MATH 2551-K MIDTERM 3
VERSION A
FALL 2023
COVERS SECTIONS 15.5-15.8, 16.1-16.8

Full name: ______________________________ **GT ID:** ____________________

Honor code statement: I will abide strictly by the Georgia Tech honor code at all times. I will not use a calculator. I will not reference any website, application, or other CAS-enabled service. I will not consult with my notes or anyone during this exam. I will not provide aid to anyone else during this exam.

(   ) I attest to my integrity.

Read all instructions carefully before beginning.

- Print your name and GT ID neatly above.
- You have 75 minutes to take the exam.
- You may not use aids of any kind.
- Please show your work.
- Good luck! Write yourself a message of encouragement on the front page!

Question	Points
1	2
2	2
3	2
4	4
5	8
6	12
7	8
8	12
Total:	50

For problems 1-3 choose whether each statement is true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false. Be sure to neatly fill in the bubble corresponding to your answer choice.

1. (2 points) Suppose $\mathbf{F}(x, y)$ is a curl-free vector field ($\nabla \times \mathbf{F} = \mathbf{0}$) defined in all of $\mathbb{R}^2$ except $(0, 0)$. Then $\mathbf{F}$ is conservative.

   ○ **TRUE** ○ **FALSE**

2. (2 points) There exists a non-constant vector field $\mathbf{F}(x, y)$ with both $\nabla \cdot \mathbf{F} = 0$ and $\nabla \times \mathbf{F} = \mathbf{0}$.

   ○ **TRUE** ○ **FALSE**

3. (2 points) If $\mathbf{F}$ is any vector field and C is a circle, then the circulation of $\mathbf{F}$ around C traversed clockwise is the negative of the circulation of $\mathbf{F}$ around C traversed counterclockwise.

   ○ **TRUE** ○ **FALSE**

4. (4 points) Consider the double integral

$$\iint_R \frac{2x - y}{3y - x}\, dA,$$

where R is the region bounded by the lines $2x - y = 0$, $2x - y = 4$, $3y - x = 0$, $3y - x = 2$. Which of the following coordinate transformations would be appropriate to use for change of variables on this integral?

○ $u = \dfrac{y}{x}$
  $v = \dfrac{x}{3y}$

○ $x = u$
  $y = v$

○ $u = 2x - y$
  $v = 3y - x$

○ $x = 2u - v$
  $y = 3v - y$

5. (8 points) Compute the flux of the vector field $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j}$ through the open-ended circular cylinder of radius 4 and height 7 with its base on the xy-plane and centerered about the z-axis, oriented away from the z-axis.

6. In this problem, you will work with the path C which is the ellipse $\dfrac{x^2}{2^2} + \dfrac{y^2}{5^2} = 1$ oriented counterclockwise.

   (a) (3 points) Give a parameterization $\mathbf{r}(t)$ of the ellipse.

   (b) (4 points) Use any method to find the flux of the vector field $\mathbf{F} = \langle e^y, 2\cos(x) \rangle$ out of the interior of C.

   (c) (5 points) Use any method to find the circulation of the vector field $\mathbf{G} = \langle 2y, x \rangle$ around C.

7. Let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x, y, z) = \langle 3x + e^{z^2 y}, y + z, \arctan(x) - z \rangle.$$

(a) (6 points) Let S be the surface defined by

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{6}\right)^2 + \left(\frac{z}{2}\right)^2 = 1,$$

oriented with outward pointing normal vectors. Compute the flux of $\mathbf{F}$ across S.

Hint: The volume of the ellipsoid $(\frac{x}{a})^2 + (\frac{y}{b})^2 + (\frac{z}{c})^2 \leq 1$ *is* $\frac{4}{3}\pi abc$.

(b) (2 points) Is there a vector field $\mathbf{G}$ such that $\mathbf{F} = \nabla \times \mathbf{G}$? Explain your answer.

8. Consider the volume D in the second octant of $\mathbb{R}^3$ ($x \leq 0, y \geq 0, z \geq 0$) which is bounded above by the sphere $x^2 + y^2 + z^2 = 2$ and below by the paraboloid $z = x^2 + y^2$.

   (a) (4 points) Write an integral for the volume of D using Cartesian coordinates in the order $dz\ dy\ dx$. Show your work. **Do not evaluate your integral**.

   (b) (4 points) Write an integral for the volume of D using cylindrical coordinates. Show your work. **Do not evaluate your integral**.

