MATH 2551-K MIDTERM 2
VERSION A
FALL 2023
COVERS SECTIONS 14.2-14.8, 15.1-15.4

Full name: ______________________________ **GT ID:**____________________

Honor code statement: I will abide strictly by the Georgia Tech honor code at all times. I will not use a calculator. I will not reference any website, application, or other CAS-enabled service. I will not consult with my notes or anyone during this exam. I will not provide aid to anyone else during this exam.

(   ) I attest to my integrity.

Read all instructions carefully before beginning.

- Print your name and GT ID neatly above.
- You have 75 minutes to take the exam.
- You may not use aids of any kind.
- Please show your work.
- Good luck! Write yourself a message of encouragement on the front page!

Question	Points
1	2
2	2
3	2
4	4
5	10
6	10
7	10
8	10
Total:	50

For problems 1-3 choose whether each statement is true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false. Be sure to neatly fill in the bubble corresponding to your answer choice.

1. (2 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$. If $\lim_{(x,y)\to(0,0)} f(x, y) = 2$ along the x-axis and $\lim_{(x,y)\to(0,0)} f(x, y) = -2$ along the line $y = 2x$, then the overall limit $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist.

   √ **TRUE** ○ **FALSE**

2. (2 points) Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (a, b). Then f is also continuous at (a, b).

   √ **TRUE** ○ **FALSE**

3. (2 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$. Suppose $Df(1, 1) = \begin{bmatrix} 0 & 0.0000000001 \end{bmatrix}$ and $Hf(1, 1) = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$. Then $(1, 1)$ is the location of a local minimum of f.

   ○ **TRUE** √ **FALSE**

4. (4 points) Set up but do not solve the system of equations that results from applying the method of Lagrange multipliers to find the pair of numbers x and y whose sum is constrained to be 20 that have the largest product. Clearly indicate your objective function and the constraint equation.

Solution: Our objective function is $f(x, y) = xy$ and our constraint is $g(x, y) = x+y = 20$. Applying the method of Lagrange multipliers results in

$$\begin{aligned} y &= \lambda \\ x &= \lambda \\ x + y &= 20 \end{aligned}$$

5. In this problem, you will determine the largest and smallest temperatures attained on the region R bounded on the left by the y-axis and on the right by the circle $x^2 + y^2 = 4$ if temperature $T : \mathbb{R}^2 \to \mathbb{R}$ is given by $T(x, y) = x^2 + y^2 - 2x + 10$ degrees C.

   (a) (2 points) Find the critical points of T inside the region R.

   Solution: $DT = \begin{bmatrix} 2x - 2 & 2y \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$
   Solving this gives $x = 1$ and $y = 0$, which is in the region R.

   (b) (6 points) Simplify the function on each boundary curve of R and find any critical points on the boundary segments. It may help to sketch R.

   Solution: On $x = 0$: $T(0, y) = y^2 + 10$ for $-2 \leq y \leq 2$. We have $T'(y) = 2y = 0$, so $(0, 0)$ is a test point. We also include the endpoints $(0, \pm 2)$

   On $x^2 + y^2 = 4$: $T = 4 - 2x + 10 = 14 - 2x$ for $0 \leq x \leq 2$. We have $T'(x) = -2 \neq 0$, so there are no critical points on the semicircle. We do include the endpoints where $x = 0$ or $x = 2$. The points with $x = 0$ are $(0, \pm 2)$ and the point with $x = 2$ is $(2, 0)$.

   (c) (2 points) Use your work above to find the largest and smallest temperatures attained on the region. Be sure to include units.

   Solution: We evaluate T on the points found above:

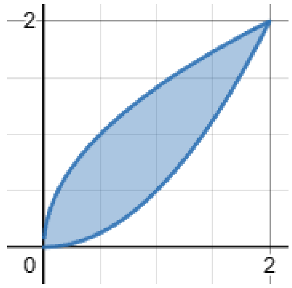
(x, y)	$T(x, y)$
$(1, 0)$	9
$(0, 0)$	10
$(0, -2)$	14
$(0, 2)$	14
$(2, 0)$	10

   So $T_{min} = 9$ degrees C and $T_{max} = 14$ degrees C.

