MATH 2551-K MIDTERM 3
VERSION A
FALL 2023
COVERS SECTIONS 15.5-15.8, 16.1-16.8

Full name: ______________________________ **GT ID:**____________________

Honor code statement: I will abide strictly by the Georgia Tech honor code at all times. I will not use a calculator. I will not reference any website, application, or other CAS-enabled service. I will not consult with my notes or anyone during this exam. I will not provide aid to anyone else during this exam.

(   ) I attest to my integrity.

Read all instructions carefully before beginning.

- Print your name and GT ID neatly above.
- You have 75 minutes to take the exam.
- You may not use aids of any kind.
- Please show your work.
- Good luck! Write yourself a message of encouragement on the front page!

Question	Points
1	2
2	2
3	2
4	4
5	8
6	12
7	8
8	12
Total:	50

For problems 1-3 choose whether each statement is true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false. Be sure to neatly fill in the bubble corresponding to your answer choice.

1. (2 points) Suppose $\mathbf{F}(x, y)$ is a curl-free vector field ($\nabla \times \mathbf{F} = \mathbf{0}$) defined in all of $\mathbb{R}^2$ except $(0, 0)$. Then $\mathbf{F}$ is conservative.

   ◯ **TRUE** √ **FALSE**

2. (2 points) There exists a non-constant vector field $\mathbf{F}(x, y)$ with both $\nabla \cdot \mathbf{F} = 0$ and $\nabla \times \mathbf{F} = \mathbf{0}$.

   √ **TRUE** ◯ **FALSE**

3. (2 points) If $\mathbf{F}$ is any vector field and C is a circle, then the circulation of $\mathbf{F}$ around C traversed clockwise is the negative of the circulation of $\mathbf{F}$ around C traversed counterclockwise.

   √ **TRUE** ◯ **FALSE**

4. (4 points) Consider the double integral

$$\iint_R \frac{2x - y}{3y - x}\, dA,$$

where R is the region bounded by the lines $2x - y = 0$, $2x - y = 4$, $3y - x = 0$, $3y - x = 2$. Which of the following coordinate transformations would be appropriate to use for change of variables on this integral?

◯ $u = \dfrac{y}{x}$, $v = \dfrac{x}{3y}$

◯ $x = u$, $y = v$

√ $u = 2x - y$, $v = 3y - x$

◯ $x = 2u - v$, $y = 3v - y$

Solution: Choice three is correct, $u = 2x - y, v = 3y - x$. Applying this transformation results in a region of integration $0 \leq u \leq 4, 0 \leq v \leq 2$ and an integrand of $\frac{u}{v}$ multiplied by the Jacobian (a constant). Choice one does not map either the integrand or the region to a nice object, choice two does nothing but rename the variables, and choice four has x, y swapped with u, v, so also works poorly.

5. (8 points) Compute the flux of the vector field $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j}$ through the open-ended circular cylinder of radius 4 and height 7 with its base on the xy-plane and centerered about the z-axis, oriented away from the z-axis.

Solution: We parameterize the cylinder as $\mathbf{r}(\theta, z) = \langle 4\cos(\theta), 4\sin(\theta), z\rangle$, $0 \le \theta \le 2\pi, 0 \le z \le 7$.

We then have

$$\mathbf{r}_\theta = \langle -4\sin(\theta), 4\cos(\theta), 0\rangle, \mathbf{r}_z = \langle 0, 0, 1\rangle \text{ and } \mathbf{r}_\theta \times \mathbf{r}_z = \langle 4\cos(\theta), 4\sin(\theta), 0\rangle.$$

We can now compute the flux $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$:

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma &= \iint_R \mathbf{F}(\mathbf{r}(\theta, z)) \cdot (\mathbf{r}_\theta \times \mathbf{r}_z)\, dA \\
&= \int_0^{2\pi} \int_0^7 \langle 4\cos(\theta), 4\sin(\theta), 0\rangle \cdot \langle 4\cos(\theta), 4\sin(\theta), 0\rangle\, dz\, d\theta \\
&= \int_0^{2\pi} \int_0^7 16\cos^2(\theta) + 16\sin^2(\theta)\, dz\, d\theta \\
&= (2\pi)(7)(16) \\
&= 224\pi
\end{aligned}$$

6. In this problem, you will work with the path C which is the ellipse $\frac{x^2}{2^2} + \frac{y^2}{5^2} = 1$ oriented counterclockwise.

   (a) (3 points) Give a parameterization $\mathbf{r}(t)$ of the ellipse.

   Solution: $\mathbf{r}(t) = \langle 2\cos(t), 5\sin(t)\rangle$, $0 \le t \le 2\pi$ is one possible parameterization.