Consider the volume D in the second octant ($x \leq 0, y \geq 0, z \geq 0$) of $\mathbb{R}^3$ which is bounded above by the sphere $x^2+y^2+z^2=2$ and below by the paraboloid $z=x^2+y^2$.

(c) (2 points) Sketch the shadow of D in the yz-plane.

(d) (2 points) Are spherical coordinates a good choice to use to compute this volume? Justify your answer.

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FORMULA SHEET

- Trig identities: $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$, $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$
- Volume$(D) = \iiint_D dV$, $\quad f_{avg} = \dfrac{\iiint_D f(x,y,z)dV}{\text{volume of } D}$, $\quad$ Mass: $M = \iiint_D \delta\ dV$
- Cylindrical coordinates: $x = r\cos(\theta)$, $\quad y = r\sin(\theta)$, $z = z$, $dV = r\ dz\ dr\ d\theta$
- Spherical coordinates: $x = \rho\sin(\phi)\cos(\theta)$, $y = \rho\sin(\phi)\sin(\theta)$, $z = \rho\cos(\phi)$, $dV = \rho^2\sin(\phi)\ d\rho\ d\phi\ d\theta$
- First moments (3D solid): $M_{yz} = \iiint_D x\delta\ dV$, $M_{xz} = \iiint_D y\delta\ dV$, $M_{xy} = \iiint_D z\delta\ dV$
- Center of mass (3D solid): $(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right)$
- Substitution for double integrals: If R is the image of G under a coordinate transformation $\mathbf{T}(u,v) = \langle x(u,v), y(u,v)\rangle$ then

$$\iint_R f(x,y)\ dx\ dy = \iint_G f(\mathbf{T}(u,v))|\det D\mathbf{T}(u,v)|\ du\ dv.$$

- Scalar line integral: $\int_C f(x,y,z)\ ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|dt$
- Tangential vector line integral: $\int_C \mathbf{F}\cdot\mathbf{T}\ ds = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\ dt$
- Normal vector line integral: $\int_C \mathbf{F}(x,y)\cdot\mathbf{n}\ ds = \int_C P\ dy - Q\ dx = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\langle y'(t), -x'(t)\rangle\ dt$.
- Fundamental Theorem of Line Integrals: $\int_C \nabla f\cdot d\mathbf{r} = f(B) - f(A)$ if C is a path from A to B
- Curl (Mixed Partials) Test: $\mathbf{F} = \nabla f$ if curl $\mathbf{F} = \mathbf{0} \Leftrightarrow P_z = R_x, Q_z = R_y$, and $Q_x = P_y$.
- $\nabla = \langle\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle$ $\qquad$ div $\mathbf{F} = \nabla\cdot\mathbf{F}$ $\qquad$ curl $\mathbf{F} = \nabla\times\mathbf{F}$
- Green's Theorem: If C is a simple closed curve with positive orientation and R is the simply-connected region it encloses, then

$$\int_C \mathbf{F}\cdot\mathbf{T}\ ds = \iint_R (\nabla\times\mathbf{F})\cdot\mathbf{k}\ dA \qquad \int_C \mathbf{F}\cdot\mathbf{n}\ ds = \iint_R (\nabla\cdot\mathbf{F})\ dA.$$

- Surface Area$=\iint_S 1\ d\sigma$
- Scalar surface integral: $\iint_S f(x,y,z)\ d\sigma = \iint_R f(\mathbf{r}(u,v))\ |\mathbf{r}_u\times\mathbf{r}_v|dA$
- Flux surface integral: $\iint_S \mathbf{F}\cdot\mathbf{n}\ d\sigma = \iint_S \mathbf{F}\cdot d\boldsymbol{\sigma} = \iint_R \mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u\times\mathbf{r}_v)\ dA$
- Stokes' Theorem: If S is a piecewise smooth oriented surface bounded by a piecewise smooth curve C and $\mathbf{F}$ is a vector field whose components have continuous partial derivatives on an open region containing S, then

$$\int_C \mathbf{F}\cdot\mathbf{T}\ ds = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\ d\sigma.$$

- Divergence Theorem: If S is a piecewise smooth closed oriented surface enclosing a volume D and $\mathbf{F}$ is a vector field whose components have continuous partial derivatives on D, then

$$\iint_S \mathbf{F}\cdot\mathbf{n}\ d\sigma = \iiint_D \nabla\cdot\mathbf{F}\ dV.$$

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