6. In this problem, you will compute the volume of the solid bounded above by $z = 3xy$, below by the xy-plane, and lying above the region R contained between the curves $2y = x^2$ and $2x = y^2$.

(a) (2 points) Sketch the region R. Label your axes and each curve.

Solution:



(b) (2 points) Is R horizontally simple, vertically simple, both, or neither?

Solution: R is both horizontally and vertically simple.

(c) (6 points) Write a double iterated integral to find the volume of the solid, then evaluate it.

Solution: We can choose either order of integration, which results in one of the following two integrals.

$$\int_0^2 \int_{x^2/2}^{\sqrt{2x}} 3xy \; dy \; dx$$

or

$$\int_0^2 \int_{y^2/2}^{\sqrt{2y}} 3xy \; dx \; dy.$$

Evaluating, we have

$$\begin{aligned}
\int_0^2 \int_{x^2/2}^{\sqrt{2x}} 3xy \; dy \; dx &= \int_0^2 \frac{3}{2}xy^2|_{y=x^2/2}^{y=\sqrt{2x}} \; dx \\
&= \int_0^2 3x^2 - \frac{3x^5}{8} \; dx \\
&= x^3 - \frac{x^6}{16}|_0^2 \\
&= 8 - \frac{64}{16} - 0 + 0 = 4
\end{aligned}$$

7. In this problem, you will work with the function $g : \mathbb{R}^3 \to \mathbb{R}$, $g(x, y, z) = x^4 + y^3 + z^2$ and the point $P = (-2, 1, 2)$ in the domain of g.

   (a) (2 points) Suppose that you are only able to travel away from P in one of the following directions. Which direction (assuming you move with unit speed) will yield the greatest instantaneous increase in g?

   (A) parallel to the x-axis, with x increasing
   (B) parallel to the y-axis, with y increasing
   (C) parallel to the z-axis, with z increasing
   (D) directly away from the origin

   Solution: (D) is correct.

   (b) (5 points) Justify your answer to part (a).

   Solution: This problem is asking in which of the given directions is the directional derivative of g greatest. So we compute.

   $Dg(x, y, z) = \begin{bmatrix} 4x^3 & 3y^2 & 2z \end{bmatrix}$, so $Dg(P) = \begin{bmatrix} -32 & 3 & 4 \end{bmatrix}$.

   We also need a unit vector in each direction; for (A)-(C) these are the standard unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ and for (D) it is the vector $\mathbf{u} = \vec{OP}/|\vec{OP}| = \frac{1}{3}\langle -2, 1, 2\rangle$.

   We then have:

$$\begin{aligned} D_{\mathbf{i}}g(P) &= Dg(P)\mathbf{i} = -32 \\ D_{\mathbf{j}}g(P) &= Dg(P)\mathbf{j} = 3 \\ D_{\mathbf{k}}g(P) &= Dg(P)\mathbf{k} = 4 \\ D_{\mathbf{u}}g(P) &= Dg(P)\mathbf{u} = \frac{1}{3}(-2(-32) + 1(3) + 2(4)) = 25 \end{aligned}$$

   Of these, 25 is the largest value, so the direction (D) yields the greatest instantaneous increase in g.

   (c) (3 points) The point P is on a level surface of $g(x, y, z)$. Find an equation for the tangent plane to this surface at P.

   Solution: The normal vector to this plane is $\nabla g(P) = \langle -32, 3, 4\rangle$. Thus the plane is:

$$-32(x + 2) + 3(y - 1) + 4(z - 2) = 0$$

   or

$$-32x + 3y + 4z = 75$$

8. (a) (6 points) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the function $f(x, y, z) = 2xy + z^2$ and $\mathbf{r}(s, t) : \mathbb{R}^2 \to \mathbb{R}^3$ be the function

$$\mathbf{r}(s,t) = \begin{bmatrix} s+t \\ s-t \\ 2st \end{bmatrix}.$$

Use the Chain Rule to compute the derivative (either the total derivative or all relevant partial derivatives) of the composite function $g : \mathbb{R}^2 \to \mathbb{R}$ where $g(s, t) = f(\mathbf{r}(s, t))$ at the point $(s, t) = (1, -1)$.