   (b) (4 points) Use any method to find the flux of the vector field $\mathbf{F} = \langle e^y, 2\cos(x)\rangle$ out of the interior of C.

   Solution: Since C is a closed curve, we can apply Green's theorem.

$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_R \nabla\cdot\mathbf{F}\,dA = \iint_R \frac{\partial}{\partial x}(e^y) + \frac{\partial}{\partial y}(2\cos(x))\,dA = \iint_R 0\,dA = 0.$$

   (c) (5 points) Use any method to find the circulation of the vector field $\mathbf{G} = \langle 2y, x\rangle$ around C.

   Solution: The easiest solution is to compute directly. We have $\mathbf{r}'(t) = \langle -2\sin(t), 5\cos(t)\rangle$.
   Thus

$$\begin{aligned}
\int_C \mathbf{G}\cdot\mathbf{T}\,ds &= \int_C \mathbf{G}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt \\
&= \int_0^{2\pi} \langle 10\sin(t), 2\cos(t)\rangle\cdot\langle -2\sin(t), 5\cos(t)\rangle\,dt \\
&= \int_0^{2\pi} -20\sin^2(t) + 10\cos^2(t)\,dt \\
&= \int_0^{2\pi} -10 + 10\cos(2t) + 5 + 5\cos(2t)\,dt \\
&= -5t + \frac{15}{2}\sin(2t)\Big|_0^{2\pi} \\
&= -10\pi
\end{aligned}$$

7. Let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x, y, z) = \langle 3x + e^{z^2 y}, y + z, \arctan(x) - z \rangle.$$

(a) (6 points) Let S be the surface defined by

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{6}\right)^2 + \left(\frac{z}{2}\right)^2 = 1,$$

oriented with outward pointing normal vectors. Compute the flux of $\mathbf{F}$ across S.

Hint: The volume of the ellipsoid $(\frac{x}{a})^2 + (\frac{y}{b})^2 + (\frac{z}{c})^2 \leq 1$ *is* $\frac{4}{3}\pi abc$.

Solution: Since S is a closed surface, we can apply the Divergence Theorem. $\nabla \cdot \mathbf{F} = 3 + 1 - 1 = 3$, so we have

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma &= \iiint_V \nabla \cdot \mathbf{F}\, dV \\ &= \iiint_V 3\, dV \\ &= 3V_{ellipsoid} \\ &= 3(\frac{4}{3}\pi)(3)(6)(2) \\ &= 144\pi \end{aligned}$$

(b) (2 points) Is there a vector field $\mathbf{G}$ such that $\mathbf{F} = \nabla \times \mathbf{G}$? Explain your answer.

Solution: No, since $\nabla \cdot \mathbf{F} = 3 \neq 0$, $\mathbf{F}$ cannot be a curl.

8. Consider the volume D in the second octant of $\mathbb{R}^3$ $(x \leq 0, y \geq 0, z \geq 0)$ which is bounded above by the sphere $x^2 + y^2 + z^2 = 2$ and below by the paraboloid $z = x^2 + y^2$.

(a) (4 points) Write an integral for the volume of D using Cartesian coordinates in the order $dz\ dy\ dx$. Show your work. **Do not evaluate your integral**.

Solution: These two surfaces intersect when $z + z^2 = 2$ and $z \geq 0$, i.e. $z = 1$. When $z = 1$, they meet in the circle $x^2 + y^2 = 1$. Therefore since we are taking only the portion of this volume in the second octant we have

$$V = \int_{-1}^{0} \int_{0}^{\sqrt{1-x^2}} \int_{x^2+y^2}^{\sqrt{2-x^2-y^2}} dz\ dy\ dx.$$

(b) (4 points) Write an integral for the volume of D using cylindrical coordinates. Show your work. **Do not evaluate your integral**.

Solution: In cylindrical coordinates, the surfaces are $r^2 + z^2 = 2$ and $z = r^2$, meeting in the circle $r = 1$ in the plane $z = 1$. Now θ runs from $\pi/2$ to π so that we get the portion in the second octant.

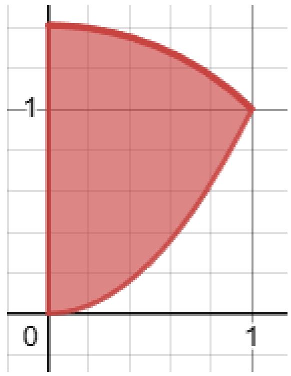
So we have

$$V = \int_{\pi/2}^{\pi} \int_{0}^{1} \int_{r^2}^{\sqrt{2-r^2}} r\ dz\ dr\ d\theta.$$

Consider the volume D in the second octant ($x \leq 0, y \geq 0, z \geq 0$) of $\mathbb{R}^3$ which is bounded above by the sphere $x^2+y^2+z^2=2$ and below by the paraboloid $z=x^2+y^2$.

