origin does not exist, since the curve is not smooth at that point. There are several ways to see this: for example, because $\mathbf{r}'(t) = \langle 0, 0 \rangle$, the tangent vector at the origin does not exist. We can also observe that the direction of the tangent vector abruptly reverses as $t \to 0$: when $t < 0$, the tangent vector is approaching $-\mathbf{i}$, but when $t > 0$ it is approaching $\mathbf{i}$.

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FORMULA SHEET

- $\langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3$
- $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos(\theta)$
- $\langle u_1, u_2, u_3 \rangle \times \langle v_1, v_2, v_3 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$
- $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}||\sin(\theta)|$
- $L = \int_a^b |\mathbf{r}'(t)| \; dt$
- $s(t) = \int_{t_0}^t |\mathbf{r}'(T)| \; dT$
- $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{d\mathbf{r}}{ds}$
- $\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$
- $\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$

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