Solution: To apply the Chain Rule, we need to compute the total derivatives of both functions.

$$Df = \begin{bmatrix} 2y & 2x & 2z \end{bmatrix} \qquad D\mathbf{r} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2t & 2s \end{bmatrix}$$

Now we evaluate Df at $\mathbf{r}(1, -1)$ and $D\mathbf{r}$ at $(1, -1)$.

$$\mathbf{r}(1,-1) = \langle 0, 2, -2 \rangle, \qquad Df(\mathbf{r}(1,-1)) = \begin{bmatrix} 4 & 0 & -4 \end{bmatrix} \qquad D\mathbf{r}(1,-1) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 2 \end{bmatrix}$$

Thus

$$Dg(1,-1) = Df(\mathbf{r}(1,-1))D\mathbf{r}(1,-1) = \begin{bmatrix} 4 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 12 & -4 \end{bmatrix}$$

(b) (4 points) Convert the integral $\displaystyle\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} xy \; dy \; dx$ to an equivalent integral in polar coordinates.

Solution: We have $-\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2}$ and $0 \leq x \leq 2$, so this region of integration is the right half of a circle of radius 2 centered at the origin.
Thus the integral becomes

$$\int_{-\pi/2}^{\pi/2} \int_0^2 (r\cos(\theta))(r\sin(\theta))r \; dr \; d\theta.$$

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FORMULA SHEET

- Total Derivative: For $\mathbf{f}(x_1, \ldots, x_n) = \langle f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n) \rangle$

$$D\mathbf{f} = \begin{bmatrix} (f_1)_{x_1} & (f_1)_{x_2} & \cdots & (f_1)_{x_n} \\ (f_2)_{x_1} & (f_2)_{x_2} & \cdots & (f_2)_{x_n} \\ \vdots & \ddots & \cdots & \vdots \\ (f_m)_{x_1} & (f_m)_{x_2} & \cdots & (f_m)_{x_n} \end{bmatrix}$$

- Linearization: Near $\mathbf{a}$, $L(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a})$

- Chain Rule: If $h = g(f(\mathbf{x}))$ then $Dh(\mathbf{x}) = Dg(f(\mathbf{x}))Df(\mathbf{x})$

- Implicit Differentiation: $\dfrac{\partial z}{\partial x} = \dfrac{-F_x}{F_z}$ and $\dfrac{\partial z}{\partial y} = \dfrac{-F_y}{F_z}$.

- Directional Derivative: If $\mathbf{u}$ is a unit vector, $D_{\mathbf{u}}f(P) = Df(P)\mathbf{u} = \nabla f(P) \cdot \mathbf{u}$

- The tangent line to a level curve of $f(x, y)$ at (a, b) is $0 = \nabla f(a, b) \cdot \langle x - a, y - b \rangle$

- The tangent plane to a level surface of $f(x, y, z)$ at (a, b, c) is

$$0 = \nabla f(a, b, c) \cdot \langle x - a, y - b, z - c \rangle.$$

- Hessian Matrix: For $f(x, y)$, $Hf(x, y) = \begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix}$

- Second Derivative Test: If (a, b) is a critical point of $f(x, y)$ then
  1. If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) < 0$ then f has a local maximum at (a, b)
  2. If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) > 0$ then f has a local minimum at (a, b)
  3. If $\det(Hf(a, b)) < 0$ then f has a saddle point at (a, b)
  4. If $\det(Hf(a, b)) = 0$ the test is inconclusive

- Area/volume: $\text{area}(R) = \iint_R dA$, $\quad \text{volume}(D) = \iiint_D dV$

- Trig identities: $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$, $\quad \cos^2(x) = \frac{1}{2}(1 + \cos(2x))$

- Average value: $f_{avg} = \dfrac{\iint_R f(x, y)dA}{\text{area of } R}$

- Polar coordinates: $x = r\cos(\theta)$, $\quad y = r\sin(\theta)$, $\quad dA = r\ dr\ d\theta$

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