(c) (2 points) Sketch the shadow of D in the yz-plane.

Solution:



(d) (2 points) Are spherical coordinates a good choice to use to compute this volume? Justify your answer.

Solution: They are not, because this region is not simple in the ρ-direction. This can be seen in the picture in part (c); a ray outward from the origin cuts through the sphere for $0 \leq \phi \leq \pi/4$ and through the paraboloid for $\pi/4 \leq \phi \leq \pi/2$.

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FORMULA SHEET

- Trig identities: $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$, $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$
- Volume$(D) = \iiint_D dV$, $\quad f_{avg} = \dfrac{\iiint_D f(x,y,z)dV}{\text{volume of } D}$, $\quad$ Mass: $M = \iiint_D \delta \, dV$
- Cylindrical coordinates: $x = r\cos(\theta)$, $\quad y = r\sin(\theta)$, $z = z$, $dV = r \, dz \, dr \, d\theta$
- Spherical coordinates: $x = \rho\sin(\phi)\cos(\theta)$, $y = \rho\sin(\phi)\sin(\theta)$, $z = \rho\cos(\phi)$, $dV = \rho^2\sin(\phi) \, d\rho \, d\phi \, d\theta$
- First moments (3D solid): $M_{yz} = \iiint_D x\delta \, dV$, $M_{xz} = \iiint_D y\delta \, dV$, $M_{xy} = \iiint_D z\delta \, dV$
- Center of mass (3D solid): $(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right)$
- Substitution for double integrals: If R is the image of G under a coordinate transformation $\mathbf{T}(u,v) = \langle x(u,v), y(u,v)\rangle$ then

$$\iint_R f(x,y) \, dx \, dy = \iint_G f(\mathbf{T}(u,v))|\det D\mathbf{T}(u,v)| \, du \, dv.$$

- Scalar line integral: $\int_C f(x,y,z) \, ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|dt$
- Tangential vector line integral: $\int_C \mathbf{F}\cdot\mathbf{T} \, ds = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t) \, dt$
- Normal vector line integral: $\int_C \mathbf{F}(x,y)\cdot\mathbf{n} \, ds = \int_C P \, dy - Q \, dx = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\langle y'(t), -x'(t)\rangle \, dt$.
- Fundamental Theorem of Line Integrals: $\int_C \nabla f\cdot d\mathbf{r} = f(B) - f(A)$ if C is a path from A to B
- Curl (Mixed Partials) Test: $\mathbf{F} = \nabla f$ if curl $\mathbf{F} = \mathbf{0} \Leftrightarrow P_z = R_x, Q_z = R_y$, and $Q_x = P_y$.
- $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle$ $\qquad$ div $\mathbf{F} = \nabla\cdot\mathbf{F}$ $\qquad$ curl $\mathbf{F} = \nabla\times\mathbf{F}$
- Green's Theorem: If C is a simple closed curve with positive orientation and R is the simply-connected region it encloses, then

$$\int_C \mathbf{F}\cdot\mathbf{T} \, ds = \iint_R (\nabla\times\mathbf{F})\cdot\mathbf{k} \, dA \qquad \int_C \mathbf{F}\cdot\mathbf{n} \, ds = \iint_R (\nabla\cdot\mathbf{F}) \, dA.$$

- Surface Area$=\iint_S 1 \, d\sigma$
- Scalar surface integral: $\iint_S f(x,y,z) \, d\sigma = \iint_R f(\mathbf{r}(u,v)) \, |\mathbf{r}_u\times\mathbf{r}_v|dA$
- Flux surface integral: $\iint_S \mathbf{F}\cdot\mathbf{n} \, d\sigma = \iint_S \mathbf{F}\cdot d\boldsymbol{\sigma} = \iint_R \mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u\times\mathbf{r}_v) \, dA$
- Stokes' Theorem: If S is a piecewise smooth oriented surface bounded by a piecewise smooth curve C and $\mathbf{F}$ is a vector field whose components have continuous partial derivatives on an open region containing S, then

$$\int_C \mathbf{F}\cdot\mathbf{T} \, ds = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n} \, d\sigma.$$

- Divergence Theorem: If S is a piecewise smooth closed oriented surface enclosing a volume D and $\mathbf{F}$ is a vector field whose components have continuous partial derivatives on D, then

$$\iint_S \mathbf{F}\cdot\mathbf{n} \, d\sigma = \iiint_D \nabla\cdot\mathbf{F} \, dV.$$

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