MATH 2551-K MIDTERM 2
VERSION A
FALL 2023
COVERS SECTIONS 14.2-14.8, 15.1-15.4

Full name: ______________________________ **GT ID:**____________________

Honor code statement: I will abide strictly by the Georgia Tech honor code at all times. I will not use a calculator. I will not reference any website, application, or other CAS-enabled service. I will not consult with my notes or anyone during this exam. I will not provide aid to anyone else during this exam.

(   ) I attest to my integrity.

Read all instructions carefully before beginning.

- Print your name and GT ID neatly above.
- You have 75 minutes to take the exam.
- You may not use aids of any kind.
- Please show your work.
- Good luck! Write yourself a message of encouragement on the front page!

Question	Points
1	2
2	2
3	2
4	4
5	10
6	10
7	10
8	10
Total:	50

For problems 1-3 choose whether each statement is true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false. Be sure to neatly fill in the bubble corresponding to your answer choice.

1. (2 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$. If $\lim_{(x,y)\to(0,0)} f(x,y) = 2$ along the x-axis and $\lim_{(x,y)\to(0,0)} f(x,y) = -2$ along the line $y = 2x$, then the overall limit $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

   ◯ **TRUE** ◯ **FALSE**

2. (2 points) Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (a, b). Then f is also continuous at (a, b).

   ◯ **TRUE** ◯ **FALSE**

3. (2 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$. Suppose $Df(1,1) = \begin{bmatrix} 0 & 0.0000000001 \end{bmatrix}$ and $Hf(1,1) = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$. Then $(1,1)$ is the location of a local minimum of f.

   ◯ **TRUE** ◯ **FALSE**

4. (4 points) Set up but do not solve the system of equations that results from applying the method of Lagrange multipliers to find the pair of numbers x and y whose sum is constrained to be 20 that have the largest product. Clearly indicate your objective function and the constraint equation.

5. In this problem, you will determine the largest and smallest temperatures attained on the region R bounded on the left by the y-axis and on the right by the circle $x^2 + y^2 = 4$ if temperature $T : \mathbb{R}^2 \to \mathbb{R}$ is given by $T(x, y) = x^2 + y^2 - 2x + 10$ degrees C.

   (a) (2 points) Find the critical points of T inside the region R.

   (b) (6 points) Simplify the function on each boundary curve of R and find any critical points on the boundary segments. It may help to sketch R.

   (c) (2 points) Use your work above to find the largest and smallest temperatures attained on the region. Be sure to include units.

6. In this problem, you will compute the volume of the solid bounded above by $z = 3xy$, below by the xy-plane, and lying above the region R contained between the curves $2y = x^2$ and $2x = y^2$.

   (a) (2 points) Sketch the region R. Label your axes and each curve.

   (b) (2 points) Is R horizontally simple, vertically simple, both, or neither?

   (c) (6 points) Write a double iterated integral to find the volume of the solid, then evaluate it.

7. In this problem, you will work with the function $g : \mathbb{R}^3 \to \mathbb{R}$, $g(x, y, z) = x^4 + y^3 + z^2$ and the point $P = (-2, 1, 2)$ in the domain of g.

   (a) (2 points) Suppose that you are only able to travel away from P in one of the following directions. Which direction (assuming you move with unit speed) will yield the greatest instantaneous increase in g?

   (A) parallel to the x-axis, with x increasing
   (B) parallel to the y-axis, with y increasing
   (C) parallel to the z-axis, with z increasing
   (D) directly away from the origin

   (b) (5 points) Justify your answer to part (a).

   (c) (3 points) The point P is on a level surface of $g(x, y, z)$. Find an equation for the tangent plane to this surface at P.

8. (a) (6 points) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the function $f(x, y, z) = 2xy + z^2$ and $\mathbf{r}(s, t) : \mathbb{R}^2 \to \mathbb{R}^3$ be the function

$$\mathbf{r}(s,t) = \begin{bmatrix} s+t \\ s-t \\ 2st \end{bmatrix}.$$

Use the Chain Rule to compute the derivative (either the total derivative or all relevant partial derivatives) of the composite function $g : \mathbb{R}^2 \to \mathbb{R}$ where $g(s, t) = f(\mathbf{r}(s, t))$ at the point $(s, t) = (1, -1)$.

(b) (4 points) Convert the integral $\displaystyle\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} xy \; dy \; dx$ to an equivalent integral in polar coordinates.

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FORMULA SHEET

- Total Derivative: For $\mathbf{f}(x_1, \ldots, x_n) = \langle f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)\rangle$

$$D\mathbf{f} = \begin{bmatrix} (f_1)_{x_1} & (f_1)_{x_2} & \cdots & (f_1)_{x_n} \\ (f_2)_{x_1} & (f_2)_{x_2} & \cdots & (f_2)_{x_n} \\ \vdots & \ddots & \cdots & \vdots \\ (f_m)_{x_1} & (f_m)_{x_2} & \cdots & (f_m)_{x_n} \end{bmatrix}$$

- Linearization: Near $\mathbf{a}$, $L(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a})$

- Chain Rule: If $h = g(f(\mathbf{x}))$ then $Dh(\mathbf{x}) = Dg(f(\mathbf{x}))Df(\mathbf{x})$

- Implicit Differentiation: $\dfrac{\partial z}{\partial x} = \dfrac{-F_x}{F_z}$ and $\dfrac{\partial z}{\partial y} = \dfrac{-F_y}{F_z}$.

- Directional Derivative: If $\mathbf{u}$ is a unit vector, $D_{\mathbf{u}}f(P) = Df(P)\mathbf{u} = \nabla f(P) \cdot \mathbf{u}$

- The tangent line to a level curve of $f(x, y)$ at (a, b) is $0 = \nabla f(a, b) \cdot \langle x - a, y - b\rangle$

- The tangent plane to a level surface of $f(x, y, z)$ at (a, b, c) is

$$0 = \nabla f(a, b, c) \cdot \langle x - a, y - b, z - c\rangle.$$

- Hessian Matrix: For $f(x, y)$, $Hf(x, y) = \begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix}$

- Second Derivative Test: If (a, b) is a critical point of $f(x, y)$ then
  1. If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) < 0$ then f has a local maximum at (a, b)
  2. If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) > 0$ then f has a local minimum at (a, b)
  3. If $\det(Hf(a, b)) < 0$ then f has a saddle point at (a, b)
  4. If $\det(Hf(a, b)) = 0$ the test is inconclusive

- Area/volume: $\text{area}(R) = \iint_R dA$, $\quad \text{volume}(D) = \iiint_D dV$

- Trig identities: $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$, $\quad \cos^2(x) = \frac{1}{2}(1 + \cos(2x))$

- Average value: $f_{avg} = \dfrac{\iint_R f(x, y)dA}{\text{area of } R}$

- Polar coordinates: $x = r\cos(\theta)$, $\quad y = r\sin(\theta)$, $\quad dA = r\ dr\ d\theta